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STABILITY AND UNIFORM ULTIMATE BOUNDEDNESS OF SOLUTIONS OF SOME THIRD-ORDER DIFFERENTIAL EQUATIONS

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Abstract. The paper is concerned with the stability and uniform ultimate boundedness for all solutions of a third order nonlinear differential equations (1.1). Sufficient conditions under which all solutions $x(t)$, its first and second derivatives tend to zero as $t \to \infty$ are given.

1. Introduction

Nonlinear differential equations of second-, third-, fourth- and higher order have been extensively studied with higher degree of generality. In particular there have been interesting works on asymptotic behavior, boundedness, periodicity, almost periodicity and stability of solutions of third-order nonlinear differential equations. For over four decades many authors have dealt with stability and boundedness of solutions of third order differential equations and obtained many interesting results, see for instance Reissig *et. al.,* [10], a survey book and Ademola *et. al.,* [1, 2, 3], Afuwape [4], Chukwu [5], Ezeilo [6], Hara [7], Omeike [9], Swick [11], Tunç[12] and the references cited therein. These works were done with the aid of Liapunov functions except in [4] were frequency domain approach was used. With respect to our observation in the relevant literature, works on the stability and uniform ultimate boundedness of solution of nonlinear non-autonomous third order differential equation (1.1) using a complete Liapunov function are scarce.

In this paper, we consider the following third order differential equation

$$\dddot{x} + f(t, x, \dot{x}, \ddot{x})\ddot{x} + q(t)g(\dot{x}) + r(t)h(x) = p(t, x, \dot{x}, \ddot{x}), \tag{1.1}$$

or its equivalent system derived by setting $\dot{x} = y$ and $\ddot{x} = z$ to get

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = p(t, x, y, z) - f(t, x, y, z)z - q(t)g(y) - r(t)h(x), \tag{1.2}$$

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where: $f, p \in C(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R})$; $g, h \in C(\mathbb{R}, \mathbb{R})$; $q, r \in C(\mathbb{R}^+, \mathbb{R})$; $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R} = (-\infty, \infty)$. It is supposed that the functions f, g, h, q, r and p depend only on the arguments displayed explicitly, and the dots, as elsewhere, denote differentiation with respect to the independent variable t. However we shall require that the derivatives $\frac{\partial}{\partial t} f(t, x, y, z) = f_t(t, x, y, z)$, $\frac{\partial}{\partial x} f(t, x, y, z) = f_x(t, x, y, z)$, $\frac{d}{dx} h(x) = h'(x)$, $\frac{d}{dt} q(t) = q'(t)$ and $\frac{d}{dt} r(t) = r'(t)$ exist and are continuous and uniqueness of solutions of (1.1) will be assumed. We shall use Liapunov's second (or direct) method as our basic tool to achieve the desired results. The results obtained in this investigation improves the existing results on the third order non-linear differential equations in the literature.

2. Some Preliminaries

The following results will be basic to the proofs of our results. We do not give the proofs since they are found in [13]. Consider the equation

$$\dot{X} = F(t, X), \tag{2.1}$$

where $X = (x_1, \cdots, x_n)$, $F \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the $n-$dimensional Euclidean space.

Theorem 1. *Suppose that there exists a Liapunov function $V(t, X)$ defined on $\mathbb{R}^+$, $\|X\| \geq K$, where K may be large, which satisfies the following conditions:*

(i) *$a(\|X\|) \leq V(t, X) \leq b(\|X\|)$, where $a(r) \in CI$ (i.e continuous and increasing), $a(r) \to \infty$ as $r \to \infty$ and $b(r) \in CI$.*

(ii) *$\dot{V}_{(2.1)}(t, X) \leq 0$.*

Then the solutions of (2.1) *are uniformly bounded.*

Theorem 2. *Suppose that $F(t, X)$ of the system* (2.1) *is bounded for all t when X belongs to an arbitrary compact set in $\mathbb{R}^n$. Moreover, suppose that there exists a non-negative Liapunov function $V(t, X)$ such that*

$$\dot{V}_{(2.1)}(t, X) \leq -W(X),$$

where $W(X)$ is positive definite with respect to a closed set Ω in the space $\mathbb{R}^n$. Then every bounded solution of (2.1) *approaches Ω as $t \to \infty$.*

Theorem 3. *Under the assumptions in Theorem 2.1, if*

$$\dot{V}_{(2.1)}(t, X) \leq -c(\|X\|)$$

where $c(r) \in CI$, then the solutions of (2.1) *are uniformly ultimately bounded.*

3. Main Results

In the case $p(t, x, \dot{x}, \ddot{x}) \equiv 0$, (1.1) becomes

$$\dddot{x} + f(t, x, \dot{x}, \ddot{x})\ddot{x} + q(t)g(\dot{x}) + r(t)h(x) = 0, \tag{3.1}$$

or its equivalent system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -f(t, x, y, z)z - q(t)g(y) - r(t)h(x), \tag{3.2}$$

with the following results.

Theorem 4. *In addition to the fundamental assumptions on f, g, h, q and r, suppose that $\delta_0, \delta_1, a, a_1, b, b_1, c,$ are positive constants and that:*

(i) $h(0) = 0$, $\delta_0 \leq h(x)/x \quad (x \neq 0)$;
(ii) $g(0) = 0$, $b \leq g(y)/y \leq b_1 \quad (y \neq 0)$;
(iii) $h'(x) \leq c$ *for all* x;
(iv) $\delta_1 \leq r(t) \leq q(t)$, $r'(t) \leq q'(t) \leq 0$ *for all* $t \geq 0$;
(v) $a \leq f(t, x, y, z) \leq a_1$ *for all* $x, y, z, t \geq 0$ *and* $ab > c$;
(vi) $f_t(t, x, y, 0) \leq 0$, $y f_x(t, x, y, 0) \leq 0$ *for all* x, y *and* $t \geq 0$.

Then every solution $(x(t), y(t), z(t))$ of (3.2) is uniformly bounded and satisfies

$$x(t) \to 0, \quad y(t) \to 0, \quad z(t) \to 0 \tag{3.3}$$

as $t \to \infty$.

Theorem 5. *Suppose that $\delta_0, \delta_1, a, b, c$ are positive constants and that:*

(i) *conditions (i), (iii) and (vi) of Theorem 4 hold;*
(ii) $g(0) = 0, \quad g(y)/y \geq b \quad (y \neq 0)$;
(iii) $f(t, x, y, z) \geq a$ *for all* x, y, z *and* $t \geq 0$.

Then every solution $(x(t), y(t), z(t))$ of (3.2) *is bounded and satisfies* (3.3) *as $t \to \infty$.*

Theorem 6. *Further to the basic assumptions on f, g, h, p, q and r, suppose that $\delta_0, \delta_1, a, b, c, P_0$ are positive constants and that:*

(i) *conditions (i), (iii)-(vi), of Theorem 4 hold;*
(ii) $b \leq g(y)/y \leq b_1 \quad (y \neq 0)$;
(iii) $|p(t, x, y, z)| \leq P_0 < \infty$ *for all* x, y, z *and* $t \geq 0$.

Then every solution $(x(t), y(t), z(t))$ of (1.2) *is uniformly ultimately bounded.*

Remark 7. It should be noted that in the special case where $f(t, x, \dot{x}, \ddot{x}) = a$, $g(\dot{x}) = by$, $h(x) = cx$, $q(t) \equiv 1$, $r(t) \equiv 1$ and $p(t, x, \dot{x}, \ddot{x}) = 0$ in (1.1), then assumptions (i)-(vi) reduce to $a > 0$, $b > 0$, $ab - c > 0$, $c > 0$ which is the Routh-Harwitz condition for the global asymptotic stability of the zero solution of the equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0.$$

Remark 8. (i) In the special case $f(t, x, y, z) = f(x, y)$, $q(t) \equiv 1$, and $r(t) \equiv 1$ the assumptions of Theorem 4 are less restrictive than those established by Ezeilo ([6],Theorem 1), our result therefore improves [6].

(ii) Also, whenever $f(t, x, y, z) \equiv p(t)$ system (3.2) specializes to that studied by Swick [11] Theorem 1 and our hypotheses and conclusion coincide with that of Swick except that:

(a) the hypothesis on $\dot{q}(t)$ and $\dot{r}(t)$ in [11] required that they be bounded below by $-L$ for some $L > 0$, which is not necessary in this case.

(b) the condition on $p'(t)$ in [11] Theorem 1 was that $\dfrac{1}{2}p'(t) \le \delta_3 \le \delta_1(b - \alpha c)$, this condition is too strong. It is replaced by a considerably weaker one on the equivalent function $f_t(t, x, y, z)$ in Theorem 4.

Hence, our result in Theorem 4 generalizes Theorem 1 in [11].

(iii) Whenever $f(t, x, y, z) \equiv p(t)$ and $p(t, x, y, z) \equiv e(t)$ system (1.2) reduces to that studied by Swick [11] Theorem 5. All hypotheses of Theorem 5 in [11] are contained in Theorem 6. In addition, the assumption on the functions $\dot{r}(t)$ and $\dot{q}(t)$ for all $t \ge 0$ are weaker than that of [11]. Finally, the main tool (the Lyapunov function) used in Theorem 6 is complete (see [5]) compared to an incomplete Lyapunov function used in [11].

(iv) In the case with $f(t, x, y, z) \equiv a(t)f(y)$ and $p(t, x, y, z) \equiv e(t)$ Eq. (1.2) reduces to that discussed by Mehri and Shadman [8]. In spite of the application of the energy function they used as the main tool, our result in Theorem 6 generalizes theirs.

(v) In the case $f(t, x, y, z) \equiv f(x, y)$ the hypotheses and conclusion of Theorem 6 coincide with that of Ademola and Arawomo [1] Theorem 2. Hence, Theorem 6 is an extension of [1] Theorem 2.

(vi) The situation when $f(t, x, y, z) \equiv f(z)$, $q(t) \equiv 1$ and $r(t) \equiv 1$, the hypotheses and conclusion of Theorem 4 coincide with that of Ademola *et. al.,* [2], but the hypothesis that $H(x) = \int_0^x h(\xi)d\xi \to \infty$ as $|x| \to \infty$ is not necessary here.

For the rest of this article, δ_i $(i = 0, 1, \cdots, 11), P_0$ and D_i $(i = 0, 1, 2, 3)$ stand for positive constants. Their identities are preserved throughout this paper. Let $H(x) = \int_0^x h(\xi)d\xi$, and $G(y) = \int_0^y g(\tau)d\tau$.

The main tool in the proofs of our results is the continuously differentiable function $V = V(t, x, y, z)$

$$
\begin{aligned}
(3.4) \quad 2V = {} & 2(\alpha + a)r(t)H(x) + 4q(t)G(y) + 2(\alpha + a)\int_0^y \tau f(t, x, \tau, 0)d\tau \\
& + 4r(t)h(x)y + b\beta x^2 + \beta y^2 + 2z^2 + 2a\beta xy + 2\beta xz + 2(\alpha + a)yz,
\end{aligned}
$$

where α and β are positive constants such that

$$(3.5a) \qquad b^{-1}c < \alpha < a$$

and

$$(3.5b) \qquad 0 < \beta < \min\left\{(ab - c)a^{-1},\ (ab - c)\delta_1\eta_1,\ (a - \alpha)\eta_2\right\},$$

where

$$\eta_1 = \left[1 + a + \delta_0^{-1}\delta_1^{-1}[q(t)\frac{g(y)}{y} - b]^2\right]^{-1}$$

and

$$\eta_2 = \left[2\left[1 + \delta_0^{-1}\delta_1^{-1}[f(t, x, y, z) - a]^2\right]\right]^{-1}.$$

Lemma 9. *Subject to the hypotheses of Theorem 4* $V(t,0,0,0)=0$ *and there exist constants* $D_0 = D_0(\alpha,\beta,\delta_0,\delta_1,a,b,c) > 0$ *and* $D_1 = D_1(\alpha,\beta,a,a_1,b,b_1,c,q_0,r_0) > 0$ *such that for all* $t \geq 0$

$$D_0(x^2(t)+y^2(t)+z^2(t)) \leq V(t,x,y,z) \leq D_1(x^2(t)+y^2(t)+z^2(t)), \tag{3.6a}$$

and

$$V(t,x(t),y(t),z(t)) \to \infty \quad \text{as} \quad x^2(t)+y^2(t)+z^2(t) \to \infty. \tag{3.6b}$$

Furthermore, there exist a finite constant $D_2(\alpha,\beta,\delta_0,\delta_1,a,b,c) > 0$ *such that along a solution* $(x(t),y(t),z(t))$ *of* (3.2),

$$\dot{V} \equiv \frac{d}{dt}V(t,x(t),y(t),z(t)) \leq -D_2(x^2(t)+y^2(t)+z^2(t)) \tag{3.6c}$$

Proof. It is clear from (3.4)), $V(t,0,0,0)=0$. Indeed we can recast the terms in (3.4) to obtain

$$2V = 2V_1 + 2V_2, \tag{3.7}$$

where

$$2V_1 = r(t)\left\{2b^{-1}\int_0^x [\alpha b - h'(\xi)]h(\xi)d\xi + 2\int_0^y \left[\frac{q(t)}{r(t)}\frac{g(\tau)}{\tau} - b\right]\tau d\tau \right.$$
$$\left. + b^{-1}[by+h(x)]^2\right\} + 2\alpha\int_0^y [f(t,x,\tau,0)-\alpha]\tau d\tau + (z+\alpha y)^2$$

and

$$2V_2 =$$
$$r(t)\left\{2b^{-1}\int_0^x [ab - h'(\xi)]h(\xi)d\xi + 2\int_0^y \left[\frac{q(t)}{r(t)}\frac{g(\tau)}{\tau} - b\right]\tau d\tau + b^{-1}[by+h(x)]^2\right\}$$
$$+ 2a\int_0^y [f(t,x,\tau,0)-a]\tau d\tau + \beta y^2 + \beta(b-\beta)x^2 + (\beta x + ay + z)^2.$$

In view of hypotheses (i) and (ii) of Theorem 4, the terms $2b^{-1}\int_0^x[\alpha b - h'(\xi)]h(\xi)d\xi$ and $2b^{-1}\int_0^x[ab-h'(\xi)]h(\xi)d\xi$ in the rearrangement for $2V_1$ and $2V_2$ satisfy

$$2b^{-1}\int_0^x [(\alpha+a)b - 2h'(\xi)]\,h(\xi)d\xi \geq \delta_0\,[(\alpha+a)b-2c]\,b^{-1}x^2.$$

Since $q(t) \geq r(t) \geq \delta_1$ for all $t \geq 0$, $g(y)/y \geq b$ $(y \neq 0)$, $h(x)/x \geq \delta_0$ $(x \neq 0)$ and $f(t,x,y,z) \geq a$ for all x,y,z and $t \geq 0$, combining these estimates, (3.7) yields

$$2V \geq \left[\beta(b-\beta) + \delta_0\delta_1[(\alpha+a)b-2c]b^{-1}\right]x^2 + [\alpha(a-\alpha)+\beta]y^2$$
$$+ 2\delta_1(\delta_0 x + by)^2 + (z+\alpha y)^2 + (\beta x + ay + z)^2.$$

By (3.5a) and (3.5b) $\alpha b > c$, $ab > c$, $a > \alpha$ and $b > \beta$. Thus, there exists a constant $\delta_2 = \delta_2(\alpha, \beta, \delta_0, \delta_1, a, b, c) > 0$ small enough such that

$$V \geq \delta_2(x^2 + y^2 + z^2) \tag{3.8}$$

for all x, y, z.

Furthermore, since $g(y)/y \leq b_1$ $(y \neq 0)$, $h'(x) \leq c$ for all x, $h(0) = 0$, $r'(t) \leq q'(t) \leq 0$ for all $t \geq 0$, $f(t, x, y, z) \leq a_1$ for all x, y, z and using the inequality $2|x||y| \leq x^2 + y^2$, there exist constants $\delta_3 > 0, \delta_4 > 0$ and $\delta_5 > 0$ such that

$$2|V| \leq \delta_3 x^2 + \delta_4 y^2 + \delta_5 z^2,$$

where $\delta_3 = (\alpha+a+2)cr_0+\beta(a+b+1)$, $\delta_4 = 2(b_1q_1+cr_0)+(\alpha+a)(a_1+1)+\beta(a+1)$ and $\delta_5 = (\alpha + \beta + a + 2)$. Hence, there exists a positive constant $\delta_6 = \frac{1}{2}\max\{\delta_3, \delta_4, \delta_5\}$ such that

$$V \leq \delta_6(x^2 + y^2 + z^2). \tag{3.9}$$

On combining the inequalities in (3.8) and (3.9), we obtain (3.6a), and from (3.8) it is clear that

$$V \equiv V(t, x, y, z) \to \infty \quad \text{as} \quad x^2 + y^2 + z^2 \to \infty. \tag{3.10}$$

To deal with the other half of the lemma, let $(x(t), y(t), z(t))$ be any solution of (3.2). Using (3.4) and the system (3.2), a direct computation of $\frac{dV}{dt}|_{(3.2)} = \dot{V}_{(3.2)}$ gives after simplification

$$\dot{V}_{(3.2)} = W_1 + W_2 - W_3 - \beta\left[q(t)\frac{g(y)}{y} - b\right]xy - \beta[f(t, x, y, z) - a]xz, \tag{3.11}$$

where

$$W_1 = (\alpha + a)r'(t)H(x) + 2q'(t)G(y) + 2r'(t)yh(x)$$

$$W_2 = (\alpha + a)\left[\int_0^y \tau f_t(t, x, \tau, 0)d\tau + y\int_0^y \tau f_x(t, x, \tau, 0)d\tau\right] + a\beta y^2 + 2\beta yz.$$

and

$$W_3 = \beta r(t)h(x)x + r(t)\left[(\alpha + a)\frac{q(t)}{r(t)}\frac{g(y)}{y} - 2h'(x)\right]y^2 + [2f(t, x, y, z) - (\alpha + a)]z^2.$$

Since $r'(t) \leq q'(t) \leq 0$, we consider the following cases for W_1:

(i) If $r'(t) = 0$, we have $q'(t) = 0$ so that $W_1 = 0$.

(ii) For those t's such that $r'(t) < 0$, we have

$$W_1 = r'(t)\left[(\alpha + a)H(x) + 2\frac{q'(t)}{r'(t)}G(y) + 2yh(x)\right]. \tag{3.12}$$

Since $h(0) = 0$, $h(x)/x \geq \delta_0$ $(x \neq 0)$, $g(y)/y \geq b$ $(y \neq 0)$, $h'(x) \leq c$ for all x, $\dfrac{q'(t)}{r'(t)} \geq 1$ and (3.5a), the terms in the square bracket of (3.12) satisfies

$$b^{-1}\int_0^x \left[(\alpha+a)b - 2h'(\xi)\right] h(\xi)d\xi + 2\int_0^y \left[\frac{g(y)}{y} - b\right]\tau d\tau + b^{-1}(by + h(x))^2 \geq 0,$$

so that $W_1 \leq 0$.

Hence, in both cases, we have

$$W_1 \leq 0. \tag{3.13a}$$

From condition (vi) of Theorem 4 and the fact that $2yz \leq y^2 + z^2$, we have

$$W_2 \leq (a+1)\beta y^2 + \beta z^2. \tag{3.13b}$$

Also since $h(x)/x \geq \delta_0$ $(x \neq 0)$, $q(t) \geq r(t) \geq \delta_1$ for all $t \geq 0$, $g(y)/y \geq b$ $(y \neq 0)$, $h'(x) \leq c$ for all x and $f(t,x,y,z) \geq a$ for all x, y, z and $t \geq 0$, it follows that

$$W_3 \geq \beta\delta_0\delta_1 x^2 + \delta_1[(\alpha+a)b - 2c]y^2 + (a-\alpha)z^2. \tag{3.13c}$$

Combining (3.13a), (3.13b) and (3.13c) with (3.11) and simplify to get

$$\begin{aligned}
\dot{V}_{(3.2)} \leq & -\frac{1}{2}\beta\delta_0\delta_1 x^2 - \frac{\beta}{4\delta_0\delta_1}\left[\delta_0\delta_1 x + 2\left[q(t)\frac{g(y)}{y} - b\right]y\right]^2 \\
& - \left\{\delta_1[(\alpha+a)b - 2c] - \beta\left[(a+1) + \frac{1}{\delta_0\delta_1}\left[q(t)\frac{g(y)}{y} - b\right]^2\right]\right\} y^2 \\
& - \left\{(a-\alpha) - \beta\left[1 + \frac{1}{\delta_0\delta_1}[f(t,x,y,z) - a]^2\right]\right\} z^2 \\
& - \frac{\beta}{4\delta_0\delta_1}\left[\delta_0\delta_1 x + 2[f(t,x,y,z) - a]z\right]^2.
\end{aligned}$$

From (3.5a) and (3.5b) the above inequality becomes

$$\dot{V}_{(3.2)} \leq -\frac{1}{2}\beta\delta_0\delta_1 x^2 - \delta_1(\alpha b - c)y^2 - \frac{1}{2}(a-\alpha)z^2.$$

Hence, there exists a positive constant $\delta_7 = \delta_7(\alpha, \beta, \delta_0, \delta_1, a, b, c)$ such that

$$\dot{V}_{(3.2)} \leq -\delta_7(x^2 + y^2 + z^2) \tag{3.14}$$

for all x, y and z. This completes the proof of Lemma 9 □

Proof of Theorem 4. In view of (3.8), (3.9), (3.10) and (3.14) conditions (i) and (ii) of Theorem 1 hold, that is all solutions of (3.2) are uniformly bounded. From (3.14), we have $W(X) = \delta_7(x^2 + y^2 + z^2)$ a positive definite function with respect to a closed set $\Omega \equiv \{(x,y,z) : W(x,y,z) = W(X) = 0\}$ and $\dot{V}(t,X) \leq -W(X)$.

Furthermore, from conditions (ii) and (v) of Theorem 4, the functions $g(y)$ and $f(t,x,y,z)$ are bounded above respectively. Also, $h(x), q(t)$ and $r(t)$ are continuous functions, it follows that $F(t, X)$ is bounded for all t when X belongs to any compact set of $\mathbb{R}^3$. Thus, by Theorem 2, the solutions of (3.2) approach $\Omega = \{(0,0,0)\}$, hence $x(t) \to 0$, $y(t) \to 0$, $z(t) \to 0$, as $t \to \infty$ which completes the proof of Theorem 4. □

Proof of Theorem 5. The main tool in the proof of Theorem 5 is the function V defined in (3.4). Clearly $V(t,0,0,0) = 0$ and in view of the hypotheses of Theorem 5, estimate (3.8) holds. Next we shall show that any solution $(x(t), y(t), z(t))$ of the system (3.2) is bounded. To see this we claim that the inequality

$$V \equiv V(t, x(t), y(t), z(t)) \leq k < \infty, \quad t > 0, \tag{3.15}$$

where k is a constant necessarily implies the boundedness of $x(t), y(t)$ and $z(t)$ for all $t \geq 0$. To see this, since $g(y)/y \geq b$ $(y \neq 0)$, $r(t) \geq q(t)$ for all $t \geq 0$, $h(x)/x$ $(x \neq 0)$ and $f(t,x,y,0) \geq a$ for all x, y and $t \geq 0$, so that (3.8) holds and if the inequality in (3.15) holds, then

$$x^2 + y^2 + z^2 \leq 2\delta_2^{-1} V \leq 2\delta_2^{-1} k.$$

Thus, there exists a positive constant δ_8 such that

$$|x(t)| \leq \delta_8, \quad |y(t)| \leq \delta_8, \quad |z(t)| \leq \delta_8,$$

where $\delta_8 = 2\delta_2^{-1}k$. This completes the verification of the assertion in (3.15). Following the procedure in the proof of Theorem 4, (3.3) holds as $t \to \infty$. This completes the proof of Theorem 5. □

Lemma 10. *Subject to the hypotheses of Theorem 6 there exists a constant $D_3 > 0$ depending only on $\alpha, \beta, \delta_0, \delta_1, a, b, c$ and P such that along a solution $(x(t), y(t), z(t))$ of (1.2)*

$$\dot{V} \leq -D_3(x^2 + y^2 + z^2)$$

for all x, y, z and $t \geq 0$.

Proof. Let $(x(t), y(t), z(t))$ be any solution of (1.2). Using (3.4) and the system (1.2) an elementary calculation yields

$$\dot{V}_{(1.2)} = \dot{V}_{(3.2)} + [\beta x + (\alpha + a)y + 2z]p(t,x,y,z).$$

By (3.14), hypothesis (iii) of Theorem 6 and the inequality $(|x| + |y| + |z|)^2 \leq 3(x^2 + y^2 + z^2)$, we have

$$\dot{V}_{(1.2)} \leq -\delta_7(x^2 + y^2 + z^2) + \delta_9(x^2 + y^2 + z^2)^{1/2},$$

where $\delta_9 = 3^{1/2} P_0 \max(\beta;\ \alpha + a;\ 2)$. Choose $(x^2 + y^2 + z^2)^{1/2} \geq \delta_{10} = 2\delta_7^{-1}\delta_9$, the above inequality becomes

$$\dot{V}_{(1.2)} \leq -\delta_{11}(x^2 + y^2 + z^2), \tag{3.16}$$

where $\delta_{11} = \dfrac{1}{2}\delta_7$. This completes the proof of Lemma 10. □

Proof of Theorem 6. From (3.8), (3.9), (3.10) and (3.16) hypotheses of Theorem 3 are satisfied. This completes the proof of Theorem 6. □

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