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COMMEMORATION ON OTTÓ VARGA ON THE CENTENARY OF HIS BIRTH

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Ottó Varga was an outstanding researcher, an architect of Finsler geometry in the 40-th and 50-th years of the last century, and initiator of the study of this geometry in Hungary. The greatest number of citations relate to his works both in the first and second monographs on Finsler geometry, written by Hanno Rund in 1959, and a generation later, in 1986, by Makoto Matsumoto.

He was born in 1909 in Szepetnek, a small village in western Hungary as a son of a Lutheran priest. Soon the family moved to Poprad (now in Slovakia). Varga attended his secondary school in the nearby town Kezmarok, a picturesque place of old historic tradition at the foot of the Tatra mountains. Here he became perfect in the Czech and German languages. He started his university studies at the Architecture Faculty of Vienna Polytechnic, but after a year he changed for the German University in Prague. Here he became influenced by the work of Ludwig Berwald, and started studies in Finsler geometry at its early stage. He received his Ph.D. degree in 1934 under Berwald's, supervision and he acquired his *Habilitation* in 1937 at the German University at a young age. In the meantime he spent a year in Hamburg at Wilhelm Blaschke. At the same time was a postdoctoral fellow there the well known Chinese-American geometer Shiing-Shen Chern, who passed away a few years ago in a high age. They never could meet each other later. After the German occupation of Czechoslovakia Varga left Prague, and after a short stay in Kolozsvár (Cluj), he moved to Debrecen. At that time he was the single mathematician at Debrecen University. This was not an exceptional phenomenon. Between the two world wars a chair usually meant a single professor and not more. Only the chairs with laboratories, as the chairs for physics or chemistry were exceptions, where one could find a first (senior) assistant. After the war the number of the students increased considerable, a new university structure was set up, and at the end of the 1950s years, when out of family reason Varga left Debrecen for Budapest, he left behind

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a new institute with about 20 well trained young mathematicians, and a group of Finsler geometers consisting of A. Rapcsák, A. Moór and Gy. Soós. I also consider him as my master. Since then a group of Finsler geometers exists and works in Debrecen. Also he founded the journal Publicationes Mathematicae Debrecen, which is a widely known journal even now. In Budapest he worked on the Technical University and in the Mathematical Research Institute of the Hungarian Academy of Sciences. However, he spent the most productive 20 years of his carrier in Debrecen. Budapest was not so appropriate place for him. He was separated from his collaborators, and also his health has impaired. He died in 1969 at a relatively young age in heart decease.

Many of his results, in other form, in modern context and notation emerge in papers of the last decades. In most of his articles he starts from a property of Riemannian geometry, and asks for the analogue one in Finsler geometry, a method often used in the modern investigations.

Nearly all of his papers are written in German. Today this is not fortunate if somebody wants to look up the origin of a problem. Nevertheless before the Second World War German was at least so widely used in science as French or English. The change started during the war, and the victory of the English became complete, when Varga's carrier came to its end.

This commemoration is not the right place for a comprehensive survey on Varga's scientific achievements. (His list of publication is added to this article.) So I present only a few details of his scientific work and ideas.

If somebody wants to develop Finsler geometry on the analogy of Riemannian geometry, the existence of a linear metrical connection is indispensable. A connection of this kind was created by Elie Cartan. First it was published in his short Comptes Rendus article (1933), and then in his booklet Les espaces de Finsler (1934). Varga finished his first work at the time of the publication of the Comptes Rendus article. Varga introduced and studied in his work the concept of an affine connection and its curvatures in the line element manifold of a Finsler space. His work had a considerable overlapping with Cartan's article and booklet. Therefore it became published only in a local journal in 1936. A few years later Varga resumed the theme, and gave an elegant, geometrical construction for the metrical parallel translation, and thus for the metrical linear connection in Finsler spaces. These ideas were applicable also for the introduction of the Cartan connection and of the flag curvature.

Invariant differential [8]. Let us consider a Finsler space $F^n = (M, \mathcal{F})$ over a base manifold M^n with fundamental function $\mathcal{F}$ (in local coordinates $\mathcal{F}(x, y)$, $x \in M$, $y \in T_xM$). Let $g(x, y)$ be the Finsler metric tensor, and $L : (x(t), y(t))$, $t_1 \leq t \leq t_2$ a line-element sequence along the curve $x(t) \subset M$. Varga considered the 1-parameter family of Finsler geodesics $\gamma(s; x(t)) = \gamma(s, t)$ emanating from $x(t)$ in the direction of $y(t)$. In $\gamma(s, t)$ t is the parameter of the family, and s is the arc length parameter on the geodesics $\gamma(s, t_0)$, $t_1 \leq t_0 \leq t_2$. $\{\gamma(s, t)\}$ can be extended in a narrow tube $\mathcal{T}$ around $x(t)$ to a family $\gamma(s,a)$, $a = a_1, a_2, \dots, a_n$ covering $\mathcal{T}$ 1-folded, such that the correspondence $(s,a) \to x \in \mathcal{T}$ is one-to-one. Thus

$$\frac{d}{ds}\gamma(s,a) =: r(x)$$

is a vector field on $\mathcal{T}$, and

$$\overline{g}(x) := g(x, r(x)) \Rightarrow V^n = (\mathcal{T}, \overline{g})$$

determines a Riemannian space V^n on $\mathcal{T}$. Let $\xi_0 \in T_{x_0}M$, and $\xi(x(t))$ be its parallel translate along $x(t)$ in V^n:

$$\mathcal{P}^{V^n}_{x(t)}\xi_0 = \xi(x(t)). \tag{0.1}$$

Then, defining this $\xi(x,(t))$ as the parallel translate of ξ_0 in F^n along L, i.e., by the prescription

$$\mathcal{P}^{F^n}_{(x(t),y(t))}\xi_0 = \xi(x(t), y(t)) := \xi(x(t)).$$

Varga obtains a metrical linear connection, which turns out to be the Cartan connection.

This construction can also be considered as a geometric interpretation of the Cartan connection. The method applied here is the method of osculation of a Finsler space by a Riemannian one along a line-element sequence.

Flag curvature [22]. The above method of osculation could be applied also in case of the flag curvature. It is well known that the sectional curvature $\overline{R}(x_0, p_0)$ of a Riemannian space $V^n = (M, g)$ at $x_0 \in M$ and at a 2-dimensional plane position p_0 in $T_{x_0}M$ is the Gauss curvature $K_{V^2}(x_0)$ of a 2-dimensional Riemannian space $V^2 = (\Phi^2, \overline{g})$ at $x_0 \in \Phi^2$, where $\Phi^2 = \{\gamma(x_0, \dot{x}_0 \mid \dot{x}_0 \in p_0\}$ consists of the geodesics of V^n emanating from x_0 in directions $\dot{x}_0$ tangent to p_0, and $\overline{g}$ means the Riemann metric induced on Φ^2 by the original V^n:

$$\overline{R}(x_0, p_0) = K_{V^2}(x_0), \quad V^2 = V^n(\Phi^2). \tag{0.2}$$

Does a similar relation exist in Finsler spaces F^n? Varga gave a positive answer to the question. He proved that

$$B(x, y, X) = S_{F^2}(x, y), \tag{0.3}$$

where B is the flag curvature with flag pole $X(x,y)$. Varga called it Berwald curvature, and it was called Riemann curvature by H. Rund. $S(x,y)$ is the "interior curvature" of a 2-dimensional Finsler space F^2 defined and used by P. Finsler in his dissertation (Kurven and Flächen in allgemeinen Räumen. Diss. Göttingen 1918, pp. 104–106). The F^2 in (0.3) is constructed by the use of $X(x,y)$.

Let us consider a Finsler space F^n with metric tensor $g(x,y)$, and a geodesic $\gamma(s)$ (denoted also by $C_0(s)$) related to the arc length parameter s. C_0 can be extended again to a congruence of curves $\mathbb{C} = \{C(s,a)\}$, $a = a_1, \dots, a_n$;

$C(s,0) = C_0$ in a tube $\mathcal{T}$ around $C_0 = \gamma$, such that the correspondence $(s,a) \to x \in \mathcal{T}$ is one-to-one. Then

$$\frac{d}{ds}C(s,a) = r(x)$$

yields a vector field on $\mathcal{T}$, and by

$$g(x, r(x)) =: \overline{g}(x) \Longrightarrow V_F^n = (\mathcal{T}, \overline{g}) \tag{0.4}$$

$r(x)$ induces a Riemannian space V^n on $\mathcal{T}$. Varga proved that along γ

$$\overline{\Gamma}_j{}^i{}_k(x) = \Gamma^*{}_j{}^i{}_k(x, r(x)),$$

where $\overline{\Gamma}_j{}^i{}_k$ are the coefficients of the Levi-Civita connection of the constructed V^n, and $\Gamma^*{}_j{}^i{}_k$ are the connection coefficients appearing in the Cartan connection of F^n. Then along γ

$$\overline{R}_{ijkl}(x)r^j(x)r^k(x) = R_{ijkl}(x, r(x))r^j(x)r^k(x),$$

where $\overline{R}$ is the curvature tensor of V^n, and R is a curvature tensor of F^n. Let now $X(\gamma(s))$ be a vector field (flag poles) along γ. Then

$$p(s) := (r(\gamma(s)), X(\gamma(s))$$

are plane positions along γ. Then

$$B(x, r, X) = \overline{R}(x, p) = K_{V^2}(x),$$

and at (x_0, p_0) we obtain

$$B_0 = K_0. \tag{0.5}$$

This is very similar to (0.2). Yet B_0 is a curvature of a Finsler space, and K_0 is the curvature of a Riemannian, not of a 2-dimensional Finslerian space.

By making use of the interior curvature Varga went a step further. The notion of the interior curvature S of an F^2 is related to the parallel curves. Let $\gamma(t)$ be a geodesic of F^2, and $\gamma^\Psi(s,t)$ a family of geodesics emanating from the points $\gamma(t)$ and making an angle Ψ_0 with $\frac{d}{dt}\gamma(t)$. Then $\gamma^{\Psi_0}(d_0, t)$ is a parallel curve. If $s = d_0 \sin\Psi$ then Ψ is a function of s, and S is defined by

$$S := \lim_{\Psi \to 0} \frac{d^2\Psi}{ds^2}.$$

Finsler proved that in case of a Riemannian space V^2 the interior curvature is independent of y, and equals the sectional curvature of V^2:

$$F^2 = V^2 \Longrightarrow S(x,y) = K_{V^2}(x).$$

If $r(x)$ is tangent to ϕ^2, then from (0.4) one obtains $K_0 = S_0$ at any (x_0, p_0), $x_0 \in \gamma$. Then by (0.5)

$$B_0 = S_0,$$

that is the flag curvature of F^n equals the interior curvature of an F^2.

Invariant basis [24]. According to Felix Klein each classical geometry is the invariant theory of a transformation group. For example notions and theorems of Euclidean, affine or projective geometries are invariants of groups of certain linear transformations. Moreover, for each of them there exists an invariant notion (an invariant basis) by which one can express any other invariant notion of the related geometry. At the mentioned geometries these are (in succession) distance, affine ratio and cross ratio. Riemannian and Finsler geometries are not classical geometries. There exists no transformation group such that every theorem of these geometries could be characterized as a statement invariant under the transformations of a group. Nevertheless there may exist geometric objects, such that any geometric object of a Riemannian or of a Finsler space, or of their non-metric version can be expressed by several given objects. These form an invariant basis of the concerned geometry. In case of a Riemannian or affinelly connected space such geometric objects (called invariant basis) was found by T. Y. Thomas and O. Veblen. The basic tool to this yielded the normal coordinates. These are certain geodesic polar coordinates, where the geodesics emanating from the origin cover a domain 1-folded.

The problem of finding an invariant basis for line-element spaces was solved by Varga. The basic difficulty was that geodesics (autoparallel curves) emanating from a point (x_0, y_0) of a line-element space do not cover 1-folded a domain of the line-element space. Varga surmounted this difficulty by introducing a new type of curves consisting not of points, but of line-elements, and called them quasi-geodesics. A quasi-geodesic is given by a line-element (x_0, y_0) and a vector ξ_0, and it is by definition a curve of line-elements $L(t) = (x(t), y(t))$ such that $\dot{x}(t)$ is a parallel vector field along $L(t)$:

$$\dot{x}(t) = \mathcal{P}_{L(t)}\dot{x}_0, \quad \dot{x}_0 = \xi_0,$$

while $y(t)$ is also parallel along $L(t)$:

$$y(t) = \mathcal{P}_{L(t)} y_0.$$

The quasi-geodesic curves belonging to a given line-element (x_0, y_0) and to varying ξ form a (normal) coordinate system. Using these coordinates Varga proved that by the connection coefficients $C_j{}^i{}_k(x, y)$ and $\Gamma^*{}_j{}^i{}_k(x, y)$ and their derivatives one can express every geometric objects of a line-element space.

Angular metric [36]. Varga revealed the geometric role of the v-curvature tensor S of a Finsler spaces. It is known that F^n induces a Riemannian metric on the indicatrix $I(x)$. He found that S can be expressed in terms of the curvature tensor R of the Riemann space induced on the indicatrix. He proved that

$$S(y, p) = \frac{S_{hkij} p^{hk} p^{ij}}{p^{rs} p_{rs}} = c(\text{const.}) \Longleftrightarrow$$

$$S_{khij} = c \begin{vmatrix} g_{kj - l_k l_j} & g_{ki} - l_k l_i \\ g_{hj} - l_k l_j & g_{hi} - l_h l_0 \end{vmatrix},$$

where p means a plane position, (g_{ik}) is the metric tensor of F^n, and $l(y)$ is the unit vector in the direction of y. From these it follows that S vanishes if and only if the sectional curvature c vanishes:

$$S(y) = 0 \Longleftrightarrow c = 0,$$

and for the curvatures we obtain that

$$R = (1 + S)k,$$

where $k = \frac{1}{F^2(y)}$, which has the value 1 on the indicatrix $I(x)$.

On the indicatrix we have $ds = d\varphi$ (ds means the infinitesimal distance on $I(x)$, and $d\varphi$ means the corresponding infinitesimal angle). Then the angular metric is Euclidean if and only if $S = 0$, which is a result having already appeared also at Cartan without proof.

These results are of basic importance.

Spaces of constant curvature [43], [46]. According to the plane criterion of F. Schur and E. Cartan, a Riemannian space $V^n = (M, g)$ is of constant curvature iff to any $(n-1)$ dimensional plane position p in $T_x M$ there exists a totally geodesic hypersurface ϕ^{n-1} tangent to p. Varga characterized the V^n of constant positive and of constant negative curvature separately. He proved that a V^n is of constant negative curvature iff through any p there exist two ϕ^{n-1} with Euclidean metric, and of constant positive curvature iff there exists one ϕ^{n-1} with the same constant curvature for each p. According to his result, the Finsler spaces of constant curvature can also be characterized by this property: F^n is of constant curvature iff through any plane position p there exists a total geodesic ϕ^{n-1}.

His further results can be found in his papers, a list of which we present in what follows.

Ottó Varga's list of publication

1. Beiträge zur Theorie der Finslerschen Räume und der affinzusammenhängenden Räume von Linienelementen. Lotos, Prague **84** (1936), 1–4.
2. Integralgeometrie 3. Croftons Formeln für den Raum. Math. Z. **40** (1935), 384–405.
3. Integralgeometrie 8. Über Masse von Paaren linearer Mannigfaltigkeiten im projektiven Raum P_n. Rev. Mat Hispano-Americana (1935), 241–279.
4. Integralgeometrie 9. Über Mittelwerte an Eikörpern, Mathematica **12**, 65–80. – with W. Blaschke.

5. Integralgeometrie 19. Mittelwerte an dem Durchschnitt bewegter Flächen. Math. Z. **41** (1936), 768–784.
6. Integralgeometrie 24. Über die Schiebungen im Raum. Math. Z. **42** (1937), 710–736.
7. Über die Integralinvarianten die zu einer Kurve in der Hermiteschen Geometrie gehören. Acta Sci. Math. Szeged **9** (1939), 88–102.
8. Zur Herleitung des invarianten Differentials in Finslerschen Räumen. Monatshefte f. Math. und Phys. **50** (1941), 165–175.
9. Zur Differentialgeometrie der Hyperflächen in Finslerschen Räumen. Deutsche Math. **6** (1941), 1992–212.
10. The establishment of the invariant differential in Finsler spaces. (Hungarian) Mat és Fiz. Lapok (1941), 423–435.
11. Zur Begründung der Minkowskischen Geometrie. Acta Sci. Math. Szeged **10** (1943), 149–163.
12. The construction of Finsler geometry with the aid of the osculating Minkowski metric. (Hungarian) Mat. és Term. Ért. **61** (1942), 14–21.
13. On a way of characterizing the Riemannian spaces of constant curvature. (Hungarian) Mat. és Fiz. Lapok **50** (1943), 34–39.
14. Über eine Klasse von Finslerschen Räumen, die die nichteuklidischen verallgemeinern. Comm. Math. Helv. **19** (1946), 367–380.
15. Linielementräume, deren Zusammenhang durch eine beliebige Transformationsgruppe bestimmt ist. Acta Sci. Math. Szeged **11** (1946), 55–62.
16. Über die Lösung differentialgeometrischen Fragen in der nichteuklidischen Geometrie unter gleichzeitiger Verwendung homogener und inhomogener Koordinaten. Hung. Acta Math. **14** (1947), 35–52.
17. Vektorfelder, deren kovariante Ableitung längs einer vorgegebenen Kurve verschwindet. Hung. Acta Math. **1/4** (1949), 1–3.
18. Über affinzusammenhängende Mannigfaltigkeiten von Linienelementen insbesondere deren Aquivalenz. Publ. Math. Debrecen **1** (1949). 7–17.
19. Bemerkung zur Arbeit des Herrn A. Dinghas ,,Zur Metrik nichteuklidischer Räume." Math. Nachrichten **2** (1949), 386–388.
20. Affinzusammenhängende Mannigfaltigkeiten von Linienelementen, die ein Inhaltsmass besitzen. Proc. Acad. Amsterdam **52** (1949), 316–322.
21. Über den Zusammenhang der Krümmungsaffinoren in zwei eineindeutig aufeinander abgebildeten Finslerschen Räumen. Acta Sci. Math. Szeged **12** (1950), 132–135.
22. Über das Krümmungsmass in Finslerschen Räumen. Publ. Math. Debrecen **1** (1949), 116–122.
23. Applications of integral geometry in geometric optics. (Hungarian) MTA III. Oszt. Kzleményei **1** (1951), 192–201.

24. Normalkoordinaten in allgemeinen differentialgeometrischen Räumen und ihre Verwendung zur Bestimmung sämtlicher Differentialinvarianten.
Comptes Rendus de I. Congrés des Math. Hongrois, (1950), 147–162.
25. Anwendung von p-Vektoren auf derivierte Matrizen. Publ. Math. Debrecen **2** (1951), 137–145.
26. Soviet results in differential geometry. (Hungarian) Mat. Lapok **2** (1951), 190–218.
27. Eine geometrische Charakterisierung der Finslerschen Räume skalarer und konstanter Krümmung. Acta Math. Acad. Sci. Hung. **2** (1951), 143–155.
28. A rewiew of the ,,Appendix”’s new edition. (Hungarian) MTA III. Oszt. Közl. **3** (1953), 281–283.
29. The effect of the geometry of Bolyai–Lobacevskii on the development of geometry. (Hungarian) MTA III. Oszt. Közl. **3** (1953), 151–171.
30. Eine Charakterisierung der Finslerschen Räume mit absolutem Parallelismus der Linienelemente. Archiv der Math. **5** (1953), 128–131.
31. Bedigungen für die Metrisierbarkeit von affinzusammenhängenden Linienelementmannigfaltigkeiten. Acta Math. Acad. Sci. Hung. **5** (1954), 7–16.
32. Bemerkung zur Cayley–Kleinschen Massbestimmung. Publ. Math. Debrecen **4** (1955), 3–15. – with J. Aczél
33. A characterization of Finsler spaces having an absolute parallelism of the line elements. Acta Univ. Debrecen **1** (1954), 105–108.
34. Über Riemannsche Räume, die freie Beweglichkeit besitzen. Schriftenreihe des Forschungsinstituts für Math. **1** (1957), 124–130.
35. L’influence de la géométrie de Bolyai–Lobatchevsky sur le développement de la géométrie. Acta Math. Acad. Sci. Hung. **5** (1954), 71–94.
36. Die Krümmung der Eichfläche des Minkowskischen Raumes und die geometrische Deutung des einen Krümmungstensors des Finslerschen Räumes. Abh. Math. Sem. Univ. Hamburg **20** (1955), 41–51.
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39. Verallgemeinerte Riemannsche Normalkoordinaten und einige Anwendungen derselben. Izvestija Szofia **4** (1959), 61–69.
40. Ein elementargeometrischer Beweis des Sylvester–Frankeschen Determinantensatzes. Izvestija Szofia **4** (1959), 105–107.

41. Hilbertsche verallgemeinerte nicht-euklidishe Geometrie und Zusammenhang derselben mit der Minkowskischen Geometrie. Internat. Congress of Math. Abstracts, Edinburg (1958), 111.
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44. Bemerkung zur Winkelmetrik in Finslerschen Räumen. Ann. Univ. Sci. Budapest, Secttio Math. **3–4** (1960/1961) 379–382.
45. Über den inneren und induzierten Zusammenhang für Hyperflächen in Finslerschen Räumen. Publ. Math. Debrecen **8** (1961) 208–217.
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50. Über Hyperflächen konstanter Normalkrümmung in Minkowskischen Räumen. Tensor N. S. **13** (1963), 246–250.
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