Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
26 (2010), 31–34
www.emis.de/journals
ISSN 1786-0091

ON REAL POLYNOMIALS WITHOUT NONNEGATIVE ROOTS

HORST BRUNOTTE

Abstract. An elementary proof of a result of D. Handelman on real Laurent polynomials without nonnegative roots is presented.

Among other things D. Handelman established the following result as part of his profound theory.

Theorem 1 ([2, Theorem A]). *Let $F, P \in \mathbb{R}[X, X^{-1}]$ be non-constant Laurent polynomials such that P has only positive coefficients and $F(r) > 0$ for all $r \in \mathbb{R}_{>0}$. Then there exists a positive integer m such that $P^m F$ has only positive coefficients.*

In this short note we provide an elementary proof of this statement. Moreover, we exhibit an effectively computable bound for m. Elementary proofs for a special case of Theorem 1 were given in [3] and independently in [1].

We start with a series of Lemmas. Apart from Lemma 4 they in fact treat special cases of our assertion. The essential contents of Lemmas 1 and 2 are well known for the choice $r = 1$ (see [3, 1]).

Lemma 1. *Let $b, c, r \in \mathbb{R},\ r > 0$ and assume $b^2 < 4c$. Then there exists a nonnegative integer n such that $(X + r)^n \cdot (X^2 + bX + c)$ has only positive coefficients. One can choose*

$$n \leq \max\left\{ \lceil \frac{1-\beta}{\alpha} \rceil,\ \lceil 1-\gamma \rceil,\ 0 \right\}$$

where

$$\alpha = 4\delta c - (2c - br)^2, \qquad \beta = 12\delta c - 2\sigma(2c - br), \qquad \gamma = 8\delta c - \sigma^2$$

and

$$\delta = r^2 - br + c, \qquad \sigma = 3c - 2br + r^2.$$

2000 *Mathematics Subject Classification.* 11C08.
Key words and phrases. real polynomials.

Proof. Pick an integer $n > \max\left\{\frac{1-\beta}{\alpha}, 1-\gamma, 0\right\}$ and write

$$(X+r)^n \cdot (X^2+bX+c) = r^n c + r^{n-1}(cn+br)X + \sum_{k=2}^{n} p_k X^k + (nr+b)X^{n+1} + X^{n+2}$$

where

$$p_k = \binom{n}{k-2} r^{n-k+2} + b\binom{n}{k-1} r^{n-k+1} + c\binom{n}{k} r^{n-k} = \binom{n}{k-1} r^{n-k} f(k) \qquad (2 \le k \le n)$$

with

$$f(k) = \frac{n+1-k}{k}c + br + \frac{k-1}{n+2-k}r^2.$$

Now we have

$$k(n-k+2)f(k) = g(k)$$

with

$$g(x) = \delta x^2 - ((2c-br)n+\sigma)x + c(n^2+3n+2).$$

Clearly, $\delta > 0$ and $\alpha = r^2(4c-b^2) > 0$. In view of

$$4\delta c(n^2+3n+2) - ((2c-br)n+\sigma)^2 = n(n\alpha+\beta)+\gamma > 0$$

this implies $g(x) > 0$ for all $x \in \mathbb{R}$, and by our choice of n the claim is completely proved. □

Lemma 2. *Let $f \in \mathbb{R}[X]$ be a monic polynomial having no nonnegative roots and $r \in \mathbb{R}_{>0}$. Then there exists some*[1] *$m \in \mathbb{N}$ bounded by an effectively computable constant such that $(X+r)^m f$ has only positive coefficients.*

Proof. We observe that f can be written as a product of monic quadratic polynomials with negative discriminants and of linear polynomials with positive coefficients. Then the statement is clear by a straightforward induction on the degree of f and Lemma 1. □

Lemma 3. *Let $f \in \mathbb{R}[X]$ be a monic polynomial having no nonnegative roots and $p \in \mathbb{R}[X]$ be a nonconstant polynomial with only positive coefficients. Then there exists some $m \in \mathbb{N}$ bounded by an effectively computable constant such that $p^m f$ has only positive coefficients.*

The proof of Lemma 3 is based on the following trivial, but useful statement which was also applied in the proof of [2, Theorem V1 A].

Lemma 4. *Let R be a commutative unital ring such that 2 is a unit in R. Then we have*

$$\sum_{i=0}^{n} a_i X^i = X^{n-1}\big(a_n X + \frac{a_{n-1}}{2}\big) + \sum_{i=1}^{n-2} X^i\big(\frac{a_{i+1}}{2}X + \frac{a_i}{2}\big) + \frac{a_1}{2}X + a_0$$

for all $a_0, \ldots, a_n \in R$ $(n \ge 1)$.

[1] We denote by $\mathbb{N}$ the set of nonnegative integers.

Proof of Lemma 3. By Lemma 4 there exist linear polynomials $p_0, \dots, p_n \in \mathbb{R}_{>0}[X]$ such that

$$p = \sum_{i=0}^{n} X^i p_i.$$

Using Lemma 2 we find nonnegative integers $m_0, \dots, m_n$ bounded by an effectively computable constant such that

$$p_i^{m_i} f \in \mathbb{R}_{>0}[X] \qquad (i = 0, \dots, n). \tag{1}$$

Let $m = m_0 + \cdots + m_n$ and

$$K = \left\{(k_0, \dots, k_n) \in \mathbb{N}^{n+1} \; : \; k_0 + \cdots + k_n = m\right\}.$$

Using multinomial coefficients $c_{(k_0,\dots,k_n)} \in \mathbb{N}_{>0}$ we can write

$$p^m f = \Big(\sum_{i=0}^{n} p_i X^i\Big)^m f = \sum_{(k_0,\dots,k_n)\in K} c_{(k_0,\dots,k_n)} \Big(\prod_{i=0}^{n} \big(p_i X^i\big)^{k_i}\Big) f. \tag{2}$$

For every $\mathbf{k} = (k_0, \dots, k_n) \in K$ the polynomial

$$g_{\mathbf{k}} = \Big(\prod_{i=0}^{n} p_i^{k_i}\Big) f$$

has degree

$$\sum_{i=0}^{n} k_i + \deg(f) = m + \deg(f),$$

and by (1) we find $g_{\mathbf{k}} \in \mathbb{R}_{>0}[X]$ because for some $i \in \{0, \dots, n\}$ we must have $k_i \geq m_i$. Thus, all polynomials

$$h_{\mathbf{k}} = c_{\mathbf{k}} g_{\mathbf{k}} \in \mathbb{R}_{>0}[X] \qquad (\mathbf{k} \in K)$$

have equal degree $m + \deg(f)$, and by (2)

$$p^m f = \sum_{(k_0,\dots,k_n)\in K} h_{(k_0,\dots,k_n)} \, X^{\sum_{i=1}^{n} i k_i} \in \mathbb{R}_{>0}[X],$$

because it can easily be verified that for each integer $s \in [0, n^2]$ there are some integers $k_1, \dots, k_n \in [0, n]$ such that $s = \sum_{i=1}^{n} i k_i$. □

Proof of Theorem 1. It is easy to check that the leading coefficient c of F is positive. Following HANDELMAN we multiply F with a suitable power of X such that $f := c^{-1} X^n F$ is monic without negative exponents and $f(0) > 0$. Similarly we find $k \in \mathbb{N}$ with

$$p := X^k P \in \mathbb{R}_{>0}[X].$$

By Lemma 3 there exists a positive m bounded by an effectively computable constant such that $p^m f$ has only positive coefficients. But then also

$$P^m F = c X^{-(mk+n)} p^m f \in \mathbb{R}_{>0}[X, X^{-1}].$$

□

Remark 1. (i) Theorem 1 can be seen as a dynamical problem in the set W of univariate Laurent polynomials with positive leading coefficients having no nonnegative roots with respect to the (continuous) transformation

$$T_P(F) = P \cdot F$$

with some fixed non-constant $P \in \mathbb{R}_{>0}[X, X^{-1}]$. It asserts that the orbit of $T_P(F)$ eventually meets the open set $\mathbb{R}_{>0}[X, X^{-1}]$. Obviously, $\mathbb{R}_{>0}[X, X^{-1}]$ is contained in the cone $W \cup \{0\}$.

(ii) The upper bound for the constant m as delivered by the proof of Lemma 3 cannot expected to be sharp. However, this does not affect the obvious algorithm for the determination of the least exponent m turning $P^m F$ into a Laurent polynomial with only positive coefficients (see (i)).

(iii) The interested reader is referred to [2, 3] for historical notes on questions related to Lemma 3.

Acknowledgement. The author is indebted to Professors S. Akiyama and P. Kirschenhofer for drawing his attention on this problem and to the referee for carefully reading the first version of this paper.

References

[1] S. Akiyama. Positive finiteness of number systems. In *Number theory*, volume 15 of *Dev. Math.*, pages 1–10. Springer, New York, 2006.

[2] D. Handelman. Positive polynomials and product type actions of compact groups. *Mem. Amer. Math. Soc.*, 54(320):xi+79, 1985.

[3] M. S. Klamkin. Solution of problem 4441. *Amer. Math. Monthly*, 59:643–644, 1952.

Received July 29, 2009.

Haus-Endt-Strasse 88,
D-40593 Düsseldorf, Germany
E-mail address: brunoth@web.de