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ON SOME ALGEBRAIC PROPERTIES OF GENERALIZED GROUPS

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Abstract. Some results that are true in classical groups are investigated in generalized groups and are shown to be either generally true in generalized groups or true in some special types of generalized groups. Also, it is shown that a Bol groupoid and a Bol quasigroup can be constructed using a non-abelian generalized group.

1. Introduction

A generalized group is an algebraic structure which has a deep physical background in the unified gauge theory and has direct relation with isotopes. Mathematicians and Physicists have been trying to construct a suitable unified theory for twistor theory, isotopes theory and so on. It was known that generalized groups are tools for constructions in unified geometric theory and electroweak theory. Electorweak theories are essentially structured on Minkowskian axioms and gravitational theories are constructed on Riemannian axioms. According to Araujo et. al. [4], generalized group is equivalent to the notion of completely simple semigroup.

Some of the structures and properties of generalized groups have been studied by Vagner [23], Molaei [17], [18], Mehrabi, Molaei and Oloomi [16], Molaei and Hoseini [21] and Agboola [1]. Smooth generalized groups were introduced in Agboola [2] and later on, Agboola [3] also presented smooth generalized subgroups while Molaei [19] and Molaei and Tahmoresi [20] considered the notion of topological generalized groups. Solarin and Sharma [22] were able to construct a Bol loop using a group with a non-abelian subgroup and recently, Chein and Goodaire [6] gave a new construction of Bol loops for odd case. Kuku [14] and Jacobson [11] contain most of the results on classical groups while for more on loops and their properties, readers should check [21, 5, 7, 8, 9, 12, 13]. The aim of this study is to investigate if some results that are true in classical group theory are also true in generalized groups and

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to find a way of constructing a Bol structure (i.e Bol loop or Bol quasigroup or Bol groupoid) using a non-abelian generalized group.

It is shown that in a generalized group G, $(a^{-1})^{-1} = a$ for all $a \in G$. In a normal generalized group G, it is shown that the anti-automorphic inverse property $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$ holds under a necessary condition. A necessary and sufficient condition for a generalized group(which obeys the cancellation law and in which $e(a) = e(ab^{-1})$ if and only if $ab^{-1} = a$) to be idempotent is established. The basic theorem used in classical groups to define the subgroup of a group is shown to be true for generalized groups. The kernel of any homomorphism(at a fixed point) mapping a generalized group to another generalized group is shown to be a normal subgroup. Furthermore, the homomorphism is found to be an injection if and only if its kernel is the set of the identity element at the fixed point. Given a generalized group G with a generalized subgroup H, it is shown that the factor set G/H is a generalized group. The direct product of two generalized group is shown to be a generalized group. Furthermore, necessary conditions for a generalized group G to be isomorphic to the direct product of any two abelian generalized subgroups are shown. It is shown that a Bol groupoid can be constructed using a non-abelian generalized group with an abelian generalized subgroup. Furthermore, it is established that if the non-abelian generalized group obeys the cancellation law, then a Bol quasigroup with a left identity element can be constructed.

2. Preliminaries

Definition 2.1. A generalized group G is a non-empty set admitting a binary operation called multiplication subject to the set of rules given below.

(i) $(xy)z = x(yz)$ for all $x, y, z \in G$.
(ii) For each $x \in G$ there exists a unique $e(x) \in G$ such that $xe(x) = e(x)x = x$ (existence and uniqueness of identity element).
(iii) For each $x \in G$, there exists $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e(x)$ (existence of inverse element).

Definition 2.2. Let L be a non-empty set. Define a binary operation $(\cdot)$ on L. If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid.

If the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions relative to x and y respectively, then $(L, \cdot)$ is called a quasigroup. Furthermore, if there exists a element $e \in L$ called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop.

Definition 2.3. A loop is called a Bol loop if and only if it obeys the identity

$$((xy)z)y = x((yz)y).$$

Remark 2.1. One of the most studied type of loop is the Bol loop.

2.1. **Properties of Generalized Groups.** A generalized group G exhibits the following properties:

(i) for each $x \in G$, there exists a unique $x^{-1} \in G$.
(ii) $e(e(x)) = e(x)$ and $e(x^{-1}) = e(x)$ where $x \in G$. Then, $e(x)$ is a unique identity element of $x \in G$.

Definition 2.4. If $e(xy) = e(x)e(y)$ for all $x, y \in G$, then G is called normal generalized group.

Theorem 2.1. *For each element x in a generalized group G, there exists a unique $x^{-1} \in G$.*

The next theorem shows that an abelian generalized group is a group.

Theorem 2.2. *Let G be a generalized group and $xy = yx$ for all $x, y \in G$. Then G is a group.*

Theorem 2.3. *A non-empty subset H of a generalized group G is a generalized subgroup of G if and only if for all $a, b \in H$, $ab^{-1} \in H$.*

If G and H are two generalized groups and $f\colon G \to H$ is a mapping then Mehrabi, Molaei and Oloomi [16] called f a homomorphism if $f(ab) = f(a)f(b)$ for all $a, b \in G$.

They also stated the following results on homomorphisms of generalized groups. These results are established in this work.

Theorem 2.4. *Let $f\colon G \to H$ be a homomorphism where G and H are two distinct generalized groups. Then:*

(i) *$f(e(a)) = e(f(a))$ is an identity element in H for all $a \in G$.*
(ii) *$f(a^{-1}) = (f(a))^{-1}$.*
(iii) *If K is a generalized subgroup of G, then $f(K)$ is a generalized subgroup of H.*
(iv) *If G is a normal generalized group, then the set*
$$\{(e(g), f(g)) :\ g \in G\}$$
with the product
$$(e(a), f(a))(e(b), f(b)) := (e(ab), f(ab))$$
is a generalized group denoted by $\cup f(G)$.

3. Main Results

3.1. Results on Generalized Groups and Homomorphisms.

Theorem 3.1. *Let G be a generalized group. For all $a \in G$, $(a^{-1})^{-1} = a$.*

Proof. $(a^{-1})^{-1}a^{-1} = e(a^{-1}) = e(a)$. Post multiplying by a, we obtain

$$[(a^{-1})^{-1}a^{-1}]a = e(a)a. \tag{1}$$

From the L. H. S.,

$$(a^{-1})^{-1}(a^{-1}a) = (a^{-1})^{-1}e(a) = (a^{-1})^{-1}e(a^{-1})$$
$$= (a^{-1})^{-1}e((a^{-1})^{-1}) = (a^{-1})^{-1}. \tag{2}$$

Hence from (1) and (2), $(a^{-1})^{-1} = a$. □

Theorem 3.2. *Let G be a generalized group in which the left cancellation law holds and $e(a) = e(ab^{-1})$ if and only if $ab^{-1} = a$. G is a idempotent generalized group if and only if $e(a)b^{-1} = b^{-1}e(a)$ $\forall a, b \in G$.*

Proof. $e(a)b^{-1} = b^{-1}e(a) \Leftrightarrow (ae(a))b^{-1} = ab^{-1}e(a) \Leftrightarrow ab^{-1} = ab^{-1}e(a) \Leftrightarrow e(a) = e(ab^{-1}) \Leftrightarrow ab^{-1} = a \Leftrightarrow ab^{-1}b = ab \Leftrightarrow ae(b) = ab \Leftrightarrow a^{-1}ae(b) = a^{-1}ab \Leftrightarrow e(a)e(b) = e(a)b \Leftrightarrow e(b) = b \Leftrightarrow b = bb$. □

Theorem 3.3. *Let G be a normal generalized group in which $e(a)b^{-1} = b^{-1}e(a)$ $\forall a, b \in G$. Then, $(ab)^{-1} = b^{-1}a^{-1}$ $\forall a, b \in G$.*

Proof. Since $(ab)^{-1}(ab) = e(ab)$, then by multiplying both sides of the equation on the right by $b^{-1}a^{-1}$ we obtain

$$[(ab)^{-1}ab]b^{-1}a^{-1} = e(ab)b^{-1}a^{-1}. \tag{3}$$

So,

$$[(ab)^{-1}ab]b^{-1}a^{-1} = (ab)^{-1}a(bb^{-1})a^{-1} = (ab)^{-1}a(e(b)a^{-1}) = (ab)^{-1}(aa^{-1})e(b)$$
$$= (ab)^{-1}(e(a)e(b)) = (ab)^{-1}e(ab) = (ab)^{-1}e((ab)^{-1}) = (ab)^{-1}. \tag{4}$$

Using (3) and (4), we get $[(ab)^{-1}ab]b^{-1}a^{-1} = (ab)^{-1} \Rightarrow e(ab)(b^{-1}a^{-1}) = (ab)^{-1} \Rightarrow (ab)^{-1} = b^{-1}a^{-1}$. □

Theorem 3.4. *Let H be a non-empty subset of a generalized group G. The following are equivalent.*

(i) *H is a generalized subgroup of G.*
(ii) *For $a, b \in H$, $ab^{-1} \in H$.*
(iii) *For $a, b \in H$, $ab \in H$ and for any $a \in H$, $a^{-1} \in H$.*

Proof. (i)$\Rightarrow$ (ii) If H is a generalized subgroup of G and $b \in G$, then $b^{-1} \in H$. So by closure property, $ab^{-1} \in H$ $\forall a \in H$.

(ii)$\Rightarrow$ (iii) If $H \neq \phi$, and $a, b \in H$, then we have $bb^{-1} = e(b) \in H$, $e(b)b^{-1} = b^{-1} \in H$ and $ab = a(b^{-1})^{-1} \in H$ i.e $ab \in H$.

(iii)$\Rightarrow$ (i) $H \subseteq G$ so H is associative since G is associative. Obviously, for any $a \in H$, $a^{-1} \in H$. Let $a \in H$, then $a^{-1} \in H$. So, $aa^{-1} = a^{-1}a = e(a) \in H$. Thus, H is a generalized subgroup of G. □

Theorem 3.5. *Let $a \in G$ and $f \colon G \to H$ be an homomorphism. If $\ker f$ at a is denoted by*

$$\ker f_a = \{x \in G : \; f(x) = f(e(a))\}.$$

Then,

(i) $\ker f_a \lhd G$.

(ii) f *is a monomorphism if and only if* $\ker f_a = \{e(a) : \ \forall a \in G\}$.

Proof. (i) It is necessary to show that $\ker f_a \leq G$. Let $x, y \in \ker f_a \leq G$, then $f(xy^{-1}) = f(x)f(y^{-1}) = f(e(a))(f(e(a)))^{-1} = f(e(a))f(e(a)^{-1}) = f(e(a))f(e(a)) = f(e(a))$. So, $xy^{-1} \in \ker f_a$. Thus, $\ker f_a \leq G$. To show that $\ker f_a \lhd G$, since $y \in \ker f_a$, then by the definition of $\ker f_a$, $f(xyx^{-1}) = f(x)f(y)f(x^{-1}) = f(e(a))f(e(a))f(e(a))^{-1} = f(e(a))f(e(a))f(e(a)) = f(e(a))$ $\Rightarrow xyx^{-1} \in \ker f_a$. So, $\ker f_a \lhd G$.

(ii) $f\colon G \to H$. Let $\ker f_a = \{e(a) : \ \forall a \in G\}$ and $f(x) = f(y)$, this implies that $f(x)f(y)^{-1} = f(y)f(y)^{-1} \Rightarrow f(xy^{-1}) = e(f(y)) = f(e(y)) \Rightarrow xy^{-1} \in \ker f_y \Rightarrow$

$$xy^{-1} = e(y) \tag{5}$$

and $f(x)f(y)^{-1} = f(x)f(x)^{-1} \Rightarrow f(xy^{-1}) = e(f(x)) = f(e(x)) \Rightarrow xy^{-1} \in \ker f_x \Rightarrow$

$$xy^{-1} = e(x). \tag{6}$$

Using (5) and (6), $xy^{-1} = e(y) = e(x) \Leftrightarrow x = y$. So, f is a monomorphism.

Conversely, if f is mono, then $f(y) = f(x) \Rightarrow y = x$. Let $k \in \ker f_a \ \forall a \in G$. Then, $f(k) = f(e(a)) \Rightarrow k = e(a)$. So, $\ker f_a = \{e(a) : \ \forall a \in G\}$. □

Theorem 3.6. *Let G be a generalized group and H a generalized subgroup of G. Then G/H is a generalized group called the quotient or factor generalized group of G by H.*

Proof. It is necessary to check the axioms of generalized group on G/H.

Associativity: Let $a, b, c \in G$ and $aH, bH, cH \in G/H$. Then $aH(bH \cdot cH) = (aH \cdot bH)cH$, so associativity law holds.

Identity: If $e(a)$ is the identity element for each $a \in G$, then $e(a)H$ is the identity element of aH in G/H since $e(a)H \cdot aH = e(a) \cdot aH = aH \cdot e(a) = aH$. Therefore identity element exists and is unique for each elements aH in G/H.

Inverse: $(aH)(a^{-1}H) = (aa^{-1})H = e(a)H = (a^{-1}a)H = (a^{-1}H)(aH)$ shows that $a^{-1}H$ is the inverse of aH in G/H.

So the axioms of generalized group are satisfied in G/H. □

Theorem 3.7. *Let G and H be two generalized groups. The direct product of G and H denoted by*

$$G \times H = \{(g, h) : \ g \in G \text{ and } h \in H\}$$

is a generalized group under the binary operation $\circ$ such that

$$(g_1, h_1) \circ (g_2, h_2) = (g_1g_2, h_1h_2).$$

Proof. This is achieved by investigating the axioms of generalized group for the pair $(G \times H, \circ)$. □

Theorem 3.8. *Let G be a generalized group with two abelian generalized subgroups N and H of G such $G = NH$. If $N \subseteq COM(H)$ or $H \subseteq COM(N)$*

where $COM(N)$ and $COM(H)$ represent the commutators of N and H respectively, then $G \cong N \times H$.

Proof. Let $a \in G$. Then $a = nh$ for some $n \in N$ and $h \in H$. Also, let $a = n_1h_1$ for some $n_1 \in N$ and $h_1 \in H$. Then $nh = n_1h_1$ so that $e(nh) = e(n_1h_1)$, therefore $n = n_1$ and $h = h_1$. So that $a = nh$ is unique.

Define $f \colon G \to H$ by $f(a) = (n, h)$ where $a = nh$. This function is well defined in the previous paragraph which also shows that f is a one-one correspondence. It remains to check that f is a group homomorphism.

Suppose that $a = nh$ and $b = n_1h_1$, then $ab = nhn_1h_1$ and $hn_1 = n_1h$. Therefore, $f(ab) = f(nhn_1h_1) = f(nn_1hh_1) = (nn_1, hh_1) = (n, h)(n_1, h_1) = f(a)f(b)$. So, f is a group homomorphism. Hence a group isomorphism since it is a bijection. □

3.2. Construction of Bol Algebraic Structures.

Theorem 3.9. *Let H be a subgroup of a non-abelian generalized group G and let $A = H \times G$. For $(h_1, g_1), (h_2, g_2) \in A$, define*

$$(h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2)$$

then $(A, \circ)$ is a Bol groupoid.

Proof. Let $x, y, z \in A$. By checking, it is true that $x \circ (y \circ z) \neq (x \circ y) \circ z$. So, $(A, \circ)$ is non-associative. H is a quasigroup and a loop(groups are quasigroups and loops) but G is neither a quasigroup nor a loop(generalized groups are neither quasigroups nor a loops) so A is neither a quasigroup nor a loop but is a groupoid because H and G are groupoids.

Let us now verify the Bol identity:

$$((x \circ y) \circ z) \circ y = x \circ ((y \circ z) \circ y)$$

$$\text{L.H.S.} = ((x \circ y) \circ z) \circ y = (h_1h_2h_3h_2, h_2h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3h_2^{-1}g_2).$$

$$\begin{aligned}\text{R.H.S.} &= x \circ ((y \circ z) \circ y)\\ &= (h_1h_2h_3h_2, h_2h_3h_2g_1h_2^{-1}(h_3^{-1}h_2^{-1}h_2h_3)g_2h_3^{-1}g_3h_2^{-1}g_2)\\ &= (h_1h_2h_3h_2, h_2h_3h_2g_1h_2^{-1}g_2h_3^{-1}g_3h_2^{-1}g_2).\end{aligned}$$

So, L.H.S.=R.H.S. Hence, $(A, \circ)$ is a Bol groupoid. □

Corollary 3.1. *Let H be a abelian generalized subgroup of a non-abelian generalized group G and let $A = H \times G$. For $(h_1, g_1), (h_2, g_2) \in A$, define*

$$(h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2)$$

then $(A, \circ)$ is a Bol groupoid.

Proof. By Theorem 2.2, an abelian generalized group is a group, so H is a group. The rest of the claim follows from Theorem 3.9. □

Corollary 3.2. *Let H be a subgroup of a non-abelian generalized group G such that G has the cancellation law and let $A = H \times G$. For $(h_1, g_1), (h_2, g_2) \in A$, define*

$$(h_1, g_1) \circ (h_2, g_2) = (h_1 h_2, h_2 g_1 h_2^{-1} g_2)$$

then $(A, \circ)$ is a Bol quasigroup with a left identity element.

Proof. The proof of this goes in line with Theorem 3.9. A groupoid which has the cancellation law is a quasigroup, so G is quasigroup hence A is a quasigroup. Thus, $(A, \circ)$ is a Bol quasigroup with a left identity element since by Kunen [15], every quasigroup satisfying the right Bol identity has a left identity. □

Corollary 3.3. *Let H be a abelian generalized subgroup of a non-abelian generalized group G such that G has the cancellation law and let $A = H \times G$. For $(h_1, g_1), (h_2, g_2) \in A$, define*

$$(h_1, g_1) \circ (h_2, g_2) = (h_1 h_2, h_2 g_1 h_2^{-1} g_2)$$

then $(A, \circ)$ is a Bol quasigroup with a left identity element.

Proof. By Theorem 2.2, an abelian generalized group is a group, so H is a group. The rest of the claim follows from Theorem 3.2. □

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