Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
26 (2010), 35–43
www.emis.de/journals
ISSN 1786-0091

SUBRINGS IN TRIGONOMETRIC POLYNOMIAL RINGS

TARIQ SHAH AND EHSAN ULLAH

ABSTRACT. In this study we explore the subrings in trigonometric polynomial rings. Consider the rings T and T' of real and complex trigonometric polynomials over the fields $\mathbb{R}$ and its algebraic extension $\mathbb{C}$ respectively (see [6]). We construct the subrings T_0 of T and T'_0, T'_1 of T'. Then T_0 is a BFD whereas T'_0 and T'_1 are Euclidean domains. We also discuss among these rings the *Condition* : Let $A \subseteq B$ be a unitary (commutative) ring extension. For each $x \in B$ there exist $x' \in U(B)$ and $x'' \in A$ such that $x = x'x''$.

1. INTRODUCTION

Following Cohn [3], an integral domain say D is atomic if each nonzero nonunit of D is a product of irreducible elements *(atoms)* of D, and it is well known that UFDs, PIDs and Noetherian domains are atomic domains. An integral domain D satisfies the *ascending chain condition on principal ideals (ACCP)* if there does not exist any infinite strictly ascending chain of principal integral ideals of D. Every PID, UFD and Noetherian domain satisfy ACCP and a domain satisfying ACCP is atomic. Grams [5] and Zaks [11] provided examples of atomic domains which do not satisfy ACCP. An integral domain D is a *bounded factorization domain (BFD)* if it is atomic and for each nonzero nonunit of D, there is a bound on the length of factorization into products of irreducible elements (cf. [1]). Examples of BFDs are UFDs and Noetherian or Krull domains (cf. [1, Proposition 2.2]). By [10], an integral domain D is said to be a *half-factorial domain (HFD)* if D is *atomic* and whenever $x_1, \dots x_m = y_1, \dots y_n$, where $x_1, x_2, \dots x_m$, $y_1, y_2 \dots y_n$ are irreducibles in D, then $m = n$. A UFD is obviously an HFD, but the converse fails, since any Krull domain D with $CI(D) \cong \mathbb{Z}_2$ is an HFD [10], but not a UFD. Moreover, a polynomial extension of an HFD is not an HFD, for example $\mathbb{Z}[\sqrt{-3}][X]$ is not an HFD, as $\mathbb{Z}[\sqrt{-3}]$ is an HFD but not integrally closed [4].

2000 *Mathematics Subject Classification.* 13A05, 13B30, 12D05, 42A05.
Key words and phrases. trigonometric polynomial, HFD, condition 1, condition 2, irreducible.

In general,

$$\textbf{UFD} \Longrightarrow \textbf{HFD} \Longrightarrow \textbf{BFD} \Longrightarrow \textbf{ACCP} \Longrightarrow \textbf{Atomic}.$$

But none of the above implications is reversible.

In integral domains, factorization properties have been a common interest of algebraists, particularly for polynomial rings. In this study, we would investigate the factorization properties of the subrings of trigonometric polynomial rings T and T' (see [6]). The basic concepts, notions and terminology are as standard in [6].

For the factorization of exponential polynomials, J. F. Ritt developed: "If $1 + a_1e^{\alpha_1 x} + \cdots + a_ne^{\alpha_n x}$ is divisible by $1 + b_1e^{\beta_1 x} + \cdots + b_re^{\beta_r x}$ with no $b = 0$, then every β is a linear combination of $\alpha_1, \ldots, \alpha_n$ with rational coefficients"[8, Theorem].

Latter on getting inspired by this, G. Picavet and M. Picavet [6] investigated some factorization properties in trigonometric polynomial rings. Following [6], when we replace all α_k above by im, with $m \in \mathbb{Z}$, we obtain trigonometric polynomials. Whereas

$$T' = \{\sum_{k=0}^{n}(a_k \cos kx + b_k \sin kx) \ : \ n \in \mathbb{N}, \ a_k, b_k \in \mathbb{C}\} \text{ and}$$

$$T = \{\sum_{k=0}^{n}(a_k \cos kx + b_k \sin kx) \ : \ n \in \mathbb{N}, \ a_k, b_k \in \mathbb{R}\}$$

are the trigonometric polynomial rings.

Again following [6], $\sin^2 x = (1 - \cos x)(1 + \cos x)$ shows that two different non-associated irreducible factorizations of the same element may appear. Throughout we denote by $\cos kx$ and $\sin kx$ the two functions $x \mapsto \cos kx$ and $x \mapsto \sin kx$ (defined over $\mathbb{R}$). Also from basic trigonometric identities, it is obvious that for each $n \in \mathbb{N}\backslash\{1\}$, $\cos nx$ represents a polynomial in $\cos x$ with degree n and $\sin nx$ represents the product of $\sin x$ and a polynomial in $\cos x$ with degree $n - 1$. Conversely by linearization formulas, it follows that any product $\cos^n x \sin^p x$ can be written as:

$$\sum_{k=0}^{q}(a_k \cos kx + b_k \sin kx), \text{ where } q \in \mathbb{N} \text{ and } a_k, b_k \in \mathbb{Q}.$$

Hence $T = \mathbb{R}[\cos x, \sin x] \subseteq \mathbb{C}[\cos x, \sin x] = T'$.

Here T' is a Euclidean domain and T is a Dedekind half-factorial domain (see [6, Theorem 2.1 & Theorem 3.1]). We continue the investigations to find the factorization properties in trigonometric polynomial rings, begun in [6]. In other words we extend this study towards finding factorization properties of the subrings of trigonometric polynomial rings, by establishing T_0, T_0', and T_1' as subrings.

In this paper we explored T_0, T_0' and T_1' and demonstrated that, the ring T_0' and T_1' are Euclidean domains, whereas T_0 is a BFD. We also characterized the irreducible elements of T_0', and discussed *Condition* 1 (see [7, page 661]) among trigonometric polynomial rings.

2. The subrings of $\mathbb{C}[\cos x, \sin x]$

The Construction of T_1'. We consider

$$T_1' = \left\{ \sum_{k=0}^{n} (a_k \cos kx + i b_k \sin kx),\ n \in \mathbb{N},\ a_k, b_k \in \mathbb{R} \right\}.$$

Let

$$z = \sum_{k=0}^{n} (a_k \cos kx + i b_k \sin kx) \in T_1',$$

As $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, so

$$z = e^{-inx} \left[\sum_{k=0}^{n} \{ (\frac{a_k + b_k}{2}) e^{i(n+k)x} + (\frac{a_k - b_k}{2}) e^{i(n-k)x} \} \right],$$

where $\frac{a_k+b_k}{2}, \frac{a_k-b_k}{2} \in \mathbb{R}$. Since z is an arbitrary, therefore every element of T_1' is of the form

$$e^{-inx} P(e^{ix}),\ n \in \mathbb{N},\ \text{where } P(X) \in \mathbb{R}[X].$$

Conversely,

$$e^{-inx} P(e^{ix}) = \sum_{k=0}^{n-1} (\alpha_k e^{-i(n-k)x} + \alpha_{2n-k} e^{i(n-k)x}) + \alpha_n,$$

where $\alpha_k \in \mathbb{R}$. As $e^{ix} = \cos x + i \sin x$, so

$$\begin{aligned} e^{-inx} P(e^{ix}) = \sum_{k=0}^{n-1} \{ &(\alpha_k + \alpha_{2n-k}) \cos(n-k)x + \\ &i(\alpha_{2n-k} - \alpha_k) \sin(n-k)x \} + \alpha_n, \end{aligned}$$

where $\alpha_k + \alpha_{2n-k}$, $\alpha_{2n-k} - \alpha_k \in \mathbb{R}$. Therefore every element which is of the form $e^{-inx} P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{R}[X]$, is in T_1'.

Conclusion 1. The consequence of above construction is:

$$T_1' = \left\{ e^{-inx} P(e^{ix}),\ n \in \mathbb{N},\ \text{where } P(X) \in \mathbb{R}[X] \right\}.$$

So we have an isomorphism $f\colon (\mathbb{R}[X])_X \to T_1'$ through the substitution morphism $X \to e^{ix}$. Therefore $T_1' \backsimeq (\mathbb{R}[X])_X$.

Theorem 1. *T_1' is a Euclidean domain.*

Proof. $(\mathbb{R}[X])_X$ is a localization of $\mathbb{R}[X]$ by a multiplicative system generated by a prime X. Also $\mathbb{R}[X]$ is a Euclidean domain. Therefore $(\mathbb{R}[X])_X$ is a Euclidean domain [9, Proposition 7]. Hence the isomorphism $T_1' \backsimeq (\mathbb{R}[X])_X$ in Conclusion 1 gives the result. □

The Construction of T_0'. We define the set T_0' of all polynomials of the form

$$\sum_{k=0}^{n}(a_k \cos kx + b_k \sin kx),$$

$n \in \mathbb{N}$, $a_k, b_k \in \mathbb{C}$ and $a_n = \alpha+\gamma+i\beta$, $b_n = -\beta+i(\alpha-\gamma)$ such that $\alpha, \beta, \gamma \in \mathbb{R}$, α, β and γ are not simultaneously zero. Let $z \in T_0'$ be an arbitrary element, so we may write

$$z = a_0+\sum_{k=1}^{n-1}(a_k \cos kx+b_k \sin kx)+\{(\alpha+\gamma+i\beta)\cos nx+(-\beta+i(\alpha-\gamma))\sin nx\},$$

As $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, so

$$z = a_0 + \sum_{k=1}^{n-1}\{(\frac{a_k' + b_k'' + i(a_k'' - b_k')}{2})e^{ikx} + (\frac{a_k' - b_k'' + i(a_k'' + b_k')}{2})e^{-ikx}\}$$
$$+ (\alpha + i\beta)e^{inx} + \gamma e^{-inx},$$

where $a_k = a_k' + ia_k''$, $b_k = b_k' + ib_k''$ and a_k', a_k'', b_k', $b_k'' \in \mathbb{R}$, $a_0 \in \mathbb{C}$. Setting

$$\alpha_k' = \frac{a_k' + b_k'' + i(a_k'' - b_k')}{2} \text{ and } \beta_k' = \frac{a_k' - b_k'' + i(a_k'' + b_k')}{2},$$

we have

$$z = e^{-inx}\left[a_0 e^{inx} + \sum_{k=1}^{n-1}\{\alpha_k' e^{i(n+k)x} + \beta_k' e^{i(n-k)x}\} + (\alpha + i\beta)e^{i2nx} + \gamma\right],$$

where α_k', β_k', $a_0 \in \mathbb{C}$, and $\alpha, \beta, \gamma \in \mathbb{R}$. Since z is an arbitrary, therefore every element of T_0' is of the form

$$e^{-inx}P(e^{ix}),\ n \in \mathbb{N},\ \text{where } P(X) \in \mathbb{R} + X\mathbb{C}[X].$$

Conversely,

$$e^{-inx}P(e^{ix}) = \alpha_0 e^{-inx} + \alpha_{2n}e^{inx} + \sum_{k=1}^{n-1}(\alpha_k e^{-i(n-k)x} + \alpha_{2n-k}e^{i(n-k)x}) + \alpha_n,$$

where $\alpha_0 \in \mathbb{R}$, $\alpha_k \in \mathbb{C}$. Let

$$\alpha_k = \alpha_k' + i\alpha_k'', \alpha_{2n-k} = \alpha_{2n-k}' + i\alpha_{2n-k}'',\ \alpha_{2n} = \alpha_{2n}' + i\alpha_{2n}''.$$

So for $e^{ix} = \cos x + i \sin x$, we have

$$\begin{aligned} e^{-inx}P(e^{ix}) &= (\alpha_0 + \alpha'_{2n} + i\alpha''_{2n})\cos nx + (-\alpha''_{2n} + i(\alpha'_{2n} - \alpha_0))\sin nx \\ &\quad + \sum_{k=1}^{n-1}\{(\alpha'_k + \alpha'_{2n-k} + i(\alpha''_k + \alpha''_{2n-k}))\cos(n-k)x \\ &\quad + (\alpha''_k - \alpha''_{2n-k} + i(\alpha'_{2n-k} - \alpha'_k))\sin(n-k)x\} + \alpha_n \\ &= a_n \cos nx + b_n \sin nx \\ &\quad + \sum_{k=1}^{n-1}\{a_k \cos(n-k)x + b_k \sin(n-k)x\} + \alpha_n, \end{aligned}$$

where

$$\begin{aligned} a_n &= \alpha_0 + \alpha'_{2n} + i\alpha''_{2n},\ b_n = -\alpha''_{2n} + i(\alpha'_{2n} - \alpha_0), \\ a_k &= \alpha'_k + \alpha'_{2n-k} + i(\alpha''_k + \alpha''_{2n-k}) \\ b_k &= \alpha''_k - \alpha''_{2n-k} + i(\alpha'_{2n-k} - \alpha'_k). \end{aligned}$$

Therefore every element which is of the form $e^{-inx}P(e^{ix})$, $n \in \mathbb{N}$, where $P(X) \in \mathbb{R} + X\mathbb{C}[X]$, is in T'_0.

Conclusion 2. The consequence of above construction is:

$$T'_0 = \left\{e^{-inx}P(e^{ix}),\ n \in \mathbb{N},\ \text{where } P(X) \in \mathbb{R} + X\mathbb{C}[X]\right\}.$$

So again we have an isomorphism $f\colon (\mathbb{R} + X\mathbb{C}[X])_X \to T'_0$ through the substitution morphism $X \to e^{ix}$. Therefore $T'_0 \backsimeq (\mathbb{R} + X\mathbb{C}[X])_X$.

Theorem 2. *The integral domain T'_0 is a Euclidean domain having irreducible elements, up to units, trigonometric polynomials of the form $\cos x + i \sin x - a$, where $a \in \mathbb{C}\backslash\{0\}$.*

Proof. Since $(\mathbb{R}+X\mathbb{C}[X])_X = \mathbb{C}[X, 1/X] = \mathbb{C}[X]_X$ is a UFD (PID, Euclidean domain, etc.). Thus the domain $(\mathbb{R}+X\mathbb{C}[X])_X$ is a Euclidean domain. Now use the isomorphism $T'_0 \backsimeq (\mathbb{R}+X\mathbb{C}[X])_X$ in Conclusion 2. □

The following assertion is the analogue of [6, Corollary 2.2] and gives the factorization in T'_0 instead of T'.

Corollary 1. *Let* $z = \sum\limits_{k=0}^{n}(a_k \cos kx + b_k \sin kx)$, $n \in \mathbb{N}\backslash\{1\}$, $a_k, b_k \in \mathbb{C}$ *with* $(a_n, b_n) \neq (0,0)$, *such that* $a_n = \alpha + \gamma + i\beta$ *and* $b_n = -\beta + i(\alpha - \gamma)$, *where* $\alpha, \beta, \gamma \in \mathbb{R}$. *Let d be a common divisor of the integers k such that* $(a_k, b_k) \neq (0,0)$. *Then z has a unique factorization*

$$\lambda(\cos nx - i \sin nx)\prod_{j=1}^{\frac{2n}{d}}(\cos dx + i \sin dx - \alpha_j),\ \text{where } \lambda, \alpha_j \in \mathbb{C}\backslash\{0\}.$$

Proof. Since $T'_0 \subseteq T'$, therefore proof follows by [6, Corollary 2.2]. □

Now onwards the symbol $\cap$ in all diagrams will represent the inclusion $\subseteq$.

Remark 1. $\mathbb{R}+X\mathbb{C}[X]$ is a Noetherian HFD wedged between two Euclidean domains $\mathbb{R}[X]$ and $\mathbb{C}[X]$, that is $\mathbb{R}[X] \subseteq \mathbb{R}+X\mathbb{C}[X] \subseteq \mathbb{C}[X]$ and the localization of all these by a multiplicative system generated by X preserves their factorization properties in the following way

$$\begin{array}{ccccc} \mathbb{R}[X] & \subseteq & \mathbb{R}+X\mathbb{C}[X] & \subseteq & \mathbb{C}[X] \\ \cap & & \cap & & \cap \\ (\mathbb{R}[X])_X & \subseteq & (\mathbb{R}+X\mathbb{C}[X])_X & \subseteq & (\mathbb{C}[X])_X. \end{array}$$

Using Conclusion 1, Conclusion 2 and [6, Theorem 2.1], we have

$$\begin{array}{ccccc} \mathbb{R}[X] & \subseteq & \mathbb{R}+X\mathbb{C}[X] & \subseteq & \mathbb{C}[X] \\ \cap & & \cap & & \cap \\ T_1' & \subseteq & T_0' & \subseteq & T', \end{array}$$

where T_0' is a Euclidean domain wedged between two Euclidean domains T_1' and T'.

Remark 2. (a) Consider the domain extension $\mathbb{R}[X] \subseteq (\mathbb{R}[X])_X$. As $X\mathbb{R}[X]$ is a maximal ideal of $\mathbb{R}[X]$ and $X\mathbb{R}[X] \cap (X) \neq \phi$. Therefore the extended ideal $(X\mathbb{R}[X])^e = (\mathbb{R}[X])_X$ [12, Corollary 2]. Hence $(X\mathbb{R}[X])^e \backsimeq T_1'$ by Conclusion 1.

(b) If we consider the domain extension $\mathbb{R}+X\mathbb{C}[X] \subseteq (\mathbb{R}+X\mathbb{C}[X])_X$. We observe, that $X\mathbb{C}[X]$ is a maximal ideal of $\mathbb{R}+X\mathbb{C}[X]$ and $X\mathbb{C}[X] \cap (X) \neq \phi$. Therefore the extended ideal $(X\mathbb{C}[X])^e = (\mathbb{R}+X\mathbb{C}[X])_X$ [12, Corollary 2]. Hence $(X\mathbb{C}[X])^e \backsimeq T_0'$ by Conclusion 2.

(c) On the same lines we can apply the same result to the domain extension $\mathbb{C}[X] \subseteq (\mathbb{C}[X])_X$. In this case $X\mathbb{C}[X]$ is a maximal ideal of $\mathbb{C}[X]$ and $X\mathbb{C}[X] \cap (X) \neq \phi$. Therefore the extended ideal $(X\mathbb{C}[X])^e = (\mathbb{C}[X])_X$ [12, Corollary 2]. Hence $(X\mathbb{C}[X])^e \backsimeq T'$ by [6, Theorem 2.1].

Definition 1. Let J be a subset of T_1' defined by

$$J = \left\{ \sum_{k=0}^{n} (a_k \cos kx + i b_k \sin kx), n \in \mathbb{N},\ a_k, b_k \in \mathbb{Q} \text{ and } a_n = b_n \right\}.$$

Definition 2. Let I be a subset of T_0' defined by

$$I = \left\{ \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) \ : \ n \in \mathbb{N},\ a_k, b_k \in \mathbb{C} \right.$$
$$\left. \text{and } a_n = \alpha + i\beta,\ b_n = -\beta + i\alpha \right\}.$$

Lemma 1. *For the maximal ideal $X\mathbb{R}[X]$ (respectively $X\mathbb{C}[X]$) of $\mathbb{R}[X]$ (respectively $\mathbb{R}+X\mathbb{C}[X]$), we have $(X\mathbb{R}[X])_X \backsimeq J$ (respectively $(X\mathbb{C}[X])_X \backsimeq I$).*

Proof. Follows by Conclusion 1 (respectively Conclusion 2). $\square$

Condition 1. Let $A \subseteq B$ be a unitary (commutative) ring extension. For each $x \in B$ there exist $x' \in U(B)$ and $x'' \in A$ such that $x = x'x''$ [7, page 661].

Example 1. (a) If the ring extension $A \subseteq B$ satisfies Condition 1, then the ring extension $A + XB[X] \subseteq B[X]$ (or $A + XB[[X]] \subseteq B[[X]]$) also satisfies Condition 1.

(b) If the ring extensions $A \subseteq B$ and $B \subseteq C$ satisfy Condition 1, then so does the ring extension $A \subseteq C$.

(c) If B is a fraction ring of A, then the ring extension $A \subseteq B$ satisfies Condition 1. Hence the ring extension $A \subseteq B$ satisfies Condition 1 is the generalization of localization.

(d) If B is a field, then the ring extension $A \subseteq B$ satisfies Condition 1.

Condition 2. Let A, A_1, B, B_1 be unitary (commutative) rings such that

$$\begin{array}{ccc} A & \subseteq & B \\ \cap & & \cap \\ A_1 & \subseteq & B_1 \end{array}$$

Then for each $x \in B_1$ there exist $x' \in U(B)$ and $x'' \in A_1$ such that $x = x'x''$.

Lemma 2. *Let $A \subseteq B$ be a unitary (commutative) ring extension which satisfies Condition* 1. *If N is a multiplicative system in A then the ring extension $N^{-1}A \subseteq N^{-1}B$ satisfies Condition* 2.

Proof. Since the ring extension $A \subseteq B$ satisfies Condition 1. Therefore for each $a \in B$ there exist $b \in U(B)$ and $c \in A$ such that $a = bc$. Obviously $N^{-1}A \subseteq N^{-1}B$ and let $x = \frac{a}{s} \in N^{-1}B$. Then $x = \frac{a}{s}$, $a \in B$, $s \in N$. This implies $x = \frac{bc}{s} = b\frac{c}{s}$, where $b \in U(B)$ and $\frac{c}{s} \in N^{-1}A$. □

Example 2. (a) If the ring extensions $A \subseteq B$ and $B \subseteq C$ satisfy Condition 2, then so does the ring extension $A \subseteq C$.

(b) For $A = A_1$ and $B = B_1$ the Condition 1 and Condition 2 coincides.

(c) If the ring extension $A_1 \subseteq B_1$ satisfies Condition 2, then it does satisfies Condition 1.

(d) By Lemma 2, the ring extensions $T_1' \subseteq T_0'$ and $T_0' \subseteq T'$ satisfy Condition 2 so does the ring extension $T_1' \subseteq T'$.

Remark 3. Consider the commutative inclusion diagram made by the following domain extensions

$$\begin{array}{ccccc} \mathbb{R}[X] & \subseteq & \mathbb{R}{+}X\mathbb{C}[X] & \subseteq & \mathbb{C}[X] \\ \cap & \searrow & \cap & \searrow & \cap \\ T_1' & \subseteq & T_0' & \subseteq & T'. \end{array}$$

Among these domain extensions $\mathbb{R}{+}X\mathbb{C}[X] \subseteq \mathbb{C}[X]$, $\mathbb{R}[X] \subseteq T_1'$, $\mathbb{R}{+}X\mathbb{C}[X] \subseteq T_0'$ and $\mathbb{C}[X] \subseteq T'$ satisfy Condition 1 (see Example 1). Whereas the domain extensions $T_0' \subseteq T'$ and $T_0' \subseteq T'$ satisfy Condition 2. So by transitivity the domain extensions $\mathbb{R}[X] \subseteq T_0'$, $\mathbb{R} + X\mathbb{C}[X] \subseteq T'$ and $T_1' \subseteq T'$ also satisfy Condition 2. Also note that the domain extension $\mathbb{R}[X] \subseteq \mathbb{R} + X\mathbb{C}[X]$ does not satisfy any of Condition 1 and Condition 2.

The subring of $\mathbb{R}[\cos x, \sin x]$. Consider the substitution morphism

$$g\colon \mathbb{Z}[X,Y] \to \mathbb{Z}[\cos x, \sin x],$$

defined by $g(X) = \cos x$ and $g(Y) = \sin x$ such that

$$g(X^2 + Y^2 - 1) = g(X^2) + g(Y^2) - 1 = \cos^2 x + \sin^2 x - 1 = 0.$$

This implies $(X^2 + Y^2 - 1) = \operatorname{Ker} g$, therefore

$$\mathbb{Z}[\cos x, \sin x] \backsimeq \mathbb{Z}[X,Y]/(X^2 + Y^2 - 1).$$

Theorem 3. *The integral domain $T_0 = \mathbb{Z}[\cos x, \sin x]$ is a BFD.*

Proof. Since $\mathbb{Z}[X,Y]/(X^2+Y^2-1) \backsimeq \mathbb{Z}[\cos x, \sin x]$, with $\mathbb{Z}[X,Y]$ a Noetherian domain. Therefore $\mathbb{Z}[\cos x, \sin x]$ is Notherian, hence the result. □

Remark 4. (a) T is a Dedekind HFD [6, Theorem 3.1], whereas T_0 is a Noetherian BFD.
(b) T_0 is a free $\mathbb{Z}[\cos x]$-module and has basis $\{1, \sin x\}$.
(c) $\mathbb{Z}[\cos x]$ is a Euclidean domain because $\mathbb{Z}[\cos x] \backsimeq \mathbb{Z}[X]$, therefore the BFD T_0 lies between Euclidean domains $\mathbb{Z}[\cos x]$ and T_0'.
(d) T_0' is a free T_0-module and has basis $\{1, i\}$.
(e) T' is a T_0-module also T is a T_0-module.
(f) T' is a T_0'-module.

References

[1] D. D. Anderson, D. F. Anderson, and M. Zafrullah. Factorization in integral domains. *J. Pure Appl. Algebra*, 69(1):1–19, 1990.

[2] D. D. Anderson, D. F. Anderson, and M. Zafrullah. Factorization in integral domains. II. *J. Algebra*, 152(1):78–93, 1992.

[3] P. M. Cohn. Bezout rings and their subrings. *Proc. Cambridge Philos. Soc.*, 64:251–264, 1968.

[4] J. Coykendall. A characterization of polynomial rings with the half-factorial property. In *Factorization in integral domains (Iowa City, IA, 1996)*, volume 189 of *Lecture Notes in Pure and Appl. Math.*, pages 291–294. Dekker, New York, 1997.

[5] A. Grams. Atomic rings and the ascending chain condition for principal ideals. *Proc. Cambridge Philos. Soc.*, 75:321–329, 1974.

[6] G. Picavet and M. Picavet-L'Hermitte. Trigonometric polynomial rings. In *Commutative ring theory and applications (Fez, 2001)*, volume 231 of *Lecture Notes in Pure and Appl. Math.*, pages 419–433. Dekker, New York, 2003.

[7] N. Radu, S. O. I. Al-Salihi, and T. Shah. Ascend and descend of factorization properties. *Rev. Roumaine Math. Pures Appl.*, 45(4):659–669 (2001), 2000.

[8] J. F. Ritt. A factorization theory for functions $\sum_{i=1}^{n} a_i e^{\alpha_i x}$. *Trans. Amer. Math. Soc.*, 29(3):584–596, 1927.

[9] P. Samuel. About Euclidean rings. *J. Algebra*, 19:282–301, 1971.

[10] A. Zaks. Half-factorial-domains. *Israel J. Math.*, 37(4):281–302, 1980.

[11] A. Zaks. Atomic rings without a.c.c. on principal ideals. *J. Algebra*, 74(1):223–231, 1982.

[12] O. Zariski and P. Samuel. *Commutative algebra. Vol. 1.* Springer-Verlag, New York, 1958.

Received December 30, 2007.

Quaid-I-Azam University, Pakistan
E-mail address: stariqshah@gmail.com

Passau Universität, Germany
E-mail address: ehsanu01@stud.uni-passau.de