Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
26 (2010), 71–89
www.emis.de/journals
ISSN 1786-0091

CONTINUITY OF THE QUENCHING TIME FOR A PARABOLIC EQUATION WITH A NONLINEAR BOUNDARY CONDITION AND A POTENTIAL

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Abstract. In this paper, we consider the following initial-boundary value problem

$$\begin{cases} u_t(x,t) &= a(x)\Delta u(x,t) \text{ in } \Omega\times(0,T),\\ \frac{\partial u(x,t)}{\partial\nu} &= -b(x)g(u(x,t)) \text{ on } \partial\Omega\times(0,T),\\ u(x,0) &= u_0(x) \text{ in } \overline{\Omega}, \end{cases}$$

where $g\colon(0,\infty)\to(0,\infty)$ is a C^1 convex, nonincreasing function,

$$\lim_{s\to 0^+} g(s)=\infty,\ \int_0 \frac{ds}{g(s)}<\infty,$$

Δ is the Laplacian, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $u_0\in C^2(\overline{\Omega})$, $u_0(x)>0$, $x\in\overline{\Omega}$, $a\in C^0(\overline{\Omega})$, $a(x)>0$, $x\in\overline{\Omega}$, $b\in C^0(\partial\Omega)$, $b(x)>0$, $x\in\partial\Omega$. Under some assumptions, we show that the solution of the above problem quenches in a finite time and estimate its quenching time. We also prove the continuity of the quenching time as a function of u_0, b and a. Finally, we give some numerical results to illustrate our analysis.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem

$$u_t(x,t)=a(x)\Delta u(x,t) \text{ in } \Omega\times(0,T), \tag{1}$$

$$\frac{\partial u(x,t)}{\partial\nu}=-b(x)g(u(x,t)) \text{ on } \partial\Omega\times(0,T), \tag{2}$$

$$u(x,0)=u_0(x) \text{ in } \overline{\Omega}, \tag{3}$$

2000 *Mathematics Subject Classification.* 35B40, 35B50, 35K60, 65M06.

Key words and phrases. Quenching, parabolic equation, nonlinear boundary condition, numerical quenching time.

where $g\colon (0,\infty) \to (0,\infty)$ is a C^1 convex, nonincreasing function,

$$\lim_{s\to 0^+} g(s) = \infty, \ \int_0 \frac{ds}{g(s)} < \infty,$$

Δ is the Laplacian, $u_0 \in C^2(\overline{\Omega})$, $\Delta u_0(x) < 0$, $x \in \Omega$, $\frac{\partial u_0(x)}{\partial \nu} = 0$, $x \in \partial\Omega$, $u_0(x) > 0$, $x \in \overline{\Omega}$, $a \in C^0(\overline{\Omega})$, $a(x) > 0$, $x \in \overline{\Omega}$, $b \in C^0(\partial\Omega)$, $b(x) > 0$, $x \in \partial\Omega$, ν is the exterior normal unit vector on $\partial\Omega$.

Here $(0,T)$ is the maximal time interval on which the solution u of (1)-(3) exists. The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t\to T} u_{\min}(t) = 0,$$

where $u_{\min}(t) = \min_{x\in\overline{\Omega}} u(x,t)$. In this last case, we say that the solution u quenches in a finite time, and the time T is called the quenching time of the solution u. Thus, in this paper, by virtue of the definition of the time T, we have

$$u(x,t) > 0 \text{ in } \overline{\Omega} \times [0,T).$$

Solutions of parabolic equations with nonlinear boundary conditions which quench in a finite time have been the subject of investigations of many authors (see [6], [11], [14], [27], and the references cited therein). In particular, in [6], the problem (1)-(3) has been studied. By standard methods, it is not hard to prove the local in time existence of a classical solution which is unique (see [6]). Also in [6], Boni has proved that the solution of (1)-(3) quenches in a finite time, and its quenching set is located on the boundary of the domain Ω. In [14], Fila and Levine have considered the above problem in the case where $\Omega = (0,1)$, $a(x) = 1$, $b(0) = 0$, $b(1) = 1$, $g(u) = u^{-p}$ with $p > 0$. They have proved that the solution u quenches in a finite time at the point $x = 1$. For quenching results of other problems, one may consult the following references [2], [3], [4], [10], [13], [24], [25], [28], [29], [31]. In the present paper, we are interested in the dependence of the quenching time with respect to the initial datum, the coefficient of the Laplacian and the potential. In other words, we want to know if the quenching time as a function of the above parameters is continuous. More precisely, let us consider the solution v of the initial-boundary value problem below

$$v_t(x,t) = a_k(x)\Delta v(x,t) \text{ in } \Omega \times (0,T^h_{l,k}), \tag{4}$$

$$\frac{\partial v(x,t)}{\partial \nu} = -b_l(x) g(v(x,t)) \text{ on } \partial\Omega \times (0,T^h_{l,k}), \tag{5}$$

$$v(x,0) = u^h_0(x) \text{ in } \overline{\Omega}, \tag{6}$$

where

$$0 < a_k(x) \le a(x), \ x \in \overline{\Omega}, \ \lim_{k\to 0} a_k = a,$$

$$0 < b_l(x) \leq b(x),\ x \in \partial\Omega,\ \lim_{l \to 0} b_l = b,$$

$$u_0^h(x) \geq u_0(x),\ x \in \overline{\Omega},\ \lim_{h \to 0} u_0^h = u_0.$$

Here $(0, T_{l,k}^h)$ is the maximal time interval of existence of the solution v. This implies that

$$v(x,t) > 0 \text{ in } \Omega \times [0, T_{l,k}^h).$$

Set $w(x,t) = u_t(x,t)$, $(x,t) \in \overline{\Omega} \times [0,T)$. Take the derivative in t on both sides of (1) to obtain

$$w_t(x,t) = a(x)\Delta w(x,t) \text{ in } \Omega \times (0,T).$$

In the same manner, we also have

$$\frac{\partial w(x,t)}{\partial \nu} = -b(x)g'(u(x,t))w(x,t) \text{ on } \partial\Omega \times (0,T).$$

Using the hypotheses $\Delta u_0(x) < 0$ in Ω, we see that $w(x,0) < 0$ in Ω. We infer from the maximum principle that $w = u_t < 0$ in $\Omega \times (0,T)$, which implies that $\Delta u < 0$ in $\Omega \times (0,T)$. Taking into account the fact that $0 < a_k(x) \leq a(x)$ in Ω, $u_0^h(x) \geq u_0(x)$ in $\overline{\Omega}$, $0 < b_l(x) \leq b(x)$ on $\partial\Omega$, we discover that

$$u_t(x,t) - a_k(x)\Delta u(x,t) \leq 0 \text{ in } \Omega \times (0,T),$$

$$\frac{\partial u(x,t)}{\partial \nu} + b_l(x)g(u(x,t)) \leq 0 \text{ on } \partial\Omega \times (0,T),$$

$$u(x,0) \leq v(x,0) \text{ in } \overline{\Omega}.$$

It follows from the maximum principle that $v \geq u$ as long as all of them are defined. We deduce that $T_{l,k}^h \geq T$. In the present paper, under some assumptions, we show that the solution v of (4)-(6) quenches in a finite time $T_{l,k}^h$, and the following relation holds

$$\lim_{(h,k,l) \to (0,0,0)} T_{l,k}^h = T.$$

Similar results have been obtained in [5], [8], [12], [16], [18], [19], [17], [20], [21], where the authors have considered the phenomenon of blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time). Our paper is organized as follows. In the next section, under some assumptions, we show that the solution v of (4)-(6) quenches in a finite time and estimate its quenching time. In the third section, we prove the continuity of the quenching time and finally in the last section, we give some computational results.

2. Quenching time

In this section, under some hypotheses, we show that the solution v of (4)-(6) quenches in a finite time and estimate its quenching time.

Using an idea of Friedman and Lacey in [15], we prove the following result.

Theorem 2.1. *Let v be the solution of (4)–(6), and assume that there exists a constant $A \in (0,1]$ such that the initial datum at (6) satisfies*

$$a_k(x)\Delta u_0^h(x) \leq -Ag(u_0^h(x)) \text{ in } \overline{\Omega}. \tag{7}$$

Then, the solution v quenches in a finite time $T_{l,k}^h$ which obeys the following estimate

$$T_{l,k}^h \leq \frac{1}{A}\int_0^{u_{0min}^h} \frac{ds}{g(s)},$$

where $u_{0min}^h = \min_{x\in\overline{\Omega}} u_0^h(x)$.

Proof. Since $(0, T_{l,k}^h)$ is the maximal time interval of existence of the solution v, our purpose is to show that $T_{l,k}^h$ is finite and obeys the above inequality. Introduce the function $J(x,t)$ defined as follows

$$J(x,t) = v_t(x,t) + Ag(v(x,t)) \quad \text{in} \quad \overline{\Omega}\times[0,T_{l,k}^h).$$

A straightforward computation reveals that

$$\begin{aligned} J_t - a_k(x)\Delta J = {} & (v_t - a_k(x)\Delta v)_t \\ & + Ag'(v)v_t - Aa_k(x)\Delta g(v) \text{ in } \Omega\times(0,T_{l,k}^h). \end{aligned} \tag{8}$$

Again, by a direct calculation, it is easy to check that

$$\Delta g(v) = g''(v)|\nabla v|^2 + g'(v)\Delta v \text{ in } \Omega\times(0,T_{l,k}^h),$$

which implies that $\Delta g(v) \geq g'(v)\Delta v$ in $\Omega\times(0,T_{l,k}^h)$. Using this estimate and (8), we arrive at

$$\begin{aligned} J_t - a_k(x)\Delta J \leq {} & (v_t - a_k(x)\Delta v)_t \\ & + Ag'(v)(v_t - a_k(x)\Delta v) \text{ in } \Omega\times(0,T_{l,k}^h). \end{aligned} \tag{9}$$

It follows from (4) that

$$J_t - a_k(x)\Delta J \leq 0 \text{ in } \Omega\times(0,T_{l,k}^h).$$

We also have

$$\frac{\partial J}{\partial\nu} = \left(\frac{\partial v}{\partial\nu}\right)_t + Ag'(v)\frac{\partial v}{\partial\nu} \text{ on } \partial\Omega\times(0,T_{l,k}^h).$$

We deduce from (5) that

$$\frac{\partial J}{\partial\nu} = -b_l(x)g'(v)v_t - Ab_l(x)g'(v)g(v) \text{ on } \partial\Omega\times(0,T_{l,k}^h).$$

Due to the expression of J, we find that

$$\frac{\partial J}{\partial\nu} = -b_l(x)g'(v)J \text{ on } \partial\Omega\times(0,T_{l,k}^h).$$

Finally, we get

$$J(x,0) = v_t(x,0) + Ag(v(x,0)) \leq a_k(x)\Delta u_0^h(x) + Ag(u_0^h(x)) \text{ in } \overline{\Omega}.$$

Thanks to (7), we discover that

$$J(x,0) \le 0 \quad \text{in} \quad \overline{\Omega}.$$

It follows from the maximum principle that

$$J(x,t) \le 0 \text{ in } \overline{\Omega} \times (0, T^h_{l,k}).$$

This estimate may be rewritten in the following manner

$$\frac{dv}{g(v)} \le -Adt \text{ in } \overline{\Omega} \times (0, T^h_{l,k}). \tag{10}$$

Integrate the above inequality over $(0, T^h_{l,k})$ to obtain

$$T^h_{l,k} \le \frac{1}{A} \int_0^{v(x,0)} \frac{d\sigma}{g(\sigma)} \text{ in } x \in \overline{\Omega}, \tag{11}$$

which implies that

$$T^h_{l,k} \le \frac{1}{A} \int_0^{u^h_{0min}} \frac{d\sigma}{g(\sigma)}. \tag{12}$$

Use the fact that the quantity on the right hand side of (12) is finite to complete the rest of the proof. □

Remark 2.1. Let $t_0 \in (0, T^h_{l,k})$. Integrating the inequality (10) over $(t_0, T^h_{l,k})$, we get

$$T^h_{l,k} - t_0 \le \frac{1}{A} \int_0^{v(x,t_0)} \frac{d\sigma}{g(\sigma)} \text{ for } x \in \overline{\Omega},$$

which implies that

$$T^h_{l,k} - t_0 \le \frac{1}{A} \int_0^{v_{\min}(t_0)} \frac{d\sigma}{g(\sigma)}. \tag{13}$$

3. Continuity of the quenching time

In this section, under some assumptions, we show that the solution v of (4)–(6) quenches in a finite time, and its quenching time goes to that of the solution u of (1)–(3) when h, k and l go to zero.

Firstly, we show that the solution v approaches the solution u in $\Omega \times [0, T-\tau]$ with $\tau \in (0,T)$ when h, k and l tend to zero. This result is stated in the following theorem.

Theorem 3.1. *Let u be the solution of (1)–(3). Suppose that $u \in C^{2,1}(\overline{\Omega} \times [0, T-\tau])$ and $\min_{t \in [0,T-\tau]} u_{\min}(t) = \alpha > 0$ with $\tau \in (0,T)$. Assume that*

$$\|u^h_0 - u_0\|_\infty = o(1) \text{ as } h \to 0, \tag{14}$$

$$\|a_k - a\|_\infty = o(1) \text{ as } k \to 0, \tag{15}$$

$$\|b_l - b\|_\infty = o(1) \text{ as } l \to 0. \tag{16}$$

Then, the problem (4)-(6) admits a unique solution $v \in C^{2,1}(\overline{\Omega} \times [0, T^h_{l,k}))$, and the following relation holds

$$\sup_{t \in [0, T-\tau]} \|v(\cdot,t) - u(\cdot,t)\|_\infty = O(\|u^h_0 - u_0\|_\infty + \|b_l - b\|_\infty + \|a_k - a\|_\infty)$$

$$\text{as } (h,l,k) \to (0,0,0).$$

Proof. The problem (4)–(6) has for each h, a unique solution $v \in C^{2,1}(\overline{\Omega} \times [0, T^h_{l,k}))$. In the introduction of the paper, we have seen that $T^h_{l,k} \geq T$. Let $t^h_{l,k} \leq T$ be the greatest value of $t > 0$ such that

$$\|v(\cdot,t) - u(\cdot,t)\|_\infty \leq \frac{\alpha}{2} \text{ for } t \in (0, t^h_{l,k}). \tag{17}$$

Obviously, we see that $\|v(\cdot,0) - u(\cdot,0)\|_\infty = \|u^h_0 - u_0\|_\infty$. Due to this fact, we deduce from (14) and (17) that $t^h_{l,k} > 0$ for h sufficiently small. By the triangle inequality, we find that

$$v_{\min}(t) \geq u_{\min}(t) - \|v(\cdot,t) - u(\cdot,t)\|_\infty \text{ for } t \in (0, t^h_{l,k}),$$

which leads us to

$$v_{\min}(t) \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2} \text{ for } t \in (0, t^h_{l,k}). \tag{18}$$

Introduce the function $e(x,t)$ defined as follows

$$e(x,t) = v(x,t) - u(x,t) \text{ in } \overline{\Omega} \times [0, t^h_{l,k}). \tag{19}$$

A routine computation reveals that

$$e_t - a_k(x)\Delta e = (a_k(x) - a(x))\Delta u \text{ in } \Omega \times (0, t^h_{l,k}),$$

$$\frac{\partial e}{\partial \nu} = -b_l(x) g'(\theta) e + (b(x) - b_l(x)) g(u) \text{ on } \partial\Omega \times (0, t^h_{l,k}),$$

$$e(x,0) = u^h_0(x) - u_0(x) \text{ in } \overline{\Omega},$$

where θ is an intermediate value between u and v. Let M be such that $g(\frac{\alpha}{2}) \leq M$ and $|\Delta u| \leq M$ for $(x,t) \in \overline{\Omega} \times (0, t^h_{l,k})$. We deduce that

$$e_t - a_k(x)\Delta e \leq M\|a - a_k\|_\infty \text{ in } \Omega \times (0, t^h_{l,k}).$$

$$\frac{\partial e}{\partial \nu} \leq -b_l(x) g'(\theta) e + \|b_l - b\|_\infty M \text{ on } \partial\Omega \times (0, t^h_{l,k}),$$

$$e(x,0) = u^h_0(x) - u_0(x) \text{ in } \overline{\Omega}.$$

Let L be such that $L \geq -\|b_l\|_\infty g'(\frac{\alpha}{2}) + M$. Since the domain Ω has a smooth boundary $\partial\Omega$, there exists a function $\rho \in C^2(\overline{\Omega})$ satisfying $\rho(x) \geq 0$ in Ω and $\frac{\partial \rho(x)}{\partial \nu} = 1$ on $\partial\Omega$. Let K be a positive constant such that $K \geq L a_k \Delta\varphi + L^2 a_k |\nabla\varphi|^2$ for $x \in \overline{\Omega}$. It is not hard to see that $g'(\frac{\alpha}{2}) \geq g'(\theta)$ on $\partial\Omega \times (0, t^h_{l,k})$. Introduce the function z defined as follows

$$z(x,t) = e^{(M+K)t + L\varphi(x)}(\|u^h_0 - u_0\|_\infty + \|b_l - b\|_\infty + \|a_k - a\|_\infty) \text{ in } \overline{\Omega} \times [0, t^h_{l,k}).$$

A straightforward calculation reveals that

$$z_t - a_k\Delta z = (M + K - La_k\Delta\varphi - L^2 a_k|\nabla\varphi|^2)z \text{ in } \Omega\times(0, t^h_{l,k}),$$
$$\frac{\partial z}{\partial\nu} = Lz \text{ on } \partial\Omega\times(0, t^h_{l,k}),$$
$$z(x,0)\geq e(x,0) \text{ in } \overline{\Omega}.$$

Since $L \geq -b_l(x)g'(\theta) + M$ for $(x,t) \in \partial\Omega\times(0, t^h_{l,k})$, and $K \geq La_k\Delta\varphi + L^2 a_k|\Delta\varphi|^2$ for $x\in\overline{\Omega}$, we deduce that

$$z_t - a_k\Delta z \geq M\|a - a_k\|_\infty \text{ in } \Omega\times(0, t^h_{l,k}),$$
$$\frac{\partial z}{\partial\nu} \geq -b_l g'(\theta)z + \|b_l - b\|_\infty M \text{ on } \partial\Omega\times(0, t^h_{l,k}),$$
$$z(x,0)\geq e(x,0) \text{ in } \overline{\Omega}.$$

It follows from the maximum principle that

$$z(x,t)\geq e(x,t) \text{ in } \overline{\Omega}\times(0, t^h_{l,k}).$$

In the same way, we also prove that

$$z(x,t)\geq -e(x,t) \text{ in } \overline{\Omega}\times(0, t^h_{l,k}),$$

which implies that

$$\|e(.,t)\|_\infty \leq e^{(K+M)t+L\|\varphi\|_\infty}(\|u^h_0 - u_0\|_\infty + \|b_l - b\|_\infty + \|a_k - a\|_\infty) \quad \text{for} t\in(0, t^h_{l,k}).$$

Let us show that $t^h_{l,k} = T$. Suppose that $t^h_{l,k} < T$. From (17), we obtain

$$\frac{\alpha}{2} = \|v(\cdot, t^h_{l,k}) - u(\cdot, t^h_{l,k})\|_\infty$$
$$\leq e^{(K+M)T+L\|\varphi\|_\infty}(\|u^h_0 - u_0\|_\infty + \|b_l - b\|_\infty + \|a_k - a\|_\infty).$$

Since the term on the right hand side of the above inequality goes to zero as h k, and l go to zero, we deduce that $\frac{\alpha}{2}\leq 0$, which is impossible. Consequently, $t^h_{l,k} = T$. □

Now, we are in a position to prove the main result of the paper.

Theorem 3.2. *Suppose that the problem (1)–(3) has a solution u which quenches in a finite time at the time T and $u \in C^{2,1}(\overline{\Omega}\times[0,T))$. Assume that the conditions (14), (15) and (16) are valid. Under the assumption of Theorem 2.1, the problem (4)-(6) admits a unique solution v which quenches in a finite time $T^h_{l,k}$, and the following relation holds*

$$\lim_{(h,k,l)\to(0,0,0)} T^h_{l,k} = T.$$

Proof. Let $0 < \varepsilon < T/2$. There exists $\rho > 0$ such that

$$\frac{1}{A}\int_0^y \frac{d\sigma}{g(\sigma)} \leq \frac{\varepsilon}{2}, 0 \leq y \leq \rho. \tag{20}$$

Since u quenches in a finite time T, there exists $T_0 \in (T - \frac{\varepsilon}{2}, T)$ such that

$$0 < u_{\min}(t) < \frac{\rho}{2} \text{ for } t \in [T_0, T).$$

Set $T_1 = \frac{T_0+T}{2}$. It is not hard to see that

$$u_{\min}(t) > 0 \text{ for } t \in [0, T_2].$$

From Theorem 3.1, the problem (4)–(6) admits a unique solution v, and we get

$$\|v(\cdot,t) - u(\cdot,t)\|_\infty < \frac{\rho}{2} \text{ for } t \in [0, T_1],$$

which implies that $\|v(\cdot,T_1) - u(\cdot,T_1)\|_\infty \leq \frac{\rho}{2}$. An application of the triangle inequality leads us to

$$v_{\min}(T_1) \leq \|v(\cdot,T_1) - u(\cdot,T_1)\|_\infty + u_{\min}(T_1) \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho.$$

Invoking Theorem 2.1, we see that v quenches at the time $T^h_{l,k}$. On the other hand, we have proved in the introduction of the paper that $T^h_{l,k} \geq T$. We infer from Remark 2.1 and (19) that

$$0 \leq T^h_{l,k} - T = T^h_{l,k} - T_1 + T_1 - T \leq \frac{1}{A}\int_0^{v_{\min}(T_1)} \frac{d\sigma}{g(\sigma)} + \frac{\varepsilon}{2} \leq \varepsilon.$$

□

4. Numerical results

In this section, we give some computational experiments to confirm the theory given in the previous section. We consider the radial symmetric solution of the following initial-boundary value problem

$$u_t = a(x)\Delta u \text{ in } B \times (0,T),$$
$$\frac{\partial u}{\partial \nu} = -b(x)u^{-p} \text{ on } S \times (0,T),$$
$$u(x,0) = u_0(x) \text{ in } \overline{B},$$

where $B = \{x \in \mathbb{R}^N;\ \|x\| < 1\}$, $S = \{x \in \mathbb{R}^N; \|x\| = 1\}$. The above problem may be rewritten in the following form

$$u_t = a(r)\left(u_{rr} + \frac{N-1}{r}u_r\right),\ r \in (0,1),\ t \in (0,T), \tag{21}$$

$$u_r(0,t) = 0,\ u_r(1,t) = -b(u(1,t))^{-p},\ t \in (0,T), \tag{22}$$

$$u(r,0) = \varphi(r),\ r \in [0,1]. \tag{23}$$

Here, we take $p=1$, $\varphi(r)=1-\frac{r^2}{3}+\varepsilon(1+\cos(\pi r))$, $a(r)=2+\sin(\pi r)-\varepsilon r^2$, $b=1-\varepsilon$, with $\varepsilon\in[0,1]$. We start by the construction of some adaptive schemes as follows. Let I be a positive integer and let $h=1/I$. Define the grid $x_i=ih$, $0\le i\le I$, and approximate the solution u of (20)-(22) by the solution $U_h^{(n)}=(U_0^{(n)},\dots,U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta t_n}=Na(x_0)\frac{2U_1^{(n)}-2U_0^{(n)}}{h^2},$$

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}=a(x_i)\left(\frac{U_{i+1}^{(n)}-2U_i^{(n)}+U_{i-1}^{(n)}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n)}-U_{i-1}^{(n)}}{2h}\right),$$
$$1\le i\le I-1,$$

$$\frac{U_I^{(n+1)}-U_I^{(n)}}{\Delta t_n}=a(x_I)\left(\frac{2U_{I-1}^{(n)}-2U_I^{(n)}}{h^2}+(N-1)\frac{U_I^{(n)}-U_{I-1}^{(n)}}{h}\right)-\frac{2b}{h}(U_I^{(n)})^{-p},$$

$$U_i^{(0)}=\varphi(x_i),\ 0\le i\le I,$$

where $n\ge 0$. In order to permit the discrete solution to reproduce the properties of the continuous one when t approaches the real quenching time T, we need to adapt the size of the time step so that we choose

$$\Delta t_n=\min\{\frac{(1-h^2)h^2}{4N},h^2(U_{hmin}^{(n)})^{p+1}\}$$

with $U_{hmin}^{(n)}=\min_{0\le i\le I}U_i^{(n)}$. Let us notice that the restriction on the time step ensures the positivity of the discrete solution. We also approximate the solution u of (20)-(22) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta t_n}=Na(x_0)\frac{2U_1^{(n+1)}-2U_0^{(n+1)}}{h^2},$$

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}$$
$$=a(x_i)\left(\frac{U_{i+1}^{(n+1)}-2U_i^{(n+1)}+U_{i-1}^{(n+1)}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n+1)}-U_{i-1}^{(n+1)}}{2h}\right),$$
$$1\le i\le I-1,$$

$$\frac{U_I^{(n+1)}-U_I^{(n)}}{\Delta t_n}=a(x_I)\left(\frac{2U_{I-1}^{(n+1)}-2U_I^{(n+1)}}{h^2}+(N-1)\frac{U_I^{(n+1)}-U_{I-1}^{(n+1)}}{h}\right)$$
$$-\frac{2b}{h}(U_I^{(n)})^{-p-1}U_I^{(n+1)},$$

$$U_i^{(0)}=\varphi(x_i),\ 0\le i\le I,$$

where $n \geq 0$. As in the case of the explicit scheme, here, we also pick

$$\Delta t_n = h^2 (U_{hmin}^{(n)})^{p+1}.$$

Let us again remark that for the above implicit scheme, the existence and positivity of the discrete solution are also guaranteed using standard methods (see, for instance [7]). It is not hard to see that $u_{rr}(0,t) = \lim_{r \to 0} \frac{u_r(r,t)}{r}$. Hence, if $r = 0$, then we note that

$$u_t(0,t) = Na(x_0) u_{rr}(0,t), \ t \in (0,T).$$

This observation has been taken into account in the construction of our schemes at the first node x_0. We need the following definition.

Definition 4.1. We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if $\lim_{n \to \infty} U_{hmin}^{(n)} = 0$, and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical quenching time of the discrete solution $U_h^{(n)}$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations n, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |t_{n+1} - t_n| \leq 10^{-16}.$$

The order (s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $p = 1$, $N = 2$.

First case: $\varepsilon = 0$.

I	t_n	n	CPU time	s
16	0.055111	208	1	-
32	0.053755	671	3	-
64	0.053345	2377	18	1.73
128	0.053225	8937	132	1.79

Table 1. Numerical quenching times, numbers of iterations, *CPU* times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.055551	209	1	-
32	0.053879	673	4	-
64	0.053378	2379	24	1.74
128	0.053233	8940	751	1.79

Table 2. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

Second case: $\varepsilon = 1/10$.

I	t_n	n	CPU time	s
16	0.074882	261	1	-
32	0.073512	863	3	-
64	0.073094	3110	22	1.72
128	0.072970	11808	194	1.75

Table 3. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.075398	262	1	-
32	0.073654	865	4	-
64	0.073131	3112	35	1.74
128	0.073002	11810	220	2.01

Table 4. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

Third case: $\varepsilon = 1/50$.

I	t_n	n	CPU time	s
16	0.058519	217	1	-
32	0.057148	705	3	-
64	0.056733	2504	19	1.72
128	0.056611	9434	138	1.77

TABLE 5. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.058961	218	1	-
32	0.057272	706	4	-
64	0.056733	2504	19	1.74
128	0.056642	9434	156	2.03

TABLE 6. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

Fourth case: $\varepsilon = 1/100$.

I	t_n	n	CPU time	s
16	0.056783	218	1	-
32	0.055419	688	2	-
64	0.055007	2439	18	1.73
128	0.054886	9181	129	1.77

TABLE 7. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.057224	214	1	-
32	0.055543	689	4	-
64	0.055040	2441	29	1.74
128	0.054903	9185	152	1.88

TABLE 8. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

Fifth case: $\varepsilon = 1/1000$.

I	t_n	n	CPU time	s
16	0.055276	208	1	-
32	0.053919	673	3	-
64	0.053508	2483	17	1.72
128	0.053388	8961	107	1.78

TABLE 9. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	CPU time	s
16	0.055715	210	1	-
32	0.054042	674	4	-
64	0.053541	2385	28	1.74
128	0.053397	8965	129	1.80

TABLE 10. Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

Remark 4.1. If we consider the problem (20)-(22) in the case where $\varepsilon \in (0,1)$, then we observe from Tables 1 to 10 that if ε is small enough, then the numerical quenching time is close to that of the solution of (20)-(22) in the case where $\varepsilon = 0$. This computational result confirms the theory established in the previous section.

In Figures 1–8, we also give some plots to illustrate our analysis. In the figures we see that the discrete solution quenches in a finite, and the quenching occurs at the last node.

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Received September 20, 2008.

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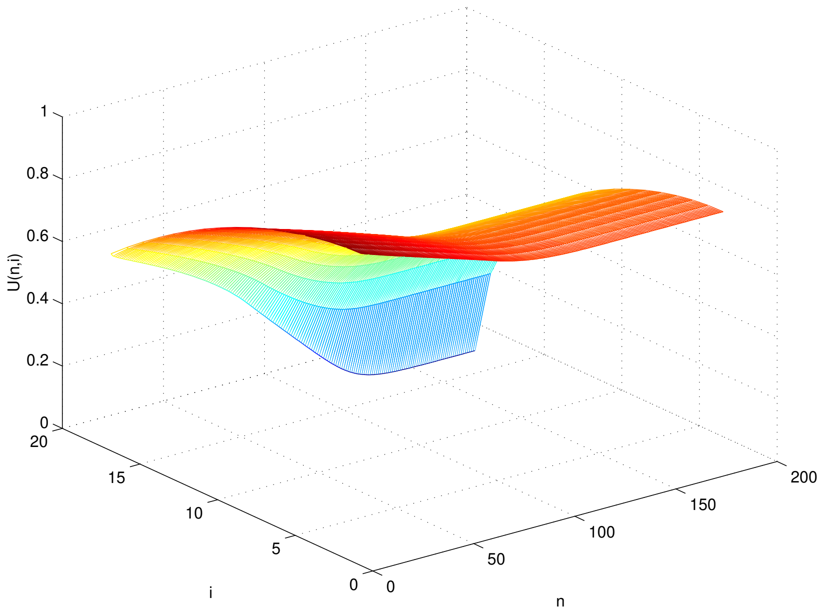

FIGURE 1. Evolution of discrete solution, $\varepsilon = 0$

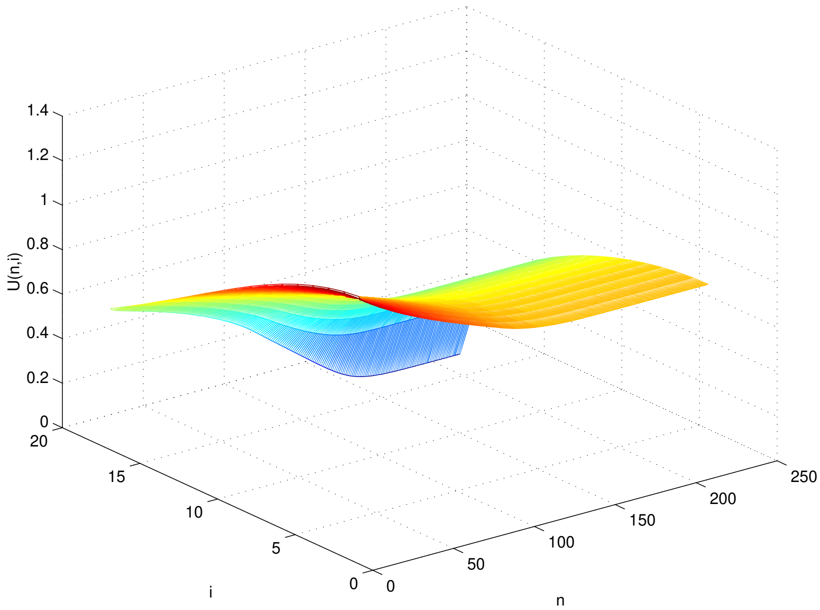

FIGURE 2. Evolution of discrete solution, $\varepsilon = 1/10$

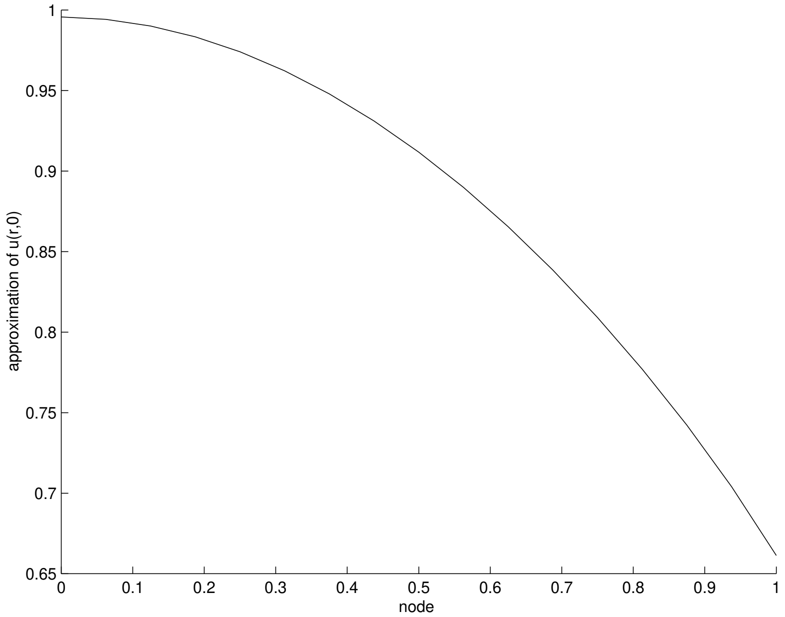

FIGURE 3. Profile of the approximation of $u(r, 0)$, $\varepsilon = 0$

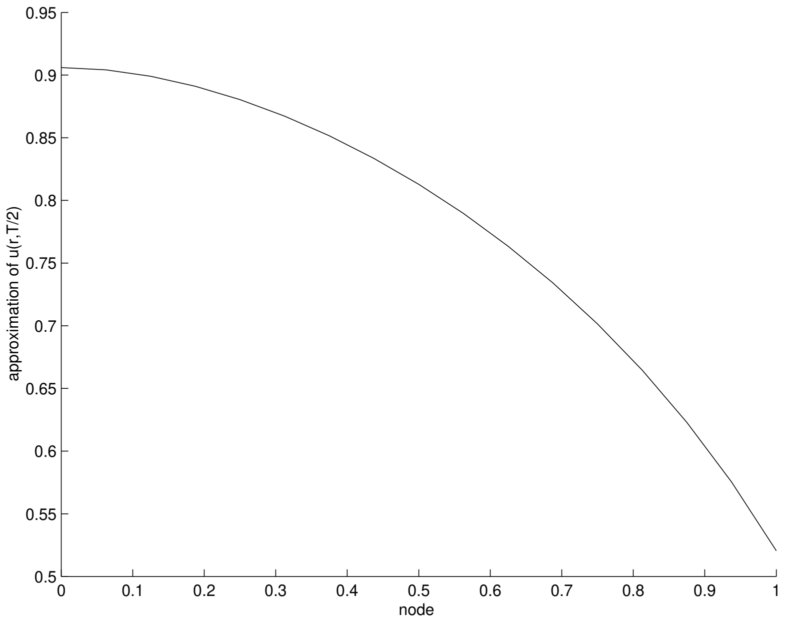

FIGURE 4. Profile of the approximation of $u(r, T/2)$, $\varepsilon = 0$

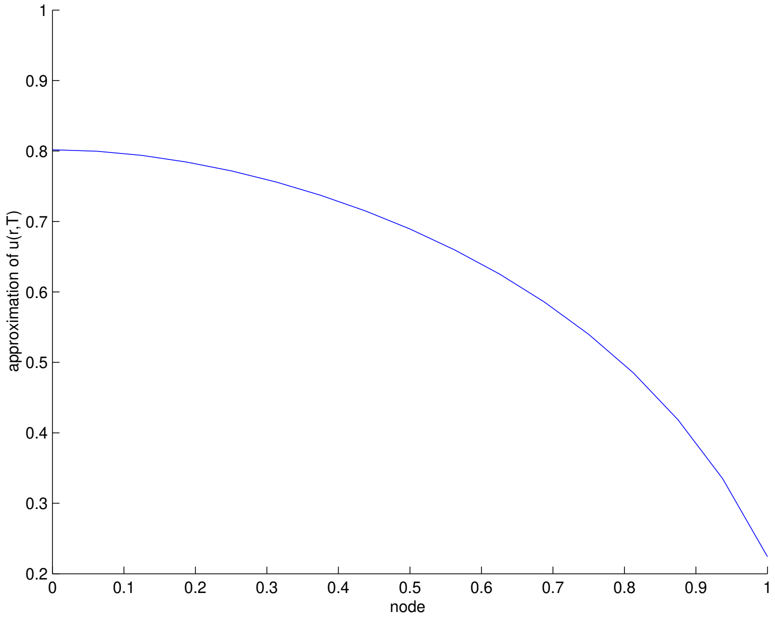

FIGURE 5. Profile of the approximation of $u(r,T)$, where T is the quenching time $\varepsilon = 0$

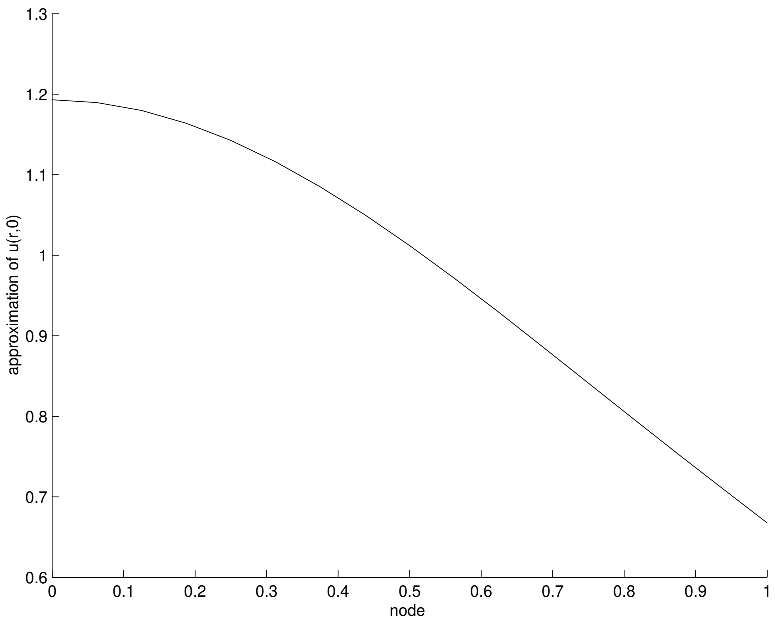

FIGURE 6. Profile of the approximation of $u(r,0)$, $\varepsilon = 1/10$

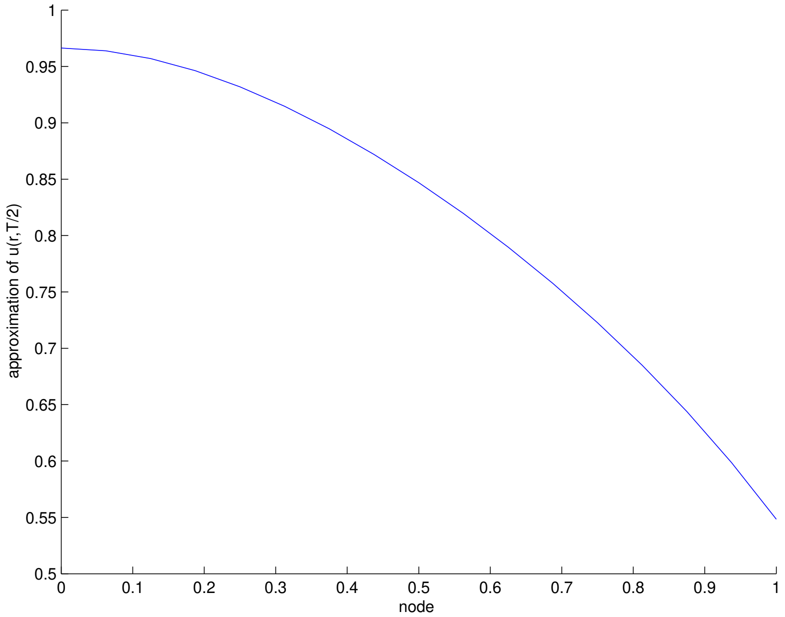

FIGURE 7. Profile of the approximation of $u(r, T/2)$, $\varepsilon = 1/10$

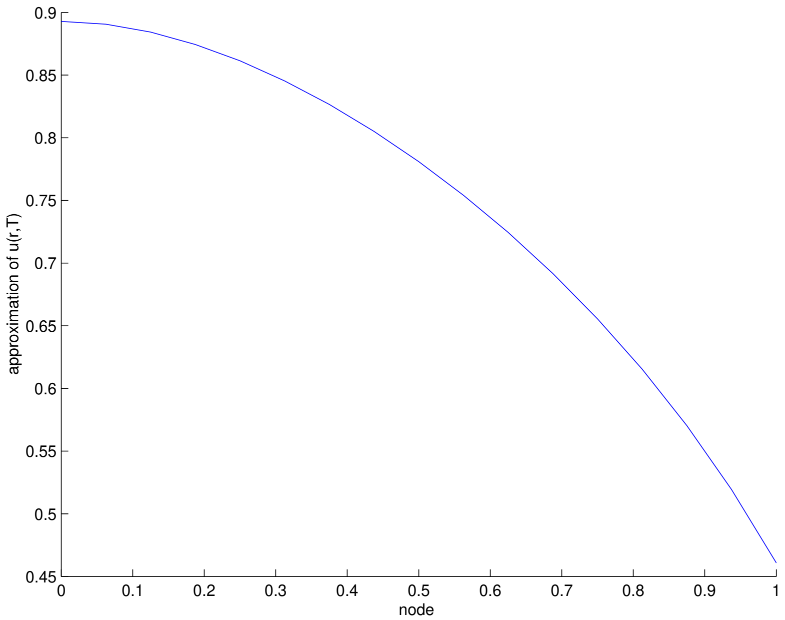

FIGURE 8. Profile of the approximation of $u(r, T)$, where T is the quenching time, $\varepsilon = 1/10$