Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
26 (2010), 343–347
www.emis.de/journals
ISSN 1786-0091

ON GEODESIC MAPPINGS OF MANIFOLDS WITH AFFINE CONNECTION

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ABSTRACT. In this paper we prove that all manifolds with affine connection are globally projectively equivalent to some manifold with equiaffine connection (equiaffine manifold). These manifolds are characterized by a symmetric Ricci tensor.

1. INTRODUCTION

Diffeomorphisms between two manifolds A_n and $\bar{A}_n$ with affine connection are called *geodesic*, if any geodesic of A_n is mapped to a geodesic of $\bar{A}_n$, see for example [2] – [16], etc.

Affine (or *trivial geodesic*) *mappings* are mappings which preserve canonical parameters of geodesics. Many papers are devoted to the metrizability or projective metrizability of manifolds with affine connection. Under metrizability of a manifold A_n we understand the existence of a metric g, which generates the affine connection ∇, such that ∇ is the Levi-Civita connection of g (for which $\nabla g = 0$), see [2, 3, 4, 9]. On the other hand, the problem of metrizability, respectively projective metrizability, is equivalent to that of affine, respectively geodesic, mappings of a manifold A_n with affine connection onto (pseudo-) Riemannian manifolds.

Equiaffine manifolds, characterized by the symmetry of the Ricci tensor, are manifolds in which the volume of an n dimensional parallelepiped is invariant under parallel transport, play an important role in the theory of geodesic mappings. J. Mikeš and V. E. Berezovski [1, 9] found fundamental equations of geodesic mappings of equiaffine manifolds onto (pseudo-) Riemannian manifolds

2000 *Mathematics Subject Classification.* 53B05, 53C05.
Key words and phrases. Geodesic mapping, manifolds with affine connection, equiaffine manifold.
Supported by the Council of the Czech Government MSM 6198959214, research & development project No. 0021630511, and Czech-Hungarian Cooperation Project MEB 040907.

in the form of systems of linear partial differential equations of Cauchy type in terms of covariant derivatives. These results were used for further studies by M. G. Eastwood and V. Matveev [2].

In the paper [11] by I. Hinterleitner, J. Mikeš and V. A. Kiosak it was proved that any manifold with affine connection is locally projective equiaffine. We show that these properties hold globally, i.e. any arbitrary manifold with affine connection globally admits geodesic mappings onto some equiaffine manifold.

For this reason the solution of the problem of the projective metrizability of a manifold A_n (or equivalently of geodesic mappings of A_n onto (pseudo-) Riemannian manifolds $\bar{V}_n$) can be realized as geodesic mapping of the equiaffine manifold $\tilde{A}_n$, which is projectively equivalent to the given manifold A_n.

There results are published in arXiv [8].

2. Main properties of geodesic mappings

Let $A_n = (M, \nabla)$ be a manifold M with affine connection ∇. We suppose that ∇ is torsion-free, i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$, where $[\ ,\ ]$ is the Lie bracket: $[X, Y]f = X(Yf) - Y(Xf)$ for differentiable functions f. Here and after $X, Y, \dots$ are tangent vectors.

The assumption that the studied manifolds have torsion-free connections bases on the fact that evidently a manifold with the connection ∇ has the same geodesics as with the symmetric part of ∇.

The curvature R of a manifold A_n is a tensor field of type $(1, 3)$ defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z, \tag{1}$$

called sometimes also the Riemannian tensor of the connection.

We can introduce the Ricci tensor Ric of type $(0, 2)$ as a trace of a linear map, namely:

$$\mathrm{Ric}(X, Y) = \mathrm{trace}\{V \mapsto R(Y, V)X\}.$$

A manifold A_n with a torsion-free (symmetric) affine connection is called *equiaffine*, if the Ricci tensor is symmetric, i.e. ([9, 12, 15])

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}(Y, X).$$

It is known [9, 12, 15] that (pseudo-) Riemannian manifolds are equiaffine manifolds.

A manifold $A_n = (M, \nabla)$ with affine connection admits a geodesic mapping onto $\bar{A}_n = (M, \bar{\nabla})$, if and only if the *Levi-Civita equation* holds [2, 4, 9, 15]:

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X, \tag{2}$$

where ∇ and $\bar{\nabla}$ are the affine connections of A_n and $\bar{A}_n$, respectively, and ψ is a linear form.

Geodesic mappings with $\psi \equiv 0$ are *trivial* or *affine*.

After applying formula (1) for the curvature tensor and expression (2) to a geodesic mapping $A_n \to \bar{A}_n$ we found a relationship between the curvature tensors R and $\bar{R}$ of A_n and $\bar{A}_n$:

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z - \Psi(Z,Y)\cdot X + \Psi(Z,X)\cdot Y \\ &\quad - (\Psi(X,Y) - \Psi(Y,X))\cdot Z,\end{aligned} \tag{3}$$

where

$$\Psi(X,Y) = \nabla_Y \psi(X) - \psi(X)\psi(Y). \tag{4}$$

By contraction of (3) we obtain the following relation for the Ricci tensors

$$\overline{\mathrm{Ric}}\,(X,Y) = \mathrm{Ric}\,(X,Y) + n\,\Psi(X,Y) - \Psi(Y,X). \tag{5}$$

3. Main results

Suppose $A_n \in C^1$, i.e. the components of the affine connection ∇ of A_n are functions of type C^1 on all charts of the manifold M.

In our paper [11] we proved that any manifold A_n with affine connection is locally projectively equiaffine.

The following theorem holds generally:

Theorem 3.1. *All manifolds A_n with affine connection are projectively equivalent to equiaffine manifolds.*

Remark 3.1. In other words, an arbitrary manifold A_n ($\in C^1$) with affine connection admits a global geodesic mapping onto an equiaffine manifold $\bar{A}_n$, and moreover $\bar{A}_n \in C^1$.

Proof. We prove the existence of a geodesic mapping of $A_n = (M,\nabla) \in C^1$ onto an equiaffine manifold $\bar{A}_n = (M,\bar{\nabla})$. The manifold $\bar{A}_n$ has a connection $\bar{\nabla}$ without torsion and its Ricci tensor $\overline{\mathrm{Ric}}$ is symmetric.

As already mentioned, we can suppose that the affine connection ∇ of A_n is torsion free.

It is known that on the manifold M we can globally construct a metric $\tilde{g}$ so that $\tilde{g} \in C^2$, i.e. the components $\tilde{g}_{ij}$ of $\tilde{g}$ in a coordinate domain of M are functions of type C^2. We denote by $\tilde{\nabla}$ the Levi-Civita connection of $\tilde{g}$.

Because $(\nabla_Y X - \tilde{\nabla}_Y X)$ is a tensor of type $(1,2)$, we can construct the one-form ψ on the manifold M in the following way

$$\psi(X) = -\frac{1}{n+1}\,\mathrm{trace}(Y \mapsto (\nabla_Y X - \tilde{\nabla}_Y X)). \tag{6}$$

This form is defined globally on M.

With the help of formula (2) applied to ψ we construct globally the affine connection $\bar{\nabla}$ on M:

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X. \tag{7}$$

It is evident that A_n is geodesically mapped onto $\bar{A}_n = (M, \bar{\nabla})$, and, evidently, $\bar{A}_n \in C^1$.

Now we prove that $\bar{A}_n$ is equiaffine. It is sufficient to calculate that

$$\overline{\mathrm{Ric}}\,(X,Y) = \overline{\mathrm{Ric}}\,(Y,X),$$

where $\overline{\mathrm{Ric}}$ is the Ricci tensor of $\bar{A}_n$.

This condition is equivalent to the existence of a function f in every coordinate neighbourhood, so that

$$\bar{\Gamma}^{\alpha}_{i\alpha}(x) = \frac{\partial f(x)}{\partial x^i}, \tag{8}$$

where $\bar{\Gamma}^h_{ij}$ are components of the connection $\bar{\nabla}$ and $x = (x^1, x^2, \ldots, x^n)$ are points with local coordinates.

In local coordinates, formulas (6) and (7) have the following form

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i(x)\,\delta^h_j + \psi_j(x)\,\delta^h_i,$$
$$\psi_i(x) = -\,\frac{1}{n+1}\,(\Gamma^{\alpha}_{i\alpha}(x) - \tilde{\Gamma}^{\alpha}_{i\alpha}(x)),$$

where Γ^h_{ij}, $\tilde{\Gamma}^h_{ij}$, ψ_i are components of ∇, $\tilde{\nabla}$, ψ, and δ^h_i is the Kronecker delta.

Using these formulas we calculate $\bar{\Gamma}^{\alpha}_{i\alpha}$. We obtain

$$\bar{\Gamma}^{\alpha}_{i\alpha} = \tilde{\Gamma}^{\alpha}_{i\alpha}.$$

From the known formula by Voss-Weyl

$$\tilde{\Gamma}^{\alpha}_{i\alpha} = \frac{\partial}{\partial x^i} \ln \sqrt{|\det(\tilde{g}_{ij})|},$$

where $\tilde{g}_{ij}$ are components of the metric tensor $\tilde{g}$ evidently, follows formula (8), where $f = \ln \sqrt{|\det(\tilde{g}_{ij})|}$.

Therefore $\bar{A}_n$ is equiaffine. □

Remark 3.2. The equiaffine connection $\bar{\nabla}$ constructed in this way is constructed explicitly from the original connection ∇.

Similarly it could be shown that a manifold with projective connection admits a geodesic mapping onto a manifold with equiaffine connection.

Acknowledgments

The authors are grateful to the referees for careful reading and remarks.

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