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GROWTH OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

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Abstract. In this paper, we investigate the growth of solutions of higher order homogeneous linear differential equations with entire coefficients. We improve and extend the results of Belaïdi and Hamouda by using the estimates for the logarithmic derivative of a transcendental meromorphic function due to Gundersen and the Wiman-Valiron theory. We also consider the nonhomogeneous linear differential equations.

1. Introduction and main results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (see [12, 17]). In addition, we use the notations $\sigma(f)$ and $\lambda(f)$ to denote respectively the order of growth and exponent of convergence of zeros of a meromorphic function $f(z)$.

We define the linear measure of a set $E \subset [0, +\infty)$ by $m(E) = \int_0^{+\infty} \chi_E(t)dt$ and the logarithmic measure of a set $F \subset (1, +\infty)$ by $lm(F) = \int_1^{+\infty} (\chi_F(t)/t)\, dt$, where χ_H is the characteristic function of a set H.

For an integer $n \geqslant 2$, we consider the linear differential equation

$$A_n(z) f^{(n)} + A_{n-1}(z) f^{(n-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.1}$$

where $A_0(z), \ldots, A_{n-1}(z),\ A_n(z)$ with $A_0(z) \not\equiv 0$, $A_n(z) \not\equiv 0$ are entire functions. If $A_n \equiv 1$ and some of the coefficients $A_0(z), \ldots, A_{n-1}(z)$ of (1.1) are transcendental, then (1.1) has at least one solution of infinite order.

Thus, a natural question is: What conditions on $A_0(z), \ldots, A_{n-1}(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.1) has an infinite order in the case when $A_n \equiv 1$?

For the above question, there are many results for second and higher order linear differential equations (see for example [2, 3, 4, 5, 9, 11]). In 2001 and

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2002, Belaïdi and Hamouda have considered the following higher order linear differential equation

$$f^{(n)} + A_{n-1}(z) f^{(n-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.2}$$

where $A_0(z), \ldots, A_{n-1}(z)$ with $A_0(z) \not\equiv 0$ are entire functions and obtained the following two results:

Theorem A (see [4]). *Let $A_0(z), \ldots, A_{n-1}(z)$ with $A_0(z) \not\equiv 0$ be entire functions such that for real constants $\alpha, \beta, \mu, \theta_1$ and θ_2 satisfying $0 \leqslant \beta < \alpha, \mu > 0$ and $\theta_1 < \theta_2$, we have*

$$|A_0(z)| \geqslant \exp\{\alpha |z|^{\mu}\} \tag{1.3}$$

and

$$|A_j(z)| \leqslant \exp\{\beta |z|^{\mu}\} \quad (j = 1, 2, \ldots, n-1) \tag{1.4}$$

as $z \to \infty$ in $\theta_1 \leqslant \arg z \leqslant \theta_2$. Then every solution $f \not\equiv 0$ of equation (1.2) *has an infinite order.*

Theorem B (see [5]). *Let $A_0(z), \ldots, A_{n-1}(z)$ with $A_0(z) \not\equiv 0$ be entire functions. Suppose that there exist a sequence of complex numbers $(z_k)_{k\in\mathbf{N}}$ with $\lim\limits_{k\to+\infty} z_k = \infty$ and three real numbers α, β and μ satisfying $0 \leqslant \beta < \alpha$ and $\mu > 0$ such that*

$$|A_0(z_k)| \geqslant \exp\{\alpha |z_k|^{\mu}\} \tag{1.5}$$

and

$$|A_j(z_k)| \leqslant \exp\{\beta |z_k|^{\mu}\} \quad (j = 1, 2, \ldots, n-1) \tag{1.6}$$

as $k \to +\infty$. Then every solution $f \not\equiv 0$ of equation (1.2) *has an infinite order.*

Recently, L. Z. Yang [21], J. Xu and Z. Zhang [20] have considered equation (1.1) and obtained different results concerning the growth of its solutions. It is well-known that if $A_n \equiv 1$, then all solutions of (1.1) are entire functions but in the case when $A_n(z)$ is a nonconstant entire function, equation (1.1) can have meromorphic solutions.

Now, the question which arises is: For so many solutions of infinite order of equation (1.1), how to describe precisely the properties of their growth?

The main purpose of this paper is to extend Theorem A and Theorem B for equations of the form (1.1) by making use of the concepts of hyper-order and iterated order.

For the definition of the iterated order of a meromorphic function, we use the same definition as in [6, 14, 15]. For all $r \in \mathbf{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbf{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbf{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.1 (see [14, 15])**.** Let $p \geqslant 1$ be an integer. Then the iterated p-order $\sigma_p(f)$ of a meromorphic function $f(z)$ is defined by

$$\sigma_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r}, \tag{1.7}$$

where $T(r,f)$ is the Nevanlinna characteristic function of f (see [12, 17]). For $p=1$, this notation is called order and for $p=2$, hyper-order.

Remark 1.1. If f is an entire function, then the iterated $p-$order $\sigma_p(f)$ of f is defined by

$$\sigma_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log r}, \tag{1.8}$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Definition 1.2 (see [14, 15])**.** The finiteness degree of the order of a meromorphic function f is defined by

$$i(f) = \begin{cases} 0, & \text{for } f \text{ rational,} \\ \min\{j \in \mathbf{N} : \sigma_j(f) < +\infty\}, & \text{for } f \text{ transcendental for which some} \\ & j \in \mathbf{N} \text{ with } \sigma_j(f) < +\infty \text{ exists,} \\ +\infty, & \text{for } f \text{ with } \sigma_j(f) = +\infty \text{ for all } j \in \mathbf{N}. \end{cases} \tag{1.9}$$

Definition 1.3 (see [14])**.** Let f be a meromorphic function. Then the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log r} \quad (p \geqslant 1 \text{ is an integer }), \tag{1.10}$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| < r\}$. For $p=1$, this notation is called exponent of convergence of the sequence of distinct zeros and for $p=2$, hyper-exponent of convergence of the sequence of distinct zeros.

In this paper, we shall consider equation (1.1) and prove the following results:

Theorem 1.1. *Let $A_0(z), \ldots, A_{n-1}(z)$, $A_n(z)$ with $A_0(z) \not\equiv 0$, $A_n(z) \not\equiv 0$ be entire functions such that for real constants $\alpha, \beta, \mu, \theta_1$ and θ_2 satisfying $0 \leqslant \beta < \alpha, \mu > 0$ and $\theta_1 < \theta_2$ and for an integer $p \geqslant 1$, we have*

$$|A_0(z)| \geqslant \exp_p\{\alpha |z|^\mu\} \tag{1.11}$$

and

$$|A_j(z)| \leqslant \exp_p\{\beta |z|^\mu\} \quad (j = 1, 2, \ldots, n) \tag{1.12}$$

as $z \to \infty$ in $\theta_1 \leqslant \arg z \leqslant \theta_2$. Then every meromorphic (or entire) solution $f \not\equiv 0$ of equation (1.1) *has an infinite order and satisfies $\sigma_p(f) = +\infty$ and $\sigma_{p+1}(f) \geqslant \mu$.*

Theorem 1.2. *Let $A_0(z), \dots, A_{n-1}(z)$, $A_n(z)$ with $A_0(z) \not\equiv 0$, $A_n(z) \not\equiv 0$ be entire functions. Suppose that there exist a sequence of complex numbers $(z_k)_{k\in\mathbf{N}}$ with $\lim_{k\to+\infty} z_k = \infty$ and three real numbers α, β and μ satisfying $0 \leqslant \beta < \alpha$ and $\mu > 0$ such that*

$$|A_0(z_k)| \geqslant \exp_p\{\alpha |z_k|^{\mu}\} \tag{1.13}$$

and

$$|A_j(z_k)| \leqslant \exp_p\{\beta |z_k|^{\mu}\} \quad (j = 1, 2, \dots, n) \tag{1.14}$$

as $k \to +\infty$, where $p \geqslant 1$ is an integer. Then every meromorphic (or entire) solution $f \not\equiv 0$ of equation (1.1) *has an infinite order and satisfies $\sigma_p(f) = +\infty$ and $\sigma_{p+1}(f) \geqslant \mu$.*

Theorem 1.3. *Let $A_0(z), \dots, A_{n-1}(z)$, $A_n(z)$ with $A_0(z) \not\equiv 0$, $A_n(z) \not\equiv 0$ be entire functions such that* $\max\{\sigma(A_j) : j = 1, 2, \dots, n\} < \sigma(A_0) = \sigma < +\infty$. *Suppose that for real constants α, β, θ_1 and θ_2 satisfying $0 \leqslant \beta < \alpha$ and $\theta_1 < \theta_2$ and for $\varepsilon > 0$ sufficiently small, we have*

$$|A_0(z)| \geqslant \exp\left\{\alpha |z|^{\sigma-\varepsilon}\right\} \tag{1.15}$$

and

$$|A_j(z)| \leqslant \exp\left\{\beta |z|^{\sigma-\varepsilon}\right\} \quad (j = 1, 2, \dots, n) \tag{1.16}$$

as $z \to \infty$ in $\theta_1 \leqslant \arg z \leqslant \theta_2$. Then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicity (or entire) of equation (1.1) *has an infinite order and satisfies $\sigma_2(f) = \sigma$.*

Theorem 1.4. *Let $A_0(z), \dots, A_{n-1}(z)$, $A_n(z)$ with $A_0(z) \not\equiv 0$, $A_n(z) \not\equiv 0$ be entire functions such that* $\max\{\sigma(A_j) : j = 1, 2, \dots, n\} < \sigma(A_0) = \sigma < +\infty$. *Suppose that there exist a sequence of complex numbers $(z_k)_{k\in\mathbf{N}}$ with $\lim_{k\to+\infty} z_k = \infty$ and two real numbers α and β satisfying $0 \leqslant \beta < \alpha$ and for $\varepsilon > 0$ sufficiently small, we have*

$$|A_0(z_k)| \geqslant \exp\left\{\alpha |z_k|^{\sigma-\varepsilon}\right\} \tag{1.17}$$

and

$$|A_j(z_k)| \leqslant \exp\left\{\beta |z_k|^{\sigma-\varepsilon}\right\} \quad (j = 1, 2, \dots, n) \tag{1.18}$$

as $k \to +\infty$. Then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicity (or entire) of equation (1.1) *has an infinite order and satisfies $\sigma_2(f) = \sigma$.*

Theorem 1.5. *Let $A_0(z), \dots, A_{n-1}(z)$, $A_n(z)$ with $A_0(z) \not\equiv 0$, $A_n(z) \not\equiv 0$ be entire functions satisfying either hypotheses of Theorem 1.3 or hypotheses of Theorem 1.4 and let $F \not\equiv 0$ be an entire function of finite order. Then every meromorphic solution f whose poles are of uniformly bounded multiplicity of the linear differential equation*

$$A_n(z) f^{(n)} + A_{n-1}(z) f^{(n-1)} + \cdots + A_1(z) f' + A_0(z) f = F \tag{1.19}$$

satisfies $\overline{\lambda}_2(f) = \sigma_2(f) = \sigma$ *with at most one exceptional solution* f_0 *of finite order.*

Remark 1.2. In Theorems 1.3-1.5, the condition that the poles are of uniformly bounded multiplicity of the solution f is necessary because the growth of coefficients A_j gives only estimate for the counting function of distinct poles $\overline{N}(r,f)$, but not for $N(r,f)$. So, this condition was missing in the main results of [20].

2. Preliminary lemmas

Lemma 2.1 (see [10])**.** *Let* $f(z)$ *be a transcendental meromorphic function of finite order* σ*. Let* $\Gamma = \{(k_1,j_1),(k_2,j_2),\ldots,(k_m,j_m)\}$ *denote a set of distinct pairs of integers satisfying* $k_i > j_i \geqslant 0$ $(i=1,2,\ldots,m)$ *and let* $\varepsilon > 0$ *be a given constant. Then the following two statements hold:*

(i) There exists a set $E_1 \subset [0,2\pi)$ *that has linear measure zero such that if* $\psi_0 \in [0,2\pi) - E_1$*, then there is a constant* $R_0 = R_0(\psi_0) > 1$ *such that for all* z *satisfying* $\arg z = \psi_0$ *and* $|z| \geqslant R_0$ *and for all* $(k,j) \in \Gamma$*, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.1}$$

(ii) There exists a set $E_2 \subset (1,+\infty)$ *that has finite logarithmic measure such that for all* z *satisfying* $|z| = r \notin E_2 \cup [0,1]$ *and for all* $(k,j) \in \Gamma$*, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.2}$$

Lemma 2.2 (see [10])**.** *Let* $f(z)$ *be a transcendental meromorphic function. Let* $\alpha > 1$ *and* $\varepsilon > 0$ *be given constants. Then there exist a constant* $A > 0$ *and a set* $E_3 \subset [0,+\infty)$ *having finite linear measure such that for all* z *satisfying* $|z| = r \notin E_3$*, we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant A\left[T(\alpha r,f)r^{\varepsilon}\log T(\alpha r,f)\right]^j \quad (j \in \mathbf{N}). \tag{2.3}$$

Lemma 2.3 (see [10])**.** *Let* $f(z)$ *be a transcendental meromorphic function, and let* $\mu > 1$ *be a given constant. Then there exist a set* $E_4 \subset (1,+\infty)$ *of finite logarithmic measure and a constant* $B > 0$ *that depends only on* μ *and* (m,n) *(*m,n *positive integers with* $m < n$*) such that for all* z *satisfying* $|z| = r \notin [0,1] \cup E_4$*, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leqslant B\left[\frac{T(\mu r,f)}{r}(\log^{\mu} r)\log T(\mu r,f)\right]^{n-m}. \tag{2.4}$$

Lemma 2.4 (see [9])**.** *Let* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ *be an entire function of infinite order with the hyper-order* $\sigma_2(f) = \sigma$*,* $\mu(r)$ *be the maximum term, i.e.,* $\mu(r) =$

$\max\{|a_n| r^n;\ n = 0, 1, \ldots\}$ *and let* $\nu_f(r)$ *be the central index of* f, *i.e.*, $\nu_f(r) = \max\{m,\ \mu(r) = |a_m| r^m\}$. *Then*

$$\limsup_{r\to+\infty} \frac{\log\log \nu_f(r)}{\log r} = \sigma. \tag{2.5}$$

Lemma 2.5 (Wiman-Valiron [13, 18])**.** *Let* $f(z)$ *be a transcendental entire function, and let* z *be a point with* $|z| = r$ *at which* $|f(z)| = M(r, f)$. *Then the estimation*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)) \qquad (j \geqslant 1 \textit{ is an integer}), \tag{2.6}$$

holds for all $|z|$ *outside a set* E_5 *of* r *of finite logarithmic measure.*

It is well-known that it is very important of the Wiman-Valiron theory [13, 18] to investigate the properties of entire solutions of differential equations. In [8] Z. X. Chen has extended the Wiman-Valiron theory from entire functions to meromorphic functions with infinitely many poles. Here we prove a special form of the result given by J. Wang and H. X. Yi in [19], when meromorphic function has infinite order:

Lemma 2.6. *Let* $f(z) = g(z)/d(z)$ *be an infinite order meromorphic function with* $\sigma_2(f) = \sigma < +\infty$, $g(z)$ *and* $d(z)$ *are entire functions, where* $\sigma(d) = \rho < \sigma$. *Then there exist a sequence of complex numbers* $\left\{z_k = r_k e^{i\theta_k}\right\}_{k\in\mathbf{N}}$ *and a set* E_6 *of finite logarithmic measure such that* $r_k \notin E_6, r_k \to +\infty, \theta_k \in [0, 2\pi)\,; k \in \mathbf{N}$, $\lim\limits_{k\to+\infty} \theta_k = \theta_0 \in [0, 2\pi)\,, |g(z_k)| = M(r_k, g)$ *and for a sufficiently large* k, *we have*

$$\frac{f^{(n)}(z_k)}{f(z_k)} = \left(\frac{\nu_g(r_k)}{z_k}\right)^n (1 + o(1)) \quad (n \in \mathbf{N}) \tag{2.7}$$

and

$$\lim_{k\to+\infty} \frac{\log\log \nu_g(r_k)}{\log r_k} = \sigma_2(g) = \sigma. \tag{2.8}$$

Proof. By mathematical induction, we obtain

$$f^{(n)} = \frac{g^{(n)}}{d} + \sum_{j=0}^{n-1} \frac{g^{(j)}}{d} \sum_{(j_1\ldots j_n)} C_{jj_1\ldots j_n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n}, \tag{2.9}$$

where $C_{jj_1\ldots j_n}$ are constants and $j + j_1 + 2j_2 + \cdots + nj_n = n$. Hence,

$$\frac{f^{(n)}}{f} = \frac{g^{(n)}}{g} + \sum_{j=0}^{n-1} \frac{g^{(j)}}{g} \sum_{(j_1\ldots j_n)} C_{jj_1\ldots j_n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n}. \tag{2.10}$$

From Lemma 2.5, there exists a set $E_5 \subset (1, +\infty)$ with finite logarithmic measure such that for a point z satisfying $|z| = r \notin E_5$ and $|g(z)| = M(r, g)$, we have

$$\frac{g^{(j)}(z)}{g(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1+o(1)) \quad (j = 1, 2, \ldots, n), \tag{2.11}$$

where $\nu_g(r)$ is the central index of g. Substituting (2.11) into (2.10) yields

$$\begin{aligned} \frac{f^{(n)}(z)}{f(z)} &= \left(\frac{\nu_g(r)}{z}\right)^n (1+o(1)) \times \\ &\times \left[1 + \sum_{j=0}^{n-1} \left(\frac{\nu_g(r)}{z}\right)^{j-n} \sum_{(j_1 \cdots j_n)} C_{jj_1\cdots j_n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n}\right]. \end{aligned} \tag{2.12}$$

By Lemma 2.1 (ii), we have for any given ε $(0 < 2\varepsilon < \sigma - \rho)$

$$\left|\frac{d^{(m)}(z)}{d(z)}\right| \leqslant r^{m(\rho-1+\varepsilon)} \quad (m = 1, 2, \ldots, n), \tag{2.13}$$

where $|z| = r \notin [0,1] \cup E_2, E_2 \subset (1, +\infty)$ with $lm(E_2) < +\infty$. From this and $j_1 + 2j_2 + \cdots + nj_n = n - j$, we have

$$|z|^{n-j} \left|\left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n}\right| \leqslant |z|^{(n-j)(\rho+\varepsilon)} \tag{2.14}$$

for $|z| = r \notin [0,1] \cup E_2$. By Lemma 2.4, there exists $\{r'_k\}$ $(r'_k \to +\infty)$ satisfying

$$\lim_{r'_k \to +\infty} \frac{\log\log \nu_g(r'_k)}{\log r'_k} = \sigma. \tag{2.15}$$

Setting the logarithmic measure of $E_2 \cup E_5, lm(E_2 \cup E_5) = \delta < +\infty$, there exists a point $r_k \in [r'_k, (\delta+1) r'_k] - (E_2 \cup E_5)$. Since,

$$\frac{\log\log \nu_g(r_k)}{\log r_k} \geqslant \frac{\log\log \nu_g(r'_k)}{\log[(\delta+1) r'_k]} = \frac{\log\log \nu_g(r'_k)}{(\log r'_k)\left[1 + \frac{\log(\delta+1)}{\log r'_k}\right]}, \tag{2.16}$$

we have

$$\lim_{r_k \to +\infty} \frac{\log\log \nu_g(r_k)}{\log r_k} = \sigma. \tag{2.17}$$

From (2.17), we obtain for sufficiently large k

$$\nu_g(r_k) \geqslant \exp\left\{r_k^{\sigma-\varepsilon}\right\}. \tag{2.18}$$

This and (2.14) lead

$$\left|\left(\frac{\nu_g(r)}{z}\right)^{j-n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n}\right| \leqslant r^{(n-j)(\rho+\varepsilon)} \left[\exp\left\{r_k^{\sigma-\varepsilon}\right\}\right]^{j-n} \to 0, \tag{2.19}$$

as $r_k \to +\infty$, where $|z_k| = r_k$ and $|g(z_k)| = M(r_k, g)$. From (2.12), (2.17) and (2.19), we obtain our result. Now we take $z_k = r_k e^{i\theta_k}$, $\theta_k \in [0, 2\pi)$.

There is a subset $\{\theta_{k_j}\}$ of $\{\theta_k\}$ such that $\lim_{j\to+\infty}\theta_{k_j} = \theta_0 \in [0, 2\pi)$. Thus $\left\{z_{k_j} = r_{k_j} e^{i\theta_{k_j}}\right\}$ satisfies our assertion. □

Lemma 2.7 (see [7]). *Let $f(z)$ be a transcendental meromorphic function of order $\sigma(f) = \sigma < +\infty$. Then for any given $\varepsilon > 0$, there exists a set $E_7 \subset (1, \infty)$ that has finite logarithmic measure, such that*

$$|f(z)| \leqslant \exp\left\{r^{\sigma+\varepsilon}\right\} \tag{2.20}$$

holds for $|z| = r \notin [0, 1] \cup E_7$, $r \to +\infty$.

Remark 2.1. Applying Lemma 2.7 to $\frac{1}{f}$, it is clearly that for any given $\varepsilon > 0$, there exists a set $E_8 \subset (1, \infty)$ that has finite logarithmic measure, such that

$$\exp\left\{-r^{\sigma+\varepsilon}\right\} \leqslant |f(z)| \leqslant \exp\left\{r^{\sigma+\varepsilon}\right\} \tag{2.21}$$

holds for $|z| = r \notin [0, 1] \cup E_8$, $r \to +\infty$.

Lemma 2.8 (see [16]). *Let $f(z)$ be an entire function of infinite order. Denote $M(r, f) = \max\{|f(z)| : |z| = r\}$, then for any sufficiently large number $\rho > 0$ and any $r \in E_9 \subset (1, +\infty)$, we have*

$$M(r, f) > c_1 \exp\{c_2 r^\rho\}, \tag{2.22}$$

where $lm(E_9) = \infty$ and c_1, c_2 are positive constants.

To avoid some problems caused by the exceptional set we recall the following lemmas.

Lemma 2.9 (see [1]). *Let $g : [0, +\infty) \to \mathbf{R}$ and $h : [0, +\infty) \to \mathbf{R}$ be monotone non-decreasing functions such that $g(r) \leqslant h(r)$ outside of an exceptional set E_{10} of finite linear measure. Then for any $\lambda > 1$, there exists $r_0 > 0$ such that $g(r) \leqslant h(\lambda r)$ for all $r > r_0$.*

Lemma 2.10 (see [11]). *Let $\varphi : [0, +\infty) \to \mathbf{R}$ and $\psi : [0, +\infty) \to \mathbf{R}$ be monotone non-decreasing functions such that $\varphi(r) \leqslant \psi(r)$ for all $r \notin E_{11} \cup [0, 1]$, where $E_{11} \subset (1, +\infty)$ is a set of finite logarithmic measure. Let $\alpha > 1$ be a given constant. Then there exists an $r_1 = r_1(\alpha) > 0$ such that $\varphi(r) \leqslant \psi(\alpha r)$ for all $r > r_1$.*

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Suppose that f $(\not\equiv 0)$ is a meromorphic solution of equation (1.1) such that $\sigma(f) = \sigma < +\infty$. Then from Lemma 2.1 (i), there exists a set E_1 that has linear measure zero such that if $\psi_0 \in [\theta_1, \theta_2] - E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant |z|^{j(\sigma-1+\varepsilon)} \quad (j = 1, 2, \ldots, n) \tag{3.1}$$

as $z \to \infty$ along $\arg z = \psi_0$. Then from (1.1), (3.1) and (1.12), we obtain

$$\begin{aligned} (3.2)\quad |A_0(z)| &\leqslant |A_n(z)| \left|\frac{f^{(n)}}{f}\right| + |A_{n-1}(z)| \left|\frac{f^{(n-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right| \\ &\leqslant n\,|z|^{n(\sigma-1+\varepsilon)} \exp_p\{\beta\,|z|^{\mu}\}. \end{aligned}$$

as $z \to \infty$ along $\arg z = \psi_0$, and this contradicts (1.11). Hence, $\sigma(f) = +\infty$.

Now, we prove that $\sigma_p(f) = +\infty$ and $\sigma_{p+1}(f) \geqslant \mu$. From (1.1), it follows that

$$(3.3)\qquad |A_0(z)| \leqslant |A_n(z)| \left|\frac{f^{(n)}}{f}\right| + |A_{n-1}(z)| \left|\frac{f^{(n-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right|.$$

By Lemma 2.2, there exist a constant $A > 0$ and a set $E_3 \subset [0, +\infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E_3$, we have

$$(3.4)\qquad \left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant Ar\,[T(2r, f)]^{n+1} \quad (j = 1, 2, \ldots, n).$$

Hence, from (1.11), (1.12), (3.3) and (3.4) it follows that for all z satisfying $|z| = r \notin E_3$ and $\theta_1 \leqslant \arg z \leqslant \theta_2$, we have

$$(3.5)\qquad \exp_p\{\alpha\,|z|^{\mu}\} \leqslant Anr\,[T(2r, f)]^{n+1} \exp_p\{\beta\,|z|^{\mu}\}$$

as $|z| = r \to +\infty$. By Lemma 2.9 and (3.5), it follows that for all z satisfying $|z| = r > r_0$ and $\theta_1 \leqslant \arg z \leqslant \theta_2$, we have

$$(3.6)\qquad \exp_p\{\alpha\,|z|^{\mu}\} \leqslant A\lambda nr\,[T(2\lambda r, f)]^{n+1} \exp_p\{\beta\lambda^{\mu}\,|z|^{\mu}\},$$

where $\lambda\ (> 1)$ and $r_0\ (> 0)$ are constants. Therefore, from (3.6), we obtain that $\sigma_p(f) = +\infty$ and $\sigma_{p+1}(f) \geqslant \mu$. □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. Suppose that $f\ (\not\equiv 0)$ is a meromorphic solution of equation (1.1) such that $\sigma(f) = \sigma < +\infty$. We can rewrite (1.1) as

$$(4.1)\qquad \frac{A_n(z)}{A_0(z)}\frac{f^{(n)}}{f} + \frac{A_{n-1}(z)}{A_0(z)}\frac{f^{(n-1)}}{f} + \cdots + \frac{A_1(z)}{A_0(z)}\frac{f'}{f} = -1.$$

By Lemma 2.1 (i), there exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero such that if $\psi_0 \in [0, 2\pi) - E_1$, then there is a constant $R_0 = R_0(\psi_0) > 0$ such that if z_k satisfies $\arg z_k = \psi_0$ and $|z_k| = r_k \geqslant R_0$, we have

$$(4.2)\qquad \left|\frac{f^{(j)}(z_k)}{f(z_k)}\right| \leqslant |z_k|^{j(\sigma-1+\varepsilon)} \quad (j = 1, 2, \ldots, n).$$

Thus, from (1.13), (1.14) and (4.2) we obtain

$$(4.3)\qquad \left|\frac{A_j(z_k)}{A_0(z_k)}\right| \left|\frac{f^{(j)}(z_k)}{f(z_k)}\right| \leqslant \frac{|z_k|^{j(\sigma-1+\varepsilon)} \exp_p\{\beta\,|z_k|^{\mu}\}}{\exp_p\{\alpha\,|z_k|^{\mu}\}}$$

$$= \frac{|z_k|^{j(\sigma-1+\varepsilon)}}{\exp\left\{\exp_{p-1}\{\alpha|z_k|^{\mu}\} - \exp_{p-1}\{\beta|z_k|^{\mu}\}\right\}}$$

for $j = 1, 2, \ldots, n$. From (4.3), it follows that

$$\lim_{k\to+\infty}\left|\frac{A_j(z_k)}{A_0(z_k)}\right|\left|\frac{f^{(j)}(z_k)}{f(z_k)}\right| = 0 \quad (j = 1, 2, \ldots, n). \tag{4.4}$$

By making $k \to +\infty$ in relation (4.1), we get a contradiction. Hence $\sigma(f) = +\infty$.

Now, we prove that $\sigma_p(f) = +\infty$ and $\sigma_{p+1}(f) \geqslant \mu$. From (1.1), it follows that

$$|A_0(z)| \leqslant |A_n(z)|\left|\frac{f^{(n)}}{f}\right| + |A_{n-1}(z)|\left|\frac{f^{(n-1)}}{f}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right|. \tag{4.5}$$

By Lemma 2.2, there exist a constant $A > 0$ and a set $E_3 \subset [0, +\infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E_3$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant Ar\,[T(2r, f)]^{n+1} \quad (j = 1, 2, \ldots, n). \tag{4.6}$$

Hence, from (1.13), (1.14), (4.5) and (4.6) we have

$$\exp_p\{\alpha|z_k|^{\mu}\} \leqslant Anr_k\,[T(2r_k, f)]^{n+1}\exp_p\{\beta|z_k|^{\mu}\} \tag{4.7}$$

as $k \to +\infty$, $|z_k| = r_k \notin E_3$. Hence, from (4.7) and Lemma 2.9, we have for $|z_k| = r_k > r_0$

$$\exp_p\{\alpha|z_k|^{\mu}\} \leqslant A\lambda nr_k\,[T(2\lambda r_k, f)]^{n+1}\exp_p\{\beta\lambda^{\mu}|z_k|^{\mu}\} \tag{4.8}$$

where λ (> 1) and r_0 (> 0) are constants. Therefore, from (4.8) we obtain that $\sigma_p(f) = +\infty$ and $\sigma_{p+1}(f) \geqslant \mu$. □

5. Proof of Theorem 1.3

Proof of Theorem 1.3. Assume f $(\not\equiv 0)$ is a meromorphic solution of equation (1.1) whose poles are of uniformly bounded multiplicity. Then by taking $p = 1$ in Theorem 1.1, it follows that f has an infinite order and satisfies $\sigma_2(f) \geqslant \sigma - \varepsilon$. Since, $\varepsilon > 0$ is arbitrary, we get $\sigma_2(f) \geqslant \sigma(A_0) = \sigma$. By $A_n(z) \not\equiv 0$, we can rewrite (1.1) as

$$f^{(n)} + \frac{A_{n-1}(z)}{A_n(z)}f^{(n-1)} + \cdots + \frac{A_0(z)}{A_n(z)}f = 0 \tag{5.1}$$

Hence, the poles of f can only occur at the zeros of A_n. Note that the pole of f are of uniformly bounded multiplicity, then $\lambda(1/f) \leqslant \sigma(A_n) < \sigma < +\infty$. By Hadamard factorization theorem, we know that f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with $\lambda(d) = \sigma(d) = \lambda(1/f) \leqslant \sigma(A_n) < \sigma < \sigma_2(f) = \sigma_2(g)$. By Lemma 2.6, for any small $\varepsilon > 0$, there exists one sequence $\left\{z_k = r_k e^{i\theta_k}\right\}_{k\in\mathbf{N}}$ $(r_k \to +\infty)$ and a set E_6 of finite logarithmic measure such that $r_k \notin E_6, r_k \to +\infty, \theta_k \in [0, 2\pi)\,; k \in \mathbf{N}, \lim\limits_{k\to+\infty} \theta_k = \theta_0 \in [0, 2\pi)\,, |g(z_k)| = M(r_k, g)$ and for a sufficiently large k, we have

$$\frac{f^{(j)}(z_k)}{f(z_k)} = \left(\frac{\nu_g(r_k)}{z_k}\right)^j (1 + o(1)) \quad (j = 1, 2, \ldots, n)\,. \tag{5.2}$$

By Remark 1.1, for any given $\varepsilon > 0$ and for a sufficiently large r, we have

$$|A_j(z)| \leqslant \exp\left\{r^{\sigma+\varepsilon}\right\} \quad (j = 0, 1, \ldots, n-1)\,. \tag{5.3}$$

Also, by Remark 2.1, there exists a set $E_8 \subset (1, +\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_8$, $r \to +\infty$, we have

$$|A_n(z)| \geqslant \exp\left\{-r^{\sigma+\varepsilon}\right\}. \tag{5.4}$$

We can rewrite (1.1) as

$$-A_n(z)\frac{f^{(n)}}{f} = A_{n-1}(z)\frac{f^{(n-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z)\,. \tag{5.5}$$

Substituting (5.2) into (5.5) we obtain for the above z_k

$$\begin{aligned} -A_n(z_k)\left(\frac{\nu_g(r_k)}{z_k}\right)^n (1 + o(1)) &= A_{n-1}(z_k)\left(\frac{\nu_g(r_k)}{z_k}\right)^{n-1} \\ &\times (1 + o(1)) + \cdots + A_1(z_k)\left(\frac{\nu_g(r_k)}{z_k}\right)(1 + o(1)) + A_0(z_k)\,. \end{aligned} \tag{5.6}$$

Hence, from (5.3), (5.4) and (5.6) for the above $z_k = r_k e^{i\theta_k}$ with $r_k \notin E_6 \cup E_8 \cup [0, 1]\,, r_k \to +\infty$, we have

$$\begin{aligned} \exp\left\{-r_k^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r_k)}{z_k}\right|^n |1 + o(1)| &\leqslant \exp\left\{r_k^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r_k)}{z_k}\right|^{n-1} |1 + o(1)| \\ &+ \cdots + \exp\left\{r_k^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r_k)}{z_k}\right| |1 + o(1)| + \exp\left\{r_k^{\sigma+\varepsilon}\right\} \\ &\leqslant n\exp\left\{r_k^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r_k)}{z_k}\right|^{n-1} |1 + o(1)|\,. \end{aligned} \tag{5.7}$$

By (5.7) and Lemma 2.10, we get

$$\limsup_{r_k\to+\infty} \frac{\log\log \nu_g(r_k)}{\log r_k} \leqslant \sigma + \varepsilon. \tag{5.8}$$

Since, $\varepsilon > 0$ is arbitrary, by (5.8) and Lemma 2.4, we obtain $\sigma_2(f) \leqslant \sigma$. This and the fact that $\sigma_2(f) \geqslant \sigma$ yield $\sigma_2(f) = \sigma$. □

6. Proof of Theorem 1.4 and 1.5

Proof of Theorem 1.4. Assume f $(\not\equiv 0)$ is a meromorphic solution of equation (1.1) whose poles are of uniformly bounded multiplicity. Then by taking $p = 1$ in Theorem 1.2, it follows that f has an infinite order and satisfies $\sigma_2(f) \geqslant \sigma - \varepsilon$. Since, $\varepsilon > 0$ is arbitrary, we get $\sigma_2(f) \geqslant \sigma(A_0) = \sigma$. By using the same arguments as in the proof of Theorem 1.3, we obtain $\sigma_2(f) \leqslant \sigma$. Hence $\sigma_2(f) = \sigma$. □

Proof of Theorem 1.5. First, we show that (1.19) can possess at most one exceptional solution f_0 of finite order. In fact, if f^* is another solution of finite order of equation (1.19), then $f_0 - f^*$ is of finite order. But, $f_0 - f^*$ is a solution of the corresponding homogeneous equation (1.1) of (1.19). This contradicts Theorem 1.3 and Theorem 1.4. We assume that f is an infinite order meromorphic solution of (1.19) whose poles are of uniformly bounded multiplicity and $f_1, f_2, \ldots, f_n$ are n meromorphic solutions of the corresponding homogeneous equation (1.1) of (1.19). Then f can be expressed in the form

$$f(z) = B_1(z) f_1(z) + B_2(z) f_2(z) + \cdots + B_n(z) f_n(z), \tag{6.1}$$

where $B_1(z), \ldots, B_n(z)$ are suitable meromorphic functions determined by

$$\begin{aligned} B_1'(z) f_1(z) + B_2'(z) f_2(z) + \cdots + B_n'(z) f_n(z) &= 0 \\ B_1'(z) f_1'(z) + B_2'(z) f_2'(z) + \cdots + B_n'(z) f_n'(z) &= 0 \\ &\vdots \\ B_1'(z) f_1^{(n-1)}(z) + B_2'(z) f_2^{(n-1)}(z) + \cdots + B_n'(z) f_n^{(n-1)}(z) &= F(z). \end{aligned} \tag{6.2}$$

Since, the Wronskian $W(f_1, f_2, \ldots, f_n)$ is a differential polynomial in

$$f_1, f_2, \ldots, f_n$$

with constant coefficients, it is easy to deduce that

$$\sigma_2(W) \leqslant \max\{\sigma_2(f_j) : j = 1, 2, \ldots, n\} = \sigma(A_0) = \sigma.$$

From (6.2), we have

$$B_j' = F.G_j(f_1, f_2, \ldots, f_n).W(f_1, f_2, \ldots, f_n)^{-1} \quad (j = 1, 2, \ldots, n), \tag{6.3}$$

where $G_j(f_1, f_2, \ldots, f_n)$ are differential polynomials in

$$f_1, f_2, \ldots, f_n$$

with constant coefficients. Thus, for $j = 1, 2, \ldots, n$, we have

$$\sigma_2(G_j) \leqslant \max\{\sigma_2(f_j) : j = 1, 2, \ldots, n\} = \sigma(A_0) = \sigma \tag{6.4}$$

and

$$\sigma_2(B_j) = \sigma_2\left(B_j'\right) \leqslant \max\{\sigma_2(F), \sigma(A_0)\} = \sigma(A_0) = \sigma. \tag{6.5}$$

Then from (6.1) and (6.5) we get

$$\sigma_2(f) \leqslant \max\{\sigma_2(f_j), \sigma_2(B_j) : j = 1, 2, \ldots, n\} = \sigma(A_0) = \sigma. \tag{6.6}$$

Now, we prove that $\sigma_2(f) \geqslant \sigma(A_0) = \sigma$. From (1.19), it follows that

$$(6.7)\quad |A_0(z)| \leqslant |A_n(z)| \left|\frac{f^{(n)}}{f}\right| + |A_{n-1}(z)| \left|\frac{f^{(n-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right| + \left|\frac{F}{f}\right|.$$

By Lemma 2.3, there exist a constant $B > 0$ and a set $E_4 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_4$, we have

$$(6.8)\qquad \left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant B\,[T(2r, f)]^{n+1} \quad (j = 1, 2, \ldots, n).$$

By $A_n(z) \not\equiv 0$, we can rewrite (1.19) as

$$(6.9)\qquad f^{(n)} + \frac{A_{n-1}(z)}{A_n(z)} f^{(n-1)} + \cdots + \frac{A_0(z)}{A_n(z)} f = \frac{F}{A_n(z)}.$$

Hence, the poles of f can only occur at the zeros of A_n. Note that the pole of f are of uniformly bounded multiplicity, then $\lambda(1/f) \leqslant \sigma(A_n) < \sigma < +\infty$. By Hadamard factorization theorem, we know that f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with $\lambda(d) = \sigma(d) = \lambda(1/f) \leqslant \sigma(A_n) < \sigma < +\infty$. Set $\delta = \max\{\sigma, \sigma(F)\}$. Since $\sigma(g) = \sigma(f) = +\infty$, then by Lemma 2.8 for any sufficiently large number $\rho > \delta$, and any $r \in E_9 \subset (1, +\infty)$, we have

$$(6.10)\qquad M(r, g) > c_1 \exp\{c_2 r^{\rho}\},$$

where $lm(E_9) = \infty$ and $c_1 > 0$, $c_2 > 0$ are constants. On the other hand, for a given ε $(0 < \varepsilon < \rho - \delta)$, we have for a sufficiently large r

$$(6.11)\qquad |F(z)| \leqslant \exp\left\{r^{\delta+\varepsilon}\right\} \text{ and } |d(z)| \leqslant \exp\left\{r^{\delta+\varepsilon}\right\}.$$

From (6.10) and (6.11) we obtain

$$(6.12)\qquad \left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)|\,|d(z)|}{|g(z)|} \leqslant \frac{1}{c_1} \exp\left\{2r^{\delta+\varepsilon} - c_2 r^{\rho}\right\} \to 0$$

as $r \to +\infty$, where $|g(z)| = M(r, g)$ and $|z| = r$.

(i) If $A_0(z), \ldots, A_{n-1}(z)$ and $A_n(z)$ satisfy hypotheses of Theorem 1.3, then from (1.15) , (1.16), (6.7), (6.8) and (6.12), it follows that for all z satisfying $|z| = r \in E_9 - E_4$, $\theta_1 \leqslant \arg z \leqslant \theta_2$ and $|g(z)| = M(r, g)$

$$(6.13)\qquad \exp\left\{\alpha |z|^{\sigma-\varepsilon}\right\} \leqslant Bn\,[T(2r, f)]^{n+1} \exp\left\{\beta |z|^{\sigma-\varepsilon}\right\} + o(1)$$

as $z \to \infty$. From (6.13) and Lemma 2.10, we get $\sigma_2(f) \geqslant \sigma - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\sigma_2(f) \geqslant \sigma$. This and the fact that $\sigma_2(f) \leqslant \sigma$ yield $\sigma_2(f) = \sigma$.

(ii) If $A_0(z), \ldots, A_{n-1}(z)$ and $A_n(z)$ satisfy hypotheses of Theorem 1.4, then from (1.17), (1.18), (6.7), (6.8) and (6.12), it follows that

$$(6.14)\qquad \exp\left\{\alpha |z_k|^{\sigma-\varepsilon}\right\} \leqslant Bn\,[T(2r_k, f)]^{n+1} \exp\left\{\beta |z_k|^{\sigma-\varepsilon}\right\} + o(1)$$

as $k \to +\infty$, $|z_k| = r_k \in E_9 - E_4$ and $|g(z_k)| = M(r_k, g)$. From (6.14) and Lemma 2.10, we get $\sigma_2(f) \geqslant \sigma - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\sigma_2(f) \geqslant \sigma$. This and the fact that $\sigma_2(f) \leqslant \sigma$ yield $\sigma_2(f) = \sigma$.

Now, we prove that $\sigma_2(f) = \overline{\lambda}_2(f) = \sigma$. By (1.19) it is easy to see that if f has a zero z_0 of order α $(> n)$, then F must have a zero at z_0 of order $\alpha - n$. Hence

$$n\left(r, \frac{1}{f}\right) \leqslant n\overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right) \tag{6.15}$$

and

$$N\left(r, \frac{1}{f}\right) \leqslant n\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right). \tag{6.16}$$

We can rewrite (1.19) as

$$\frac{1}{f} = \frac{1}{F}\left(A_n(z)\frac{f^{(n)}}{f} + A_{n-1}(z)\frac{f^{(n-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0(z)\right). \tag{6.17}$$

By (6.17) we have

$$m\left(r, \frac{1}{f}\right) \leqslant \sum_{j=1}^{n} m\left(r, \frac{f^{(j)}}{f}\right) + \sum_{j=0}^{n} m(r, A_j) + m\left(r, \frac{1}{F}\right) + O(1). \tag{6.18}$$

By (6.16) and (6.18) we obtain for $|z| = r$ outside a set E of a finite linear measure

$$\begin{aligned} T(r,f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &\leqslant n\overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=0}^{n} T(r, A_j) + T(r, F) + O(\log(rT(r,f))). \end{aligned} \tag{6.19}$$

For sufficiently large r and any given $\varepsilon > 0$, we have

$$O(\log r + \log T(r,f)) \leqslant \frac{1}{2}T(r,f), \tag{6.20}$$

$$\sum_{j=0}^{n} T(r, A_j) \leqslant (n+1)\, r^{\sigma+\varepsilon} \tag{6.21}$$

and

$$T(r,F) \leqslant r^{\sigma(F)+\varepsilon}. \tag{6.22}$$

Thus, by (6.19) -(6.22), we have

$$T(r,f) \leqslant 2n\overline{N}\left(r, \frac{1}{f}\right) + 2(n+1)\, r^{\sigma+\varepsilon} + 2r^{\sigma(F)+\varepsilon}, \tag{6.23}$$

where $|z| = r \notin E$. Hence, for any f with $\sigma_2(f) = \sigma$, by (6.23) and Lemma 2.9, we have $\sigma_2(f) \leqslant \overline{\lambda}_2(f)$. Since, $\overline{\lambda}_2(f) \leqslant \sigma_2(f)$ we obtain

$$\overline{\lambda}_2(f) = \sigma_2(f) = \sigma. \qquad \square$$

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