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INTEGRABILITY OF DISTRIBUTION $D^{\perp}$ ON A NEARLY SASAKIAN MANIFOLD

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Abstract. In this paper, we give some sufficient and necessary conditions for integrability of distribution $D^{\perp}$ on a nearly Sasakian manifold, and generalize Bejancu's result.

1. Introduction

Let $\overline{M}$ be a real $(2n+1)$-dimensional almost contact metric manifold with the structure tensors (Φ, ξ, η, g), then

$$\Phi\xi = 0, \eta(\xi) = 1, \Phi^2 = -I + \eta \otimes \xi, \eta(X) = g(X, \xi), \tag{1.1}$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y), \eta \circ \Phi = 0. \tag{1.2}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.1 ([1])**.** The Nijenhuis tensor field of Φ on an almost contact metric manifold is defined by

$$[\Phi, \Phi](X, Y) = [\Phi X, \Phi Y] + \Phi^2[X, Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y], \tag{1.3}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.2 ([1])**.** An almost contact metric manifold $\overline{M}$ is called a nearly Sasakian manifold, if we have

$$(\overline{\nabla}_X \Phi)Y + (\overline{\nabla}_Y \Phi)X = 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y, \tag{1.4}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 1.3. An almost contact metric manifold $\overline{M}$ is called a Sasakian manifold, if we have

$$(\overline{\nabla}_X \Phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{1.5}$$

for any $X, Y \in \Gamma(T\overline{M})$.

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Obviously, a Sasakian manifold is a nearly Sasakian manifold.

Let M be an m-dimensional submanifold of an n-dimensional almost contact metric manifold $\overline{M}$. We denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$, denote by ∇ the induced connection on M, and denote by $\nabla^{\perp}$ the normal connection on M. Thus, for any $X, Y \in \Gamma(TM)$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.6}$$

where $h\colon \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM^{\perp})$ is a normal bundle valued symmetric bilinear form on $\Gamma(TM)$. The equation (1.6) is called the Gauss formula and h is called the second fundamental form of M.

Now, for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$,we denote by $-A_V X$ and $\nabla^{\perp}_X V$ the tangent part and normal part of $\overline{\nabla}_X V$ respectively. Then we have

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V. \tag{1.7}$$

Thus, for any $V \in \Gamma(TM^{\perp})$, we have a linear operator, satisfying

$$g(A_V X, Y) = g(X, A_V Y) = g(h(X, Y), V). \tag{1.8}$$

The equation (1.7) is called the Weingarten formula.

An m-dimensional distribution on a manifold $\overline{M}$ is a mapping D defined on $\overline{M}$, which assigns to each point x of $\overline{M}$ an m-dimensional linear subspace D_x of $T_x\overline{M}$. A vector field X on $\overline{M}$ belongs to D if we have $X_x{\in}D_x$ for each $x \in \overline{M}$. When this happens we write $X \in \Gamma(D)$. The distribution D is said to be differentiable if for any $x \in \overline{M}$ there exist m differentiable linearly independent vector fields $X_i \in \Gamma(D)$ in a neighborhood of x. From now on, all distribution are supposed to be differentiable of class C^{∞}.

The distribution D is said to be involutive if for all vector fields $X, Y \in \Gamma(D)$ we have $[X, Y] \in \Gamma(D)$. A submanifold M of $\overline{M}$ is said to be an integral manifold of D if for every point $x \in M$, D_x coincides with the tangent space to M at x. If there exists no integral manifold of D which contains M, then M is called a maximal integral manifold or a leaf of D. The distribution D is said to be integrable if for every $x \in \overline{M}$ there exists an integral manifold of D containing x.

Definition 1.4 ([1])**.** Let M be a real $(2m{+}1)$-dimensional submanifold of a real $(2n{+}1)$-dimensional almost contact metric manifold $\overline{M}$ with the structure tensors (Φ, ξ, η, g). We assume that the structure tensor ξ is tangent to M, and denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M. Then M is called a semi-invariant submanifold of $\overline{M}$, if there exist two differentiable distributions D and $D^{\perp}$ on M, satisfying

(1) $TM = D \oplus D^{\perp} \oplus \{\xi\}$, where D, $D^{\perp}$ and $\{\xi\}$ are mutually orthogonal to each other;
(2) the distribution D is invariant by Φ, that is, $\Phi(D_x) = D_x$, for each $x \in M$;
(3) the distribution $D^{\perp}$ is anti-invariant by Φ, that is, $\Phi(D^{\perp}_x) \subset T_xM^{\perp}$, for each $x \in M$.

For each vector field X tangent to M, we put

$$\Phi X = \psi X + \omega X, \tag{1.9}$$

where ψX and ωX are respectively the tangent part and the normal part of ΦX. Also, for each vector field V normal to M, we put

$$\Phi V = BV + CV, \tag{1.10}$$

where BV and CV are respectively the tangent part and the normal part of ΦV.

In paper [1], we know that the distribution $D^{\perp}$ on M is integrable if and only if $[X, Y] \in \Gamma(D^{\perp})$, for all vector fields $X, Y \in \Gamma(D^{\perp})$.

2. Main Results

Theorem 2.1. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\overline{M}$. Then the distribution $D^{\perp}$ is integrable if and only if*

$$g(A_{\Phi Y}X - A_{\Phi X}Y + 2(\overline{\nabla}_X\Phi)Y, \Phi Z) = \eta([X, Y])\eta(Z), \tag{2.1}$$

for any $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D \oplus \{\xi\})$.

Proof. By using (1.7) we obtain

$$\overline{\nabla}_X\Phi Y = -A_{\Phi Y}X + \nabla^{\perp}_X\Phi Y, \tag{2.2}$$

for any $X, Y \in \Gamma(D^{\perp})$. On the other hand, by using (1.6) we also obtain

$$\overline{\nabla}_X\Phi Y = (\overline{\nabla}_X\Phi)Y + \Phi\overline{\nabla}_X Y = (\overline{\nabla}_X\Phi)Y + \Phi\nabla_X Y + \Phi h(X, Y). \tag{2.3}$$

By comparing (2.2) and (2.3) we get

$$(\overline{\nabla}_X\Phi)Y = -A_{\Phi Y}X + \nabla^{\perp}_X\Phi Y - \Phi\nabla_X Y - \Phi h(X, Y). \tag{2.4}$$

By changing X and Y in (2.4) we have

$$(\overline{\nabla}_Y\Phi)X = -A_{\Phi X}Y + \nabla^{\perp}_Y\Phi X - \Phi\nabla_Y X - \Phi h(Y, X). \tag{2.5}$$

By using (2.4) and (2.5) we obtain

$$(\overline{\nabla}_X\Phi)Y - (\overline{\nabla}_Y\Phi)X = -A_{\Phi Y}X + A_{\Phi X}Y + \nabla^{\perp}_X\Phi Y - \nabla^{\perp}_Y\Phi X - \Phi[X, Y]. \tag{2.6}$$

By using (1.4)+(2.6) we get

$$2(\overline{\nabla}_X\Phi)Y = 2g(X, Y)\xi - A_{\Phi Y}X + A_{\Phi X}Y + \nabla^{\perp}_X\Phi Y - \nabla^{\perp}_Y\Phi X - \Phi[X, Y]. \tag{2.7}$$

That is,

$$\Phi[X, Y] = 2g(X, Y)\xi - A_{\Phi Y}X + A_{\Phi X}Y + \nabla^{\perp}_X\Phi Y - \nabla^{\perp}_Y\Phi X - 2(\overline{\nabla}_X\Phi)Y. \tag{2.8}$$

For any $Z \in \Gamma(D \oplus \{\xi\})$, then $\Phi Z \in \Gamma(D)$. Hence, we have

$$g(2g(X, Y)\xi + \nabla^{\perp}_X\Phi Y - \nabla^{\perp}_Y\Phi X, \Phi Z) = 0. \tag{2.9}$$

By using (2.8), (1.2) and (2.9) we obtain

$$g([X, Y], Z) = g(-A_{\Phi Y}X + A_{\Phi X}Y - 2(\overline{\nabla}_X\Phi)Y, \Phi Z) + \eta([X, Y])\eta(Z). \tag{2.10}$$

Thus, $[X, Y] \in \Gamma(D^{\perp})$ holds if and only if (2.1) is satisfied. □

Lemma 2.1 ([1]). *Let M be a semi-invariant submanifold of a Sasakian manifold $\overline{M}$. Then*

$$A_{\Phi X}Y = A_{\Phi Y}X \tag{2.11}$$

and

$$[X,Y] \in \Gamma(D \oplus D^{\perp}), \tag{2.12}$$

for any $X, Y \in \Gamma(D^{\perp})$.

Corollary (Bejancu-Papaghiuc [1]). *Let M be a semi-invariant submanifold of a Sasakian manifold $\overline{M}$. Then the distribution $D^{\perp}$ is integrable.*

Proof. For any $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D \oplus \{\xi\})$, then $\Phi Z \in \Gamma(D)$ holds. By using Lemma 2.1 and (1.1) we obtain

$$g(-A_{\Phi Y}X + A_{\Phi X}Y, \Phi Z) = 0, \eta([X,Y]) = 0. \tag{2.13}$$

On the other hand, by using (1.5) we get

$$g(2(\overline{\nabla}_X \Phi)Y, \Phi Z) = 0. \tag{2.14}$$

Taking into account that a Sasakian manifold is a nearly Sasakian manifold,by using (2.13) and (2.14) we have (2.1). By using theorem 2.1, then the distribution $D^{\perp}$ is integrable. □

Theorem 2.2. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\overline{M}$. Then the distribution $D^{\perp}$ is integrable if and only if*

$$g(A_{\Phi X}Y - A_{\Phi Y}X - 2\Phi\nabla_X Y, \Phi Z) = \eta([X,Y])\eta(Z), \tag{2.15}$$

for any $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D \oplus \{\xi\})$.

Proof. By using (2.4) and (2.8), we obtain

$$\begin{aligned} &\Phi[X,Y] = \\ &\quad 2g(X,Y)\xi + A_{\Phi Y}X - A_{\Phi X}Y + \nabla^{\perp}_X \Phi Y - \nabla^{\perp}_Y \Phi X + 2\Phi\nabla_X Y + 2\Phi h(X,Y), \end{aligned} \tag{2.16}$$

for any $X, Y \in \Gamma(D^{\perp})$. For any $Z \in \Gamma(D \oplus \{\xi\})$, then $\Phi Z \in \Gamma(D)$. hence, we have

$$g(2g(X,Y)\xi + \nabla^{\perp}_X \Phi Y - \nabla^{\perp}_Y \Phi X, \Phi Z) = 0. \tag{2.17}$$

From (1.2), we get

$$g(2\Phi h(X,Y), \Phi Z) = 0. \tag{2.18}$$

By using (1.2), (2.16), (2.17) and (2.18), we obtain

$$g([X,Y],Z) = -g(A_{\Phi X}Y - A_{\Phi Y}X - 2\Phi\nabla_X Y, \Phi Z) + \eta([X,Y])\eta(Z). \tag{2.19}$$

Thus, $[X,Y] \in \Gamma(D^{\perp})$ holds if and only if (2.15) is satisfied. □

Lemma 2.2 ([4]). *Let $\overline{M}$ be a nearly Sasakian Manifold. Then*

$$
\begin{aligned}
[\Phi,\Phi](X,Y) &= 4\Phi(\overline{\nabla}_Y\Phi)X + \eta(Y)\overline{\nabla}_X\xi + X\eta(Y)\xi - \eta(\overline{\nabla}_XY)\xi \\
&- 2g(X,\Phi Y)\xi + \eta(X)\Phi Y - \eta(X)\overline{\nabla}_Y\xi - Y\eta(X)\xi + \eta(\overline{\nabla}_YX)\xi \\
&+ 2g(Y,\Phi X)\xi - \eta(Y)\Phi X + 4g(X,Y)\xi - 2\eta(Y)X - 2\eta(X)Y,
\end{aligned}
\tag{2.20}
$$

for any $X, Y \in \Gamma(T\overline{M})$.

Lemma 2.3. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\overline{M}$. Then*

$$
\begin{aligned}
2\Phi[X,Y] = &-4g(X,Y)\xi - 2A_{\Phi Y}X + 2A_{\Phi X}Y + 2\nabla^{\perp}_X\Phi Y \\
&- 2\nabla^{\perp}_Y\Phi X - \Phi[\Phi,\Phi](X,Y) + 4\eta((\overline{\nabla}_Y\Phi)X)\xi,
\end{aligned}
\tag{2.21}
$$

for any $X, Y \in \Gamma(D^{\perp})$.

Proof. By using (2.20) and (1.1), we obtain

$$
\Phi[\Phi,\Phi](X,Y) = 4\Phi^2(\overline{\nabla}_Y\Phi)X = -4(\overline{\nabla}_Y\Phi)X + 4\eta((\overline{\nabla}_Y\Phi)X)\xi,
\tag{2.22}
$$

for any $X, Y \in \Gamma(D^{\perp})$. From (2.22) and (2.7), we get

$$
\begin{aligned}
\Phi[\Phi,\Phi](X,Y) = &-4g(X,Y)\xi - 2A_{\Phi Y}X + 2A_{\Phi X}Y + 2\nabla^{\perp}_X\Phi Y \\
&- 2\nabla^{\perp}_Y\Phi X - 2\Phi[X,Y] + 4\eta((\overline{\nabla}_Y\Phi)X)\xi.
\end{aligned}
\tag{2.23}
$$

By using (2.23), we have (2.21). □

Theorem 2.3. *Let M be a semi-invariant submanifold of a nearly Sasakian manifold $\overline{M}$. Then the distribution $D^{\perp}$ is integrable if and only if*

$$
g(2A_{\Phi Y}X - 2A_{\Phi X}Y + \Phi[\Phi,\Phi](X,Y), \Phi Z) = \eta([X,Y])\eta(Z),
\tag{2.24}
$$

for any $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D \oplus \{\xi\})$.

Proof. For any $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D \oplus \{\xi\})$, then $\Phi Z \in \Gamma(D)$. Hence, we obtain

$$
g(-4g(X,Y)\xi + 2\nabla^{\perp}_X\Phi Y - 2\nabla^{\perp}_Y\Phi X + 4\eta((\overline{\nabla}_Y\Phi)X)\xi, \Phi Z) = 0.
\tag{2.25}
$$

By using (2.21), (1.2) and (2.25), we get

$$
\begin{aligned}
2g([X,Y],Z) &= g(2\Phi[X,Y],\Phi Z) + 2\eta([X,Y])\eta(Z) \\
&= g(-2A_{\Phi Y}X + 2A_{\Phi X}Y - \Phi[\Phi,\Phi](X,Y), \Phi Z) + \eta([X,Y])\eta(Z).
\end{aligned}
\tag{2.26}
$$

Thus, $[X,Y] \in \Gamma(D^{\perp})$ holds if and only if (2.24) is satisfied. □

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