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EXAMPLES AND NOTES ON GENERALIZED CONICS AND THEIR APPLICATIONS

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Abstract. Let Γ be a subset of the Euclidean coordinate space. A generalized conic is a set of points with the same average distance from the points $\gamma \in \Gamma$. First of all we consider some realizations of this concept. Basic properties will be given together with an application. It is a general process to construct convex bodies which are invariant under a fixed subgroup G of the orthogonal group in $\mathbb{R}^n$. Such a body induces a Minkowski functional with the elements of G in the linear isometry group. To take the next step consider $\mathbb{R}^n$ as the tangent space at a point of a connected Riemannian manifold M and G as the holonomy group. By the help of the method presented here M can be changed into a non-Riemannian Berwald manifold with the same canonical linear connection as that of M as a Riemannian manifold. Indicatrices with respect to the Finslerian fundamental function are generalized conics with respect to the Euclidean norm induced by the Riemannian metric.

1. Examples and basic properties

Let Γ be a subset of the Euclidean coordinate space $\mathbb{R}^n$. Norm and distance of the elements of the space are defined by the help of the canonical inner product

$$< \gamma_1, \gamma_2 >:= \gamma_1^1\gamma_2^1 + \dots \gamma_1^n\gamma_2^n$$

as usual. A generalized conic is a set of points with the same average distance from the points $\gamma \in \Gamma$. First of all we consider some realizations of this concept.

Example 1. $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ is a finite set of points in $\mathbb{R}^n$ and the average distance is measured as the arithmetic mean

$$F(x) := \frac{d(x,\gamma_1) + \dots + d(x,\gamma_m)}{m}$$

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of distances from the points γ_i's. Hypersurfaces of the form $F(x) = \text{const.}$ are called polyellipses/polyellipsoids with foci $\gamma_1, \dots, \gamma_m$, see [3], [4] and [8].

Remark 1. Instead of the arithmetic mean we can use its weighted version [4] or any other types of mean.

Example 2. Let

$$e_1 := (1, 0, \dots, 0),\ e_2 := (0, 1, 0, \dots, 0),\ \dots,\ e_n := (0, \dots, 0, 1)$$

be the canonical basis and consider the hyperplanes

$$H_i := \operatorname{aff}\{e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n\}, \text{ where } i = 1 \dots n.$$

$\Gamma = \{H_1, \dots, H_n\}$ and the average distance is measured as the arithmetic mean

$$F(x) := \frac{d(x, H_1) + \dots + d(x, H_n)}{n}$$

of distances from the hyperplanes H_i's. Hypersurfaces of the form $F(x) = \text{const.}$ are just spheres with respect to the 1-norm because

$$F(x) = \frac{|x^1| + \dots + |x^n|}{n},$$

where $x = (x^1, \dots, x^n)$. They can be also considered as generalized conics in this sense. For another exotic example see Example 3.

Definition 1. Let Γ be a bounded orientable submanifold in $\mathbb{R}^n$ such that $\operatorname{vol} \Gamma < \infty$ with respect to the induced Riemannian volume form $d\gamma$. The average distance is measured as the integral

$$F(x) := \frac{1}{\operatorname{vol} \Gamma} \int_\Gamma \gamma \mapsto d(x, \gamma)\, d\gamma.$$

Hypersurfaces of the form $F(x) = \text{const.}$ are called *generalized conics* with foci $\gamma \in \Gamma$.

Remark 2. Let $\Gamma \subset \mathbb{R}^n$ be of dimension n. The integral

$$\int_\Gamma \gamma \mapsto d(x, \gamma)\, d\gamma$$

can be interpreted as the volume of the body $C(x) \subset \mathbb{R}^{n+1}$ bounded by Γ in the horizontal hyperplane $\mathbb{R}^n$ and the upper half of the right circular cone with opening angle $\frac{\pi}{2}$. It has a vertical axis to the horizontal hyperplane at the vertex x. Instead of the n-dimensional measure we can use any other types of measure of sets constructed from the set of foci and the points of the space. One of the possible idea is presented in the following example.

Example 3. If we measure the area of generalized cones with a common directrix as the set of foci then any set of vertices of cones with the same area can be also considered as a generalization of conics, see [5].

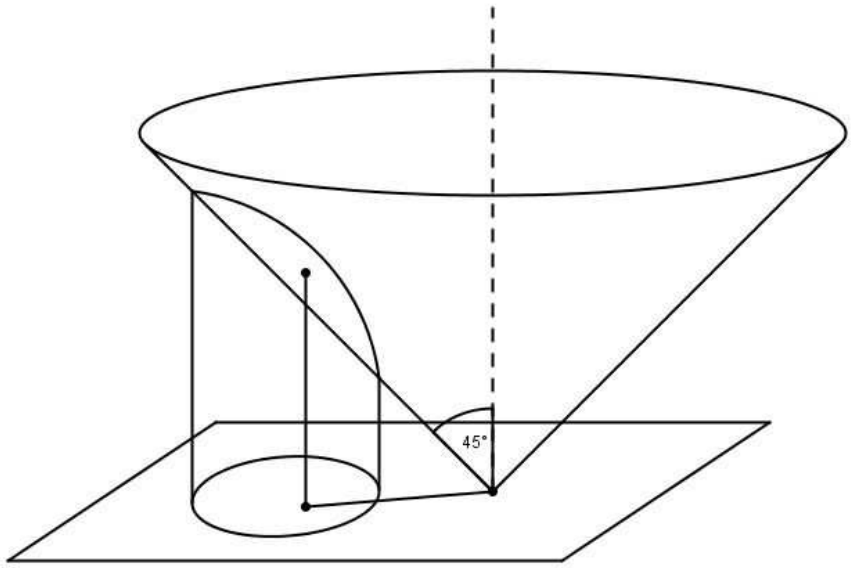

FIGURE 1. The body $C(x)$.

Theorem 1. *F is a convex function satisfying the growth condition*

$$\liminf_{|x|\to\infty} \frac{F(x)}{|x|} > 0,$$

where $|x|$ is the Euclidean norm of x.

Proof. Convexity is clear because for any fixed element $\gamma \in \Gamma$ the function

$$x \mapsto d(x, \gamma)$$

is convex. Since Γ is bounded, we can define the constant $K := \sup_{\gamma\in\Gamma} \ |\gamma|$. Then

$$K + d(x,\gamma) \geq \ |\gamma| + |x-\gamma| \ \geq \ |x|, \text{ i.e } d(x,\gamma) \geq \ |x| - K$$

and the inequality

$$1 - \frac{K}{|x|} \geq 1 - \frac{1}{n}$$

is satisfied on the neighbourhood $|x| > nK$ of ∞ for any $\gamma \in \Gamma$. Therefore

$$\liminf_{|x|\to\infty} \frac{F(x)}{|x|} \geq 1 > 0$$

as was to be stated. □

Corollary 1. *The levels of the function F is bounded.*

For a proof see [5], see also [2].

Corollary 2. *F has a global minimizer.*

Proof. The statement follows from the Weierstrass's theorem: if all the level sets of a continuous function defined on a nonempty, closed set in $\mathbb{R}^n$ are bounded then it has a global minimizer, see [2]. $\square$

Let

$$\gamma\colon [0,2\pi] \to \mathbb{R}^3,\ \gamma(t) := (\cos t, \sin t, 0)$$

be the unit circle in the xy-coordinate plane and

$$F(x,y,z) := \frac{1}{2\pi}\int_0^{2\pi} \sqrt{\big(x-\cos t\big)^2 + \big(y-\sin t\big)^2 + z^2}\, dt.$$

The surface of the form $F(x,y,z) = \frac{8}{2\pi}$ is a generalized conics with foci S_1.

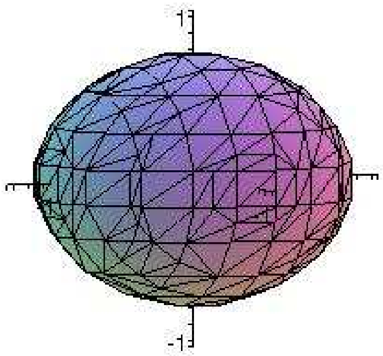

FIGURE 2. The surface of the form $F(x,y,z) = \frac{8}{2\pi}$.

It is obviously a revolution surface with generatrix

$$\int_0^{2\pi} \sqrt{\cos^2 t + \big(y-\sin t\big)^2 + z^2}\, dt = 8$$

in the yz-coordinate plane.

Lemma 1. *The surface $F(x, y, z) = \frac{8}{2\pi}$ is not an ellipsoid.*

Proof. It is enough to prove that the generatrix

$$\int_0^{2\pi} \sqrt{\cos^2 t + \big(y - \sin t\big)^2 + z^2}\, dt = 8$$

is not an ellipse in the yz-coordinate plane. If $y = 0$ then we have that

$$z = \pm\sqrt{\left(\frac{8}{2\pi}\right)^2 - 1}.$$

On the other hand, if $z = 0$ then the solutions of the equation

$$\int_0^{2\pi} \sqrt{\cos^2 t + \big(y - \sin t\big)^2}\, dt = 8$$

are just $y = \pm 1$. Therefore the only possible ellipse has the parametric form

$$y(s) = \cos s \text{ and } z(s) = \sqrt{\left(\frac{8}{2\pi}\right)^2 - 1}\,\sin s.$$

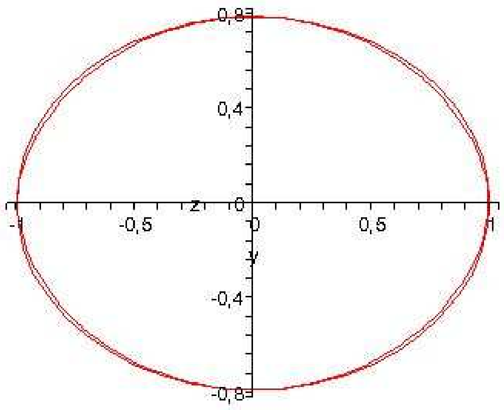

FIGURE 3. The generatrix and its approximating ellipse.

Consider the auxiliary function

$$v(s) := \int_0^{2\pi} \sqrt{\cos^2 t + \big(y(s) - \sin t\big)^2 + z^2(s)}\, dt.$$

Then

$$v\left(\frac{\pi}{3}\right)=\frac{2}{\pi}\sqrt{2}\sqrt{3}\sqrt{8+\pi^2}\ \mathrm{E}\left(\frac{2\sqrt{3}\pi}{3\sqrt{8+\pi^2}}\right),$$

where

$$\mathrm{E}(r):=\int_0^{\frac{\pi}{2}}\sqrt{1-r^2\sin^2 t}\,dt$$

is the standard elliptic integral. In 1997 the Vuorinen's conjecture

$$E(r)\geq\frac{\pi}{2}\left(\frac{1+(\sqrt{1-r^2})^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}}$$

was proved [1] and, consequently, the inequality

$$\sqrt{3}\sqrt{2}\sqrt{8+\pi^2}\left(\frac{1}{2}+\frac{1}{18}\sqrt{3}\left(9-\frac{12\pi^2}{8+\pi^2}\right)^{\frac{3}{4}}\right)^{\frac{2}{3}}>8$$

shows that $v(s)$ is not a constant function. Let

$$f:x\mapsto 9-\frac{12x^2}{8+x^2}\quad x\in[2,4]\,.$$

According to Taylor's formula there exists $\xi\in\ ]\,3,\pi\,[$ such that

$$f(\pi)=f(3)+f'(3)(\pi-3)+\frac{f''(3)}{2}(\pi-3)^2+\frac{f'''(3)}{6}(\pi-3)^3+\frac{f^{(4)}(\xi)}{24}(\pi-3)^4.$$

Further simple calculation gives that $f^{(4)}(x)$ is strictly increasing on the interval $[2,4]$ and, consequently,

$$f^{(4)}(\xi)>f^{(4)}(3)=-\frac{578304}{1419857}.$$

Therefore

$$f(\pi)>\frac{45}{17}-\frac{576}{289}\frac{1416}{10000}+\frac{1}{2}\frac{3648}{4913}\left(\frac{1415}{10000}\right)^2-\frac{1}{6}\frac{6912}{83521}\left(\frac{1416}{10000}\right)^3-$$

$$-\frac{1}{24}\frac{578304}{1419857}\left(\frac{1416}{10000}\right)^4=\frac{4111599143905029057}{1733223876953125000},$$

Since

$$\sqrt[4]{4111599143905029057}>45030,\qquad\sqrt[4]{1733223876953125000}<36284$$

and

$$\sqrt{3}>\frac{173205}{100000},$$

it follows that

$$\frac{1}{2}+\frac{\sqrt{3}}{18}f(\pi)^{\frac{3}{4}}>\frac{1}{2}+\frac{173205}{1800000}\left(\frac{45030}{36284}\right)^3=\frac{1306820588093503}{1910757030172160}.$$

Here

$$\sqrt[3]{1306820588093503} > 109329 \text{ and } \sqrt[3]{1910757030172160} < 124090.$$

On the other hand

$$6(8+\pi^2) > 6\left(8+\left(\frac{314159}{100000}\right)^2\right) = \frac{536087631843}{5000000000},$$

where

$$\sqrt{536087631843} > 732180 \text{ and } \sqrt{5000000000} < 70711.$$

Thus

$$\sqrt{6(8+\pi^2)}\left(\frac{1}{2}+\frac{\sqrt{3}}{18}f(\pi)^{\frac{3}{4}}\right)^{\frac{2}{3}} > \frac{732180}{70711}\left(\frac{109329}{124090}\right)^2 =$$

$$= \frac{437581162292769}{54441558913955} > 8$$

as was to be proved. □

Corollary 3. *The generalized conic $F(x,y,z) = \frac{8}{2\pi}$ induces a not Euclidean Minkowski functional containing the Euclidean isometries leaving the curve S_1 invariant in its linear isometry group.*

In what follows we are going to illustrate how to use generalized conics in the problem of metrizability of subgroups $G \subset O(n)$ in the sense of Corollary 3.

2. The case of reducible subgroups

If G is reducible and $n = 2$ we can always find a finite invariant set of points $\Gamma = \{\pm x_1, \pm x_2,\}$ under G. It is clear because the invariant subspace must be of dimension 1 together with its orthogonal complement. Their Euclidean unit vectors form the set of Γ. We can choose the origin as one of the foci too. Therefore any polyellipse with foci Γ induces a not Euclidean norm (in an equivalent terminology: Minkowski functional) L such that G is the subgroup of the linear isometries with respect to L.

If the dimension is great or equal than 3 then, by the reducibility of G we can take one of the Euclidean unit spheres

$$S_1 \subset S_2 \subset \ldots \subset S_{n-2}$$

as the invariant set under G. In case of S_{n-1} the conics are invariant under the whole orthogonal group because of the invariance of the set of their foci. Therefore they are spheres of dimension $n-1$. In case of S_1 at least one of the generalized conics is different from the ellipsoids centered at the origin as Lemma 1 says by taking $\mathbb{R}^3$ in $\mathbb{R}^n$ as a natural subspace. In what follows we are going to discuss the case of S_k for some special value of k. Odd and even integers give essentially different cases because elliptic integrals can be omitted if k is even.

Let $n \geq 4$ and $2 \leq k \leq n-2$ be a fixed integer. To express $S_k \subset \mathbb{R}^n$ in a parametric form consider the mapping

$$\rho_{k-1} \colon H \to S_{k-1} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$$

where $H \subset \mathbb{R}^{k-1}$ and

$$\rho_{k-1}(u) = (\rho^1(u), \dots, \rho^k(u), 0, \dots, 0)$$

gives the points of the sphere S_{k-1} by taking $\mathbb{R}^k$ in $\mathbb{R}^n$ as a natural subspace. Then

$$\rho_k \colon H \times [-\frac{\pi}{2}, \frac{\pi}{2}] \to S_k \subset \mathbb{R}^{k+1} \times \mathbb{R}^{n-(k+1)},$$
$$\rho_k(u,v) = (\rho(u)\cos(v), \sin(v), 0, \dots, 0).$$

Since the determinant of the first fundamental forms of S_{k-1} and S_k are related

$$\det g_{ij}(u,v) = (\cos^2(v))^{k-1} \det h_{ij}(u),$$

we have that for all $x \in \mathbb{R}^n$

$$F_k(x) := \int_{S_k} \gamma \mapsto d(x,\gamma)\, d\gamma = \int_{S_{k-1}} \gamma \mapsto \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{D(x,\gamma,v)} \cos^{k-1}(v)\, dv \right) d\gamma,$$

where

$$D(x,\gamma,v) := (x^1 - \gamma^1 \cos(v))^2 + \dots + (x^k - \gamma^k \cos(v))^2 + (x^{k+1} - \sin(v))^2 + (x^{k+2})^2 + \dots + (x^n)^2.$$

Consider the intersection of conics of the form $F_k(x) = \text{const.}$ with the plane

$$x^1 = \ldots = x^{k-1} = 0 \quad \text{and} \quad x^{k+3} = \ldots = x^n = 0$$

we have the niveau's of the function

$$f_k(y,z) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + y^2 + z^2 - 2y \sin t} \cos^{k-1} t\, dt$$

with variables $y := x^k$ and $z := x^{k+1}$, respectively.

For the sake of simplicity let $l := k-1$; then

$$f_k(1,0) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2(1-\sin t)} \cos^l t\, dt = \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos\frac{t}{2} - \sin\frac{t}{2} \right) \cos^l t\, dt$$
$$= \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos\frac{t}{2} - \sin\frac{t}{2} \right) \left(\cos^2\frac{t}{2} - \sin^2\frac{t}{2} \right)^l dt$$

$$= 2\sqrt{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos x - \sin x) \left(\cos^2 x - \sin^2 x\right)^l \, dx$$

$$= 2\sqrt{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \left(\cos^2 x - \sin^2 x\right)^l \, dx = 2\sqrt{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \left(1 - 2\sin^2 x\right)^l \, dx$$

because of $-\sin x = \sin(-x)$. Here

$$\begin{aligned} \int \cos x \left(1 - 2\sin^2 x\right)^l \, dx &= \int \cos x \sum_{i=0}^{l} \binom{l}{i} (-2)^{l-i} (\sin x)^{2(l-i)} \, dx \\ &= \sum_{i=0}^{l} \binom{l}{i} (-2)^{l-i} \int \cos x (\sin x)^{2(l-i)} \, dx \\ &= \sum_{i=0}^{l} \frac{1}{2(l-i)+1} \binom{l}{i} (-2)^{l-i} (\sin x)^{2(l-i)+1} \\ &= \sin x \sum_{i=0}^{l} \frac{1}{2l+1-2i} \binom{l}{i} \left(-2\sin^2 x\right)^{l-i} \end{aligned}$$

and thus

$$\begin{aligned} f_k(1,0) &= 2\sqrt{2} \left[\sin x \sum_{i=0}^{l} \frac{1}{2l+1-2i} \binom{l}{i} \left(-2\sin^2 x\right)^{l-i} \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\ &= 4 \sum_{i=0}^{l} \frac{1}{2l+1-2i} \binom{l}{i} (-1)^{l-i} = 4 \frac{1}{2l+1} \sum_{i=0}^{l} \frac{2l+1}{2l+1-2i} \binom{l}{i} (-1)^{l-i} \\ &= 4 \frac{1}{2l+1} \sum_{i=0}^{l} \binom{l}{i} (-1)^{l-i} + 4 \frac{1}{2l+1} \sum_{i=0}^{l} \frac{2i}{2l+1-2i} \binom{l}{i} (-1)^{l-i} \\ &= 4 \frac{2}{2l+1} \sum_{i=1}^{l} \frac{i}{2l+1-2i} \frac{l!}{i!(l-i)!} (-1)^{l-i} \\ &= 4 \frac{2l}{2l+1} \sum_{i=1}^{l} \frac{1}{2l+1-2i} \frac{(l-1)!}{(i-1)!(l-i)!} (-1)^{l-i} \\ &= 4 \frac{2l}{2l+1} \sum_{i=0}^{l-1} \frac{1}{2l+1-2i-2} \binom{l-1}{i} (-1)^{l-1-i} \\ &= 4 \frac{2l}{2l+1} \frac{2(l-1)}{2l+1-2} \sum_{i=0}^{l-2} \frac{1}{2l+1-2i-4} \binom{l-2}{i} (-1)^{l-2-i} = \ldots \end{aligned}$$

$$= \frac{2^{l+2} \cdot l!}{1 \cdot 3 \cdots (2l+1)}$$

Consider the curve

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1+y^2+z^2-2y\sin t}\cos^l t\,dt = \frac{2^{l+2} \cdot l!}{1 \cdot 3 \cdot \ldots \cdot (2l+1)}$$

passing through the point $y=1$ and $z=0$. We are going to prove that it is not an ellipse for some special value of l $(:= k-1)$. The proof is similar as in Lemma 1. First of all we determine the only possible ellipse by substituting $y=0$ into the equation. Then we have that

$$\sqrt{1+z^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^l t\,dt = c(l), \quad \text{where} \quad c(l) := \frac{2^{l+2} \cdot l!}{1 \cdot 3 \cdot \ldots \cdot (2l+1)}.$$

I. case. If l is odd, i.e. k is even, then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^l t\,dt = 2\frac{(l-1)!!}{l!!},$$

where

$$0!! := 1, \quad (l-1)!! := (l-1)\cdot(l-3)\cdot(l-5)\cdot\ldots\cdot 2 \quad \text{and}$$
$$l!! := l\cdot(l-2)\cdot(l-4)\cdot\ldots\cdot 1.$$

Therefore the only possible ellipse has the parametric form

$$y(s) = \cos s \quad \text{and} \quad z(s) = b(l)\sin s$$

with

$$b(l) := \sqrt{\frac{c^2(l)l!!^2}{4(l-1)!!^2} - 1}$$

Consider the auxiliary function

$$v_l(s) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1+y^2(s)+z^2(s)-2y(s)\sin t}\cos^l t\,dt.$$

Then

$$v_l\left(\frac{\pi}{3}\right) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{5}{4}+\frac{3}{4}b^2(l)-\sin t}\cos^l t\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{5}{4} + \frac{3}{4} b^2(l) - \sin t} \cos^{l-1} t \cos t \, dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{5}{4} + \frac{3}{4} b^2(l) - \sin t} \left(1 - \sin^2 t\right)^{\frac{l-1}{2}} \cos t \, dt.$$

Substituting $s^2 = 5 + 3b^2(l) - 4\sin t$

$$v\left(\frac{\pi}{3}\right) = \frac{1}{4} \int_{\sqrt{1+3b^2(l)}}^{\sqrt{9+3b^2(l)}} s^2 \left(1 - \left(\frac{5 + 3b^2(l) - s^2}{4}\right)^2\right)^{\frac{l-1}{2}} ds$$

$$= \frac{(-1)^{\frac{l-1}{2}}}{4^l} \int_{\sqrt{1+3b^2(l)}}^{\sqrt{9+3b^2(l)}} s^2 \left(s^2 - (1 + 3b^2(l))\right)^{\frac{l-1}{2}} \left(s^2 - (9 + 3b^2(l))\right)^{\frac{l-1}{2}} ds.$$

Taking $m = \frac{l-1}{2}$ we have that

$$v_l\left(\frac{\pi}{3}\right) = \frac{(-1)^m}{4^{2m+1}} \sum_{i,j=0}^{m} B_{ij}^2(l) \int_{\sqrt{1+3b^2(l)}}^{\sqrt{9+3b^2(l)}} \left(s^2\right)^{i+j+1} ds,$$

where

$$B_{ij}^2(l) := \binom{m}{i}\binom{m}{j}(-1)^{i+j}(1 + 3b^2(l))^{m-i}(9 + 3b^2(l))^{m-j}.$$

Therefore

$$v_l\left(\frac{\pi}{3}\right) = r_1\sqrt{9 + 3b^2(l)} - r_2\sqrt{1 + 3b^2(l)},$$

where r_1 and r_2 are rationals.

Lemma 2. *If* $L \equiv 1 \pmod 4$ *then* $\sqrt{L + 3b^2(l)}$ *is irrational.*

Proof. Suppose in contrary, that

$$L + 3b^2(l) = \frac{m^2}{n^2}$$

for some integers m and n. Since $L = 4K + 1$,

$$L + 3b^2(l) = 3\frac{c^2(l) l!!^2}{4(l-1)!!^2} + 4K - 2$$

$$= 3\frac{4^{l+1} \cdot l!^2 l!!^2}{1^2 \cdot 3^2 \cdots (2l+1)^2 \cdot (l-1)!!^2} + 4K - 2$$

$$= 3\frac{4^{l+1}\cdot(l-1)!!^2 l!!^4}{1^2\cdot 3^2\cdots(2l+1)^2\cdot(l-1)!!^2} + 4K - 2$$
$$= 3\frac{4^{l+1}\cdot l!!^4}{1^2\cdot 3^2\cdots(2l+1)^2} + 4K - 2$$
$$= 2\left(\frac{6\cdot 4^l\cdot l!!^4 + (2K-1)\cdot 1^2\cdot 3^2\cdots(2l+1)^2}{1^2\cdot 3^2\cdots(2l+1)^2}\right).$$

Therefore

$$2n^2\left(6\cdot 4^l\cdot l!!^4 + (2K-1)\cdot 1^2\cdot 3^2\cdots(2l+1)^2\right) = m^2\cdot 1^2\cdot 3^2\cdots(2l+1)^2$$

i. e. $2n^2(2N+1) = m^2(2M+1)$ which is obviously a contradiction because the prime factorizations of the left and the right hand side are of the form

$$2^{2\alpha+1}\cdots \text{ and } 2^{2\beta}\cdots,$$

respectively. □

Lemma 3. *For any odd integer l the hypersurface*

$$F_{l+1}(x) = c(l)$$

in $\mathbb{R}^n$, where $n \geq l+3$ is not an ellipsoid.

Proof. Suppose that

$$v_l\left(\frac{\pi}{3}\right) = c(l).$$

This means that

$$r_1\sqrt{9+3b^2(l)} - r_2\sqrt{1+3b^2(l)} = r_3,$$

where r_1, r_2 and $r_3 = c(l)$ are rationals. In view of Lemma 2 both of the coefficients r_1 and r_2 are different from zero. Taking the square of both side of the equation we have that

$$\sqrt{9+3b^2(l)} = \frac{r}{\sqrt{1+3b^2(l)}}$$

for some rational number r. Therefore

$$\frac{r_1 r - r_2(1+3b^2(l))}{r_3} = \sqrt{1+3b^2(l)}$$

which contradicts to Lemma 2. □

Corollary 4. *For any odd integer l the generalized conic*

$$F_{l+1}(x) = c(l)$$

in $\mathbb{R}^n$, where $n \geq l+3$ induces a not Euclidean Minkowski functional containing the Euclidean isometries leaving the surface S_{l+1} invariant in its linear isometry group.

II. case. If l is even, i.e. k is odd then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^l t\, dt = \pi \frac{(l-1)!!}{l!!}$$

and the only possible ellipse has the parametric form

$$y(s) = \cos s \quad \text{and} \quad z(s) = b(l) \sin s$$

with

$$b(l) := \sqrt{\frac{c^2(l) l!!^2}{\pi^2 (l-1)!!^2} - 1}.$$

The following figure shows the difference $\Delta_l := v_l\left(\frac{\pi}{3}\right) - c(l)$ in cases of $l = 2, 4, \ldots, 10$. The auxiliary function involves elliptic integrals which can not be calculated by using standard calculus.

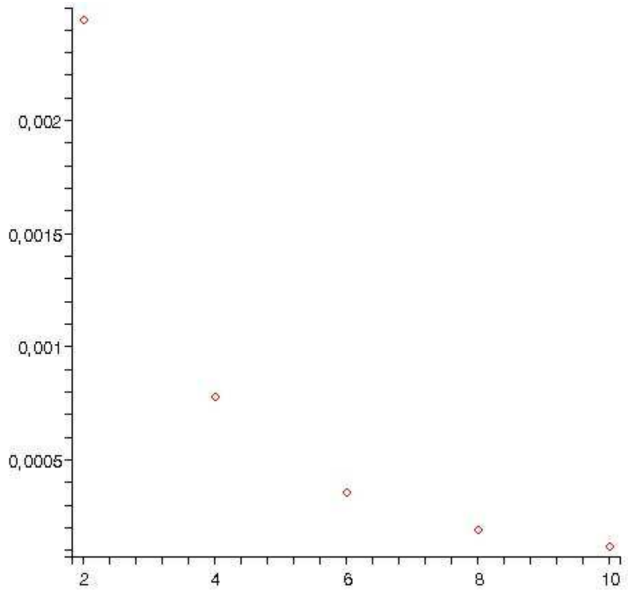

FIGURE 4. The case of $l = 2, 4, \ldots, 10$.

3. THE CASE OF IRREDUCIBLE SUBGROUPS

Surprisingly this case is almost trivial in view of the following Lemma. As we have seen above the key step of the construction is to find an invariant set under G as the foci of a generalized conic. It is natural to consider the orbits of the points with respect to G.

Lemma 4. *Let G be a closed subgroup of $O(n)$; it is irreducible if and only if the origin is the interior point of the convex hull of any non-trivial orbit (the only trivial orbit is that of the origin – it is a singleton).*

Proof. First of all note that the convex hulls of the orbits are closed and invariant under G. If G is irreducible and the origin is not a point of the convex hull of a non-trivial orbit then we can use a simple nearest-point-type argumentation as follows: taking the uniquely determined nearest point of the convex hull to the origin it can be easily seen that it must be a fixed point of any element of G. This contradicts to the irreducibility. If the origin is not in the interior of the convex hull we can consider the common part H of supporting hyperplanes at this point. It is not a singleton because the origin doesn't belong to any non-trivial orbit and thus it can not be one of the extremal points of the convex hull. H is obviously an invariant linear subspace under G which contradicts to the irreducibility. The converse is trivial. □

Note that if one of the convex hull of a non-trivial orbit is an ellipsoid (as a body) centered at the origin then it must be a ball in the Euclidean sense according to the irreducibility of G. By the Krein Milmann theorem the convex hull (as a convex compact set) K of the orbit is equal to the convex hull of the extremal points in K. Therefore they form a whole Euclidean sphere. It can be easily seen that the extremals must be in the orbit itself (because in the opposite case the punctured set $K \setminus \{p\}$ would be the convex hull instead of K) thus G is transitive on the Euclidean spheres and all of the possible Minkowski functionals must be Euclidean. In any other case Lemma 4 shows that themselves the convex hulls of the orbits induce possible Minkowski functionals.

Corollary 5. *If G is not transitive, closed and irreducible then the convex hull of any non-trivial orbit induces a not Euclidean Minkowski functional L such that G is the subgroup of the linear isometries with respect to L.*

Integration can be used to avoid singularities as the following example shows.

Example 4. Consider the group of the symmetries of the square

$$[-1,1] \times [-1,1]$$

centered at the origin in the Euclidean plane. The convex hull of all of non-trivial orbits are polygons, i.e. the boundary always has singularities. The orbit

$$\Gamma = \{(-1,-1),(1,-1),(1,1),(-1,1)\}$$

induces the supremum norm

$$|(x,y)| := \frac{1}{\sqrt{2}} \max \{|x|,|y|\}.$$

To avoid the singularities at the vertices consider the function

$$F(x,y) := \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\sqrt{(x-t)^2+(y-s)^2}\,ds\,dt.$$

The curves of the form $F(x,y) = \text{const.}$ are just generalized conics with foci $\operatorname{conv}\Gamma$. They are illustrated in the following figure in case of the constants 2, 1.5 and 1.3.

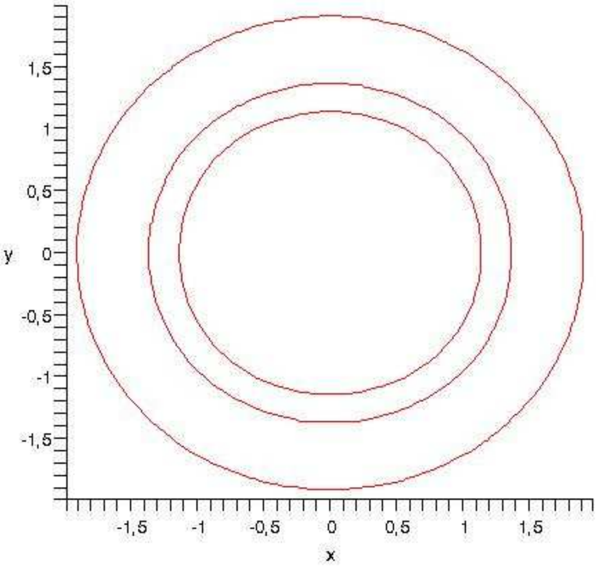

FIGURE 5. Curves of the form $F(x,y) = 2, 1.5$ and 1.3.

The following lemma states that they are not circles (according to the irreducibility of G the invariant ellipses must be circles). Therefore not Euclidean Minkowski functionals L's without singularities are induced such that G is the subgroup of the linear isometries with respect to L's.

Lemma 5. *The curve of the form $F(x,y) = c$ passing through the point $(2,1)$ is not a circle.*

Proof. According to the symmetric role of the variables x, t and y, s, respectively, we can calculate the coordinates

$$D_1F(x,y) = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\frac{x-t}{\sqrt{(x-t)^2+(y-s)^2}}\,ds\,dt,$$

$$D_2F(x,y) = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\frac{y-s}{\sqrt{(x-t)^2+(y-s)^2}}\,ds\,dt.$$

of the gradient vector field. Here

$$D_1F(x,y) = -\frac{1}{8}\Bigg[(s-y)\sqrt{(x-1)^2+(y-s)^2} + (x-1)^2\ln\left((s-y)+\sqrt{(x-1)^2+(y-s)^2}\right) + (s-y)\sqrt{(x+1)^2+(y-s)^2} + (x+1)^2\times \times\ln\left((s-y)+\sqrt{(x+1)^2+(y-s)^2}\right)\Bigg]_{-1}^{1}$$

and

$$D_2F(x,y) = D_1F(y,x).$$

Using these formulas consider the auxiliary function

$$v(x,y) := yD_1F(x,y) - xD_2F(x,y)$$

to measure the difference between the gradient vectors of the family of generalized conics and circles. We have

$$v(2,1) = -2\sqrt{13} + \frac{9}{2}\ln 3 - \frac{9}{2}\ln(-2+\sqrt{13}) + \frac{1}{2}\ln(-2+\sqrt{5}) - 8\ln 2 + 4\ln(-3+\sqrt{13}) + 4\ln(\sqrt{5}+1) + 8$$

which is obviously different from zero. □

Let now (M,g) be a connected Riemannian manifold and consider a point $p \in M$. If the holonomy group at p is not transitive on the unit sphere in T_pM we can use the technic presented here to construct a convex body (a generalized conic) containing the origin in its interior such that it is invariant under the element of the holonomy group at p. This induces a not Euclidean norm L_p in T_pM having the elements of the holonomy group as linear isometries. Extending this functional by the help of the parallel transport with respect to the Riemannian structure we have a Finsler manifold because *Finsler geometry is a non-Riemannian geometry in a finite number of dimensions. The differentiable structure is the same as the Riemannian one but distance is not uniform in all directions. Instead of the Euclidean spheres in the tangent spaces, the unit vectors form the boundary of general convex sets containing the origin in their interiors.* (M. Berger). Moreover this Finsler manifold has the same canonical connection as the original Riemannian one. In a precise terminology it is a (non-Riemannian) Berwald manifold.

Theorem 2. *If the holonomy group of a connected Riemannian manifold is closed and not transitive on the unit sphere in the tangent space then its Lvi-Civita connection is strictly Berwald metrizable.*

As it is well-known this is just the summary of Z. Szabó's result [7] on the characterization of the canonical connections of Berwald manifolds which is the first step to the classification theorem, see also [9]. He successfully used the results on the holonomy of Riemannian manifolds together with the standards of symmetric Lie algebras in the theory of Berwald manifolds. Recall them cite here some thoughts by J. Simons: *Several years ago M. Berger gave a classification of possible candidates for holonomy groups of manifolds having affine connections with zero torsion.... The most striking of his result is the list he determines of possible holonomy groups of a Riemannian manifold. These groups all turn out to be transitive on the unit sphere in the tangent space of the manifold except in the case that the manifold is a symmetric space of rank* ≥ 2. *It is natural to ask for an intrinsic proof of this rather startling fact, one which avoids the classification theorem.* Simons successfully realized the idea of such an intrinsic proof [6] established the theory of holonomy systems. Here we have made an attempt to give another way to the theory of Berwald manifolds, one which avoids both Simons's theory and the theory of symmetric Lie algebras.

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