Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
27 (2011), 105–117
www.emis.de/journals
ISSN 1786-0091

NEW ITERATIVE SCHEMES FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF NONEXPANSIVE MAPPING

ZHENHUA HE AND FENG GU

ABSTRACT. In this paper, two new iterative schemes are introduced in Hilbert space. They can be used to find a common element of the set of solutions of an equilibrium problem and the set of fixed point of the nonexpansive mapping. Under suitable conditions, some weak and strong convergence theorems are obtained.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let K be a nonempty closed convex subset of H. We first recall some definitions and conclusions:

$f\colon H\to H$ is said to be a *contraction* mapping with contraction constant $\alpha\in(0,1)$, if $\forall\ x,y\in H$, $\|f(x)-f(y)\|\le\alpha\|x-y\|$. $T\colon K\to K$ is said to be a *nonexpansive* mapping, if $\forall\ x,y\in K$, $\|Tx-Ty\|\le\|x-y\|$. The set of fixed points for T is denoted by $F(T)=\{x\in K: Tx=x\}$. $G\colon K\to K$ is said to be a *L-Lipschitzian* mapping, if $\forall\ x,y\in K$, $\|Gx-Gy\|\le L\|x-y\|$, $L>0$.

Let$F$ be a bifunction of $K\times K$ into $\mathbf{R}$, where $\mathbf{R}$ is the set of real number. The equilibrium problem for $F\colon K\times K\to\mathbf{R}$ is to find $x\in K$ such that

$$F(x,y)\ge 0,\quad \forall y\in K. \tag{1.1}$$

Let $EP(F)$ denote the set of solutions of (1.1). In (1.1), if $F(x,y)=\langle Tx,y-x\rangle$ for all $x,y\in K$, where $T\colon K\to H$ is a mapping. Obviously, $p\in EP(F)$ if and only if$\langle Tp,y-p\rangle\ge 0$ for all $y\in K$, that is, p is a solution of the variational inequality. This shows that equilibrium problem (1.1) includes

2000 *Mathematics Subject Classification.* 47H09; 47H10; 47J25.

Key words and phrases. Strong and weak convergence; Viscosity iterative method; Fixed point problem; Equilibrium problem.

The author was supported by the Scientific Research Foundation from Yunnan Province Education Committee (08Y0338) and the Natural Science Foundation of Yunnan Province (2010ZC152).

some variational inequalities as especially cases. In addition, this equilibrium problem contains also the fixed point problem, optimization problem and Nash equilibrium problem as its special cases(for example,[1]).

For finding a common element of $F(T) \cap EP(F)$, Tada and Takahashi [6] introduced the following iterative scheme by metric projection:

$$
\begin{aligned}
&u_n \in K \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \text{ for all } y \in K; \\
&w_n = (1-\alpha_n)x_n + \alpha_n S u_n, \\
&C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\
&D_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\
&x_{n+1} = P_{C_n \cap D_n}(x), \quad n \geq 1.
\end{aligned} \tag{1.2}
$$

For reducing the complexity of computation caused by the projection P_K, Yamada [9], proposed an iteration method to solve the variational inequalities $VI(A, K)$. For arbitrary $u_0 \in H$,

$$u_{n+1} = Tu_n - \lambda_{n+1}\mu A(Tu_n), \quad n \geq 0, \tag{1.3}$$

where $T \colon H \to H$ is a nonexpansive mapping, $A \colon H \to H$ is a nonlinear operator and VI(A,K) denote

$$\langle Au^*, v - u^*\rangle \geq 0 \quad \forall v \in K. \tag{1.4}$$

Under suitable conditions, Yamada [9] proved that $\{u_n\}$ converges strongly to the unique solution of the $VI(A, K)$.

Motivated by Yamada [9], in 2007, Wang [7] purposed an explicit scheme as follows:

$$u_{n+1} = \alpha_n u_n + (1-\alpha_n)(Tu_n - \lambda_{n+1}\mu A(Tu_n), \tag{1.5}$$

where $u_0 \in H$, $T \colon H \to H$ is a nonexpansive mapping, $A \colon H \to H$ is a nonlinear operator. Wang studied convergence property of the sequence u_n and obtained strong and weak convergence theorems.

Inspired by above results, in this paper, we introduce two iterative algorithms to find a common element of $F(T) \cap EP(F)$.

Algorithm 1.1.

$$
\begin{aligned}
&x_1 \in H;\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in K; \\
&x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n; \\
&y_n = (1-\sigma)x_n + \sigma T u_n - \alpha_n \lambda_{n+1}\mu G(Tu_n), \quad n \geq 1,
\end{aligned} \tag{1.6}
$$

where $\sigma \in (0, 1)$ is an arbitrarily real number(but fixed), $\mu > 0$. $\{\alpha_n\}$, $\{\lambda_n\} \subset [0, 1]$ and $r_n \subset (0, \infty)$ satisfy the following conditions:

(C1) $\lim_n \alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;
(C2) $\liminf_n r_n > 0$, $\lim_n |r_{n+1} - r_n| = 0$;
(C3) $\Sigma_{n=0}^{\infty}\lambda_n < \infty$.

Algorithm 1.2.

$$
\begin{aligned}
&x_1 \in H;\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in K;\\
&x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n;\\
&y_n = Tu_n - \lambda_{n+1}\mu G(Tu_n), \quad n \geq 1,
\end{aligned}
\tag{1.7}
$$

where $\mu > 0$. $\{\alpha_n\}$, $\{\lambda_n\} \subset [0,1]$ and $r_n \subset (0,\infty)$ satisfy conditions:

$(C1')$ $\alpha_n \subset [\alpha, \beta]$, $\alpha, \beta \in (0,1)$;
$(C2')$ $\liminf_n r_n > 0$;
$(C3')$ $\Sigma_{n=0}^{\infty}\lambda_n < \infty$.

Remark 1.1. We claim that Algorithm 1.1 and Algorithm 1.2 are two viscosity iterative schemes with L-Lipschitzian mapping error. Obviously, the conditions of coefficients in Algorithm 1.1 and Algorithm 1.2 are different.

In order to study convergence property of Algorithm 1.1-1.2, we need introduce some preliminaries.

2. Preliminaries

For the sequence $\{x_n\}$ in H, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. $\omega_w(x_n)$ denotes the weak $\omega-$limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) := \{x \in H : x_{n_j} \rightharpoonup x \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\}.$$

In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in \mathbf{R}$. Let K be a closed convex subset of H, for each point $x \in H$, there exists a unique nearest point in K, denoted by $P_K x$, such that

$$\|x - P_K x\| \leq \|x - y\|, \quad \forall\, y \in K.$$

P_K is called the *metric projection* of H into K. It is well known that P_K satisfies

$$\langle x - y, P_K x - P_K y\rangle \geq \|P_K x - P_K y\|^2$$

for every $x, y \in H$. Moreover, $P_K x$ is characterized by the properties: for $x \in H$, and $z \in K$,

$$z = P_K x \Leftrightarrow \langle x - z, z - y\rangle \geq 0, \quad \forall y \in K. \tag{2.1}$$

For solving the equilibrium problem about a bifunction $F\colon K \times K \to \mathbf{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in K$;
(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in K$;

(A3) for each $x, y, z \in K$,

$$\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) for each $x \in K, y \mapsto F(x, y)$ is convex and lower semicontinuous.

In what follows, we shall make use of the following Lemmas.

Lemma 2.1 (demicloseness principle [3]). *Let H be a real Hilbert space. K is a closed convex subset of H and $T\colon K \to H$ is a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on K, where I is the identity mapping, that is, $x_n \rightharpoonup x$ in H and $(I-T)x_n \to y$ imply that $x \in K$ and $(I-T)x = y$.*

Lemma 2.2 (Suzuki [5]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_n \beta_n \leq \limsup_n \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integers $n \geq 0$ and $\limsup_n(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then, $\lim_n \|y_n - x_n\| = 0$.*

Lemma 2.3 ([1]). *Let K be a nonempty convex subset of H and F be a bifunction of $K \times K$ into $\boldsymbol{R}$ satisfying (A1) − (A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in K$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \text{ for all } y \in K.$$

Lemma 2.4 (see [2]). *Assume that F is a bifunction of $K \times K$ onto $\boldsymbol{R}$ satisfying (A1) − (A4). For $r > 0$ and $x \in H$, define a mapping $T_r\colon H \to K$ as follows:*

$$T_r(x) = \left\{ z \in K : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall\, y \in K \right\}$$

for all $z \in H$. Then the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

Lemma 2.5. *Let T_r be a mapping defined by Lemma 2.4. Let $u_n = T_{r_n} x_n$. If $\liminf_n r_n > 0$, $\lim_n \|u_n - x_n\| = 0$ and $u_n \rightharpoonup z$, then $z \in EP(F)$.*

Proof. Since $u_n = T_{r_n} x_n$, then from Lemma 2.4 we have that

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in K.$$

By (A2), we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n).$$

Since $\lim_n \|u_n - x_n\| = 0$ and $u_n \rightharpoonup z$, from (A4) we have

$$0 \geq F(y, z) \text{ for all } y \in K.$$

Let $t \in (0,1)$ and $y \in K$, $y_t = ty + (1-t)z$. Since $y, z \in K$, $y_t \in K$ and $F(y_t, z) \leq 0$. Thus, using (A1) and (A4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, z) \leq tF(y_t, y),$$

this implies that $0 \leq F(y_t, y)$. From (A3), we have

$$0 \leq F(z, y) \text{ for all } y \in K,$$

this shows that $z \in EP(F)$. □

Next, it is well known about Lemma 2.6.

Lemma 2.6. *Let $\forall x, y \in H$, then $\|x+y\|^2 \leq \|y\|^2 + 2\langle x, x+y\rangle$.*

Lemma 2.7 (see [8])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0.$$

If

(i) $\alpha_n \in [0,1]$, $\sum \alpha_n = \infty$;
(ii) $\limsup \sigma_n \leq 0$;
(iii) $\gamma_n \geq 0$, $\sum \gamma_n < \infty$,

then $a_n \to 0$, as $n \to \infty$.

Lemma 2.8 (see [4])**.** *Let $\{a_n\}$, $\{b_n\}$, $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n.$$

If $\Sigma_{n=1}^{\infty}\delta_n < \infty$, $\Sigma_{n=1}^{\infty} b_n < \infty$, then $\lim_n a_n$ exists.

3. Main results

In this section, we study the convergence property of Algorithm 1.1-1.2.

Theorem 3.1. *Let K be a closed convex subset of a real Hilbert space H, $f\colon H \to H$ is a contraction mapping with contraction constant $\alpha \in (0,1)$, $T\colon K \to K$ is a nonexpansive mapping, F is a bifunction of $K \times K$ onto $\boldsymbol{R}$ satisfying $(A1)-(A4)$. $G\colon K \to K$ is a $L-$Lipschitzian mapping. $F(T) \cap EP(F) \neq \emptyset$, let $\{x_n\}$ and $\{u_n\}$ be defined by Algorithm 1.1, then $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $p \in F(T) \cap EP(F)$, where $p = P_{F(T)\cap EP(F)}f(p)$.*

Proof. First, we show that $\{x_n\}$ and $\{u_n\}$ are bounded. Let $p \in F(T) \cap EP(F)$, from $u_n = T_{r_n}x_n$, we have

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|, \quad n \geq 1. \tag{3.1}$$

From (1.6) and (3.1), it is easy to know

$$\begin{aligned} \|y_n - p\| &\le (1-\sigma)\|x_n - p\| + \sigma\|Tu_n - p\| + \alpha_n\lambda_{n+1}\mu\|G(Tu_n)\| \\ &\le \|x_n - p\| + \alpha_n\lambda_{n+1}\mu L\|x_n - p\| + \lambda_{n+1}\mu\|G(p)\| \\ &\le (1+\alpha_n\lambda_{n+1}\mu L)\|x_n - p\| + \lambda_{n+1}\mu\|G(p)\|. \end{aligned} \tag{3.2}$$

Again from (1.6) and (3.1)-(3.2), we have

$$\begin{aligned} \|x_{n+1} - p\| \le& \alpha_n\|f(x_n) - p\| + (1-\alpha_n)(1+\alpha_n\lambda_{n+1})\mu L\|x_n - p\| \\ &+ \lambda_{n+1}\mu\|G(p)\| \\ \le& (1-\alpha_n(1-\alpha) + \alpha_n\lambda_{n+1}\mu L)\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &+ \lambda_{n+1}\mu\|G(p)\|, \end{aligned} \tag{3.3}$$

Since $\alpha \in (0,1)$, there exists a constant $\varepsilon > 0$ such that $1-\alpha-\varepsilon > 0$. On the other hand, since $\lambda_n \to 0 (n \to \infty)$, there exists n_0 such that $n \ge n_0$, $\lambda_n\mu L < \varepsilon$. Thus, from (3.3) we have

$$\|x_{n+1} - p\| \le (1-\alpha_n(1-\alpha-\varepsilon))\|x_n - p\| + \alpha_n\|f(p) - p\| + \lambda_{n+1}\mu\|G(p)\|, \tag{3.4}$$

for $n \ge n_0$. By mathematical induction and simply computation, from(3.4) we have

$$\|x_n - p\| \le \max\{\|x_{n_0} - p\|, \frac{\|f(p)-p\|}{1-\alpha-\varepsilon}\} + \mu G(p)\Sigma_{i=n_0}^n \lambda_i, \quad n \ge n_0. \tag{3.5}$$

The inequality (3.5) shows that $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{y_n\}$.

Second, we show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Since $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ are all bounded, there exists a constant $M > 0$ such that

$$\max\{\|f(x_n)\|, \|Tu_n\|, \|x_n\|, \|x_n - p\|, \|u_n\|, \|x_n - u_n\|, \|y_n\|\} \le M, \quad n \ge 1.$$

We claim that $\|u_{n+1} - u_n\| \le \|x_{n+1} - x_n\| + \frac{1}{r_n}|r_{n+1} - r_n|\|u_n - x_n\|$. Indeed, it follows from Lemma 2.4 that

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0, \quad \forall y \in K \tag{3.6}$$

and

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in K. \tag{3.7}$$

Taking $y = u_n$ in (3.6) and $y = u_{n+1}$ in (3.7), then

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0$$

and

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \ge 0.$$

Thus, it follows from (A2) that

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \rangle \geq 0.$$

Then

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \rangle \geq 0,$$

which yields that

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|. \tag{3.8}$$

Let $\gamma_n = \sigma + (1-\sigma)\alpha_n$. Since $\alpha_n \to 0$ as $n \to \infty$, we have

$$0 < \liminf_n \gamma_n \leq \limsup_n \gamma_n < 1.$$

Let

$$v_n = \frac{x_{n+1} - x_n + \gamma_n x_n}{\gamma_n} = \frac{\alpha_n f(x_n) + (1-\alpha_n)(\sigma T u_n - \alpha_n \lambda_{n+1} \mu G(T u_n))}{\gamma_n},$$

then

$$\begin{aligned} v_{n+1} - v_n = & \frac{f(x_{n+1})\alpha_{n+1}}{\gamma_{n+1}} - \frac{f(x_n)\alpha_n}{\gamma_n} + \frac{(1-\alpha_{n+1})\sigma(Tu_{n+1} - Tu_n)}{\gamma_{n+1}} \\ & + \left(\frac{1-\alpha_{n+1}}{\gamma_{n+1}} - \frac{1-\alpha_n}{\gamma_n}\right)\sigma T u_n - \frac{(1-\alpha_{n+1})\alpha_{n+1}\lambda_{n+2}\mu G(Tu_{n+1})}{\gamma_{n+1}} \\ & + \frac{(1-\alpha_n)\alpha_n\lambda_{n+1}\mu G(Tu_n)}{\gamma_n}, \end{aligned}$$

which implies that

$$\begin{aligned} \|v_{n+1} - v_n\| \leq & \frac{\alpha_{n+1} + \alpha_n}{\sigma}2M + \frac{(1-\alpha_{n+1})\sigma\|u_{n+1} - u_n\|}{\gamma_{n+1}} \\ & + \left|\frac{1-\alpha_{n+1}}{\gamma_{n+1}} - \frac{1-\alpha_n}{\gamma_n}\right| M \\ \leq & \frac{\alpha_{n+1} + \alpha_n}{\sigma}2M + \frac{(1-\alpha_{n+1})\sigma\|x_{n+1} - x_n\|}{\gamma_{n+1}} + \frac{|r_{n+1} - r_n|M}{r_{n+1}\gamma_{n+1}} \\ & + \frac{|\alpha_n - \alpha_{n+1}|}{\sigma^2}M. \end{aligned}$$

Hence,

$$\limsup_{n\to\infty}\{\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|\} \leq 0. \tag{3.9}$$

By Lemma 2.2 and (3.9), we have that $\lim_{n\to\infty}\|v_n - x_n\| = 0$, which implies

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.10}$$

Third, we prove $\|Tu_n - u_n\| \to 0$ and $\|Tx_n - x_n\| \to 0$ as $n \to \infty$. Since $\|x_{n+1} - y_n\| = \alpha_n\|f(x_n) - y_n\| \to 0$, as $n \to \infty$, we have

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0(n \to \infty).$$

Furthermore, when $n \to \infty$, we have

$$\begin{aligned} \|Tu_n - x_n\| &= \frac{1}{\sigma}\|y_n - x_n + \alpha_n\lambda_{n+1}\mu G(Tu_n)\| \\ &\le \frac{1}{\sigma}(\|y_n - x_n\| + \lambda_{n+1}\|\mu G(Tu_n)\|) \to 0. \end{aligned} \tag{3.11}$$

On the other hand, let $p \in F(T) \cap EP(F)$, from Lemma 2.4 we have

$$\begin{aligned} \|u_n - p\|^2 = \|T_{r_n}x_n - T_{r_n}p\|^2 &\le \langle T_{r_n}x_n - T_{r_n}p, x - y\rangle \\ &= \langle u_n - p, x_n - p\rangle = \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2), \end{aligned}$$

i.e.,

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.12}$$

Since

$$\begin{aligned} \|y_n - p\|^2 &= \|(1-\sigma)(x_n - p) + \sigma(Tu_n - p) - \alpha_n\lambda_{n+1}\mu G(Tu_n)\|^2 \\ &\le (\|(1-\sigma)(x_n - p) + \sigma(Tu_n - p)\| + \alpha_n\lambda_{n+1}M)^2 \\ &\le \|(1-\sigma)(x_n - p) + \sigma(Tu_n - p)\|^2 + \alpha_n\lambda_{n+1}M' \\ &= (1-\sigma)\|x_n - p\|^2 + \sigma\|Tu_n - p\|^2 - (1-\sigma)\sigma\|Tu_n - x_n\|^2 \\ &\quad + \alpha_n\lambda_{n+1}M' \\ &\le (1-\sigma)\|x_n - p\|^2 + \sigma\|x_n - p\|^2 - (1-\sigma)\sigma\|Tu_n - x_n\|^2 \\ &\quad + \alpha_n\lambda_{n+1}M' \\ &= \|x_n - p\|^2 - (1-\sigma)\sigma\|Tu_n - x_n\|^2 + \alpha_n\lambda_{n+1}M', \end{aligned} \tag{3.13}$$

where M' is a constant such that $2M\|(1-\sigma)(x_n - p) + \sigma(Tu_n - p)\| + \alpha_n\lambda_{n+1}M^2 \le M'$, $n \ge 1$, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(f(x_n) - p) + (1-\alpha_n)(y_n - p)\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - (1-\sigma)\sigma\|x_n - u_n\|^2 + \lambda_{n+1}M', \end{aligned}$$

i.e.,

$$\begin{aligned} (1-\sigma)\sigma\|x_n - u_n\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \lambda_{n+1}M' \\ &\le \alpha_n\|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\ &\quad + \lambda_{n+1}M', \end{aligned}$$

which implies that $\|x_n - u_n\| \to 0$ as $n \to \infty$. Then from (3.11), we have

$$\|Tu_n - u_n\| \le \|Tu_n - x_n\| + \|x_n - u_n\| \to 0 \quad (n \to \infty),$$

and

$$\|Tx_n - x_n\| \leq \|Tu_n - x_n\| + \|Tx_n - Tu_n\| \leq \|Tu_n - x_n\| + \|x_n - u_n\| \to 0 \quad (n \to \infty). \tag{3.14}$$

Fourth, we prove $\{x_n\}$ and $\{u_n\}$ converge strongly to $p \in F(S) \cap EP(F)$, $p = P_{F(S)\cap EP(F)}f(p)$. Since $u_n = T_{r_n}x_n$ and $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup q$. By Lemma 2.1 and $\lim_{n\to\infty} \|Tu_n - u_n\| = 0$, we have $q = Tq$, i.e., $q \in F(T)$. On the other hand, together Lemma 2.5 with $\lim_{n\to\infty} \|x_n - u_n\| = 0$ and $u_{n_i} \rightharpoonup q$, we obtain $q \in EP(F)$. Notice that $\lim_{n\to\infty} \|x_n - u_n\| = 0$ and

$$p = P_{F(S)\cap EP(F)}f(p) \Leftrightarrow \langle f(p) - p, q - p\rangle \leq 0, \quad \forall q \in F(S) \cap EP(F),$$

(see (2.1)) this shows that

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle &= \limsup_{n\to\infty}\langle f(p) - p, x_{n_i} - p\rangle \\
&\leq \limsup_{n\to\infty}\langle f(p) - p, x_{n_i} - u_{n_i}\rangle + \limsup_{n\to\infty}\langle f(p) - p, u_{n_i} - p\rangle \\
&= \limsup_{n\to\infty}\langle f(p) - p, x_{n_i} - u_{n_i}\rangle + \langle f(p) - p, q - p\rangle \leq 0.
\end{aligned}$$

It follows from Lemma 2.6 and (1.6) and (3.13) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(f(x_n) - p) + (1 - \alpha_n)(y_n - p)\|^2 \\
&\leq (1 - \alpha_n)^2\|y_n - p\|^2 + 2\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle \\
&\leq (1 - \alpha_n)^2\|y_n - p\|^2 + 2\alpha_n\langle f(x_n) - f(p), x_{n+1} - p\rangle \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq (1 - \alpha_n)^2\|y_n - p\|^2 + 2\alpha_n\alpha\|x_n - p\|\|x_{n+1} - p\| \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq (1 - \alpha_n)^2\|y_n - p\|^2 + \alpha_n\alpha(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq (1 - \alpha_n)^2\|x_n - p\|^2 + \alpha_n\lambda_{n+1}M' + \alpha_n\alpha(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\
&\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq (1 - \frac{2\alpha_n(1-\alpha)}{1 - \alpha_n\alpha})\|x_n - p\|^2 + \frac{\alpha_n\lambda_{n+1}M'}{1 - \alpha_n\alpha} \\
+ \frac{\alpha_n^2}{1 - \alpha_n\alpha}M^2 + \frac{2\alpha_n}{1 - \alpha_n\alpha}\langle f(p) - p, x_{n+1} - p\rangle.
\end{aligned} \tag{3.15}$$

By the condition (C1) and Lemma 2.7, $\{x_n\}$ converges strongly to p. Notice that $\|u_n - p\| \leq \|x_n - p\|$, hence $\{u_n\}$ also converges strongly to p. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let K be a closed convex subset of a real Hilbert space H, $f\colon H \to H$ is a contraction mapping with contraction constant $\alpha \in (0,1)$, $T\colon K \to K$ is a nonexpansive mapping, F is a bifunction of $K \times K$ onto $\boldsymbol{R}$ satisfying (A1) − (A4). $G\colon K \to K$ is a $L-$Lipschitzian mapping. $F(T) \cap EP(F) \neq \emptyset$, let $\{x_n\}$ and $\{u_n\}$ be defined by Algorithm 1.2, then $\{x_n\}$ and $\{u_n\}$ converge weakly to a point $p \in F(T) \cap EP(F)$.*

Proof. Let $p \in F(T) \cap EP(F)$. Since $u_n = T_{r_n} x_n$, we have

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \le \|x_n - p\|.$$

It follows from (1.7) that

$$\begin{aligned} \|y_n - p\| &\le \|u_n - p\| + \lambda_{n+1}\mu\|G(Tu_n)\| \\ &\le \|x_n - p\| + \lambda_{n+1}\mu\|G(Tu_n)\| \\ &\le (1 + \lambda_{n+1}\mu L)\|x_n - p\| + \lambda_{n+1}\mu\|G(p)\|, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \|x_{n+1} - p\| &\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\| \\ &\le (1 + \lambda_{n+1}\mu L)\|x_n - p\| + \lambda_{n+1}\mu\|G(p)\| \end{aligned} \tag{3.17}$$

Based on Lemma 2.8 and (3.17), we have that $\lim_n \|x_n - p\|$ exists. This also shows that $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{y_n\}$. Let M be a constant such that

$$\max\{\|x_n\|, \|u_n\|, \mu\|G(Tu_n)\|\} \le M, \quad n \ge 1.$$

From (1.7) and (3.16), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)y_n - p\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 - \alpha_n(1-\alpha_n)\|y_n - x_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + \lambda_{n+1}M_1 \\ &\quad - \alpha_n(1-\alpha_n)\|y_n - x_n\|^2 \\ &= \|x_n - p\|^2 + \lambda_{n+1}M_1 - \alpha_n(1-\alpha_n)\|y_n - x_n\|^2, \end{aligned} \tag{3.18}$$

where M_1 is a constant such that

$$2\mu\|G(Tu_n)\|\|x_n - p\| + \lambda_{n+1}\mu^2\|G(Tu_n)\|^2 \le M_1, \quad n \ge 1.$$

Hence,

$$\alpha_n(1-\alpha_n)\|y_n - x_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \lambda_{n+1}M_1 \to 0, \tag{3.19}$$

as $n \to \infty$. i.e., $\lim_n \|y_n - x_n\| = 0$. Further, we have that

$$\|x_n - Tu_n\| \le \|x_n - y_n\| + \lambda_{n+1}M \to 0 \quad (n \to \infty). \tag{3.20}$$

By inequality (3.12) and (3.16) and convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)y_n - p\|^2\\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|y_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)(\|u_n-p\| + \lambda_{n+1}\mu\|G(Tu_n)\|)^2\\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|u_n-p\|^2 + \lambda_{n+1}M_2 \qquad (3.21)\\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|x_n-p\|^2 - (1-\alpha_n)\|x_n-u_n\|^2\\ &\quad + \lambda_{n+1}M_2\\ &= \|x_n-p\|^2 + \lambda_{n+1}M_2 - (1-\alpha_n)\|x_n-u_n\|^2,\end{aligned}$$

where M_2 is a constant such that

$$2\mu\|u_n-p\|\|G(Tu_n)\| + \lambda_{n+1}\mu^2\|G(Tu_n)\|^2 \le M_2, \quad n\ge 1,$$

then

$$(1-\alpha_n)\|x_n-u_n\|^2 \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \lambda_{n+1}M \to 0 \qquad (3.22)$$

as $n\to\infty$. i.e., $\lim_n\|x_n-u_n\|=0$, which implies that

$$\|Tu_n-u_n\| \le \|Tu_n-x_n\| + \|u_n-x_n\| \to 0 \quad (n\to\infty), \qquad (3.23)$$

Next, we show that $\omega_w(x_n)\subset F(T)\cap EP(F)$. Indeed, let $\forall z\in\omega_w(x_n)$, then there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\rightharpoonup z$.

From (3.21), we know that $u_{n_j}\rightharpoonup z$. Notice that $u_n = T_{r_n}x_n$ is bounded, hence from (3.22) and Lemma 2.5 and Lemma 2.1, we have that $u_{n_j}\rightharpoonup z\in EP(F)$ and $z\in F(T)$.

Finally, we claim that $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $F(T)\cap EP(F)$. For this purpose, we prove that $\omega_w(x_n)$ is a single-point set. Indeed, $\forall\ p_1,p_2\in\omega_w(x_n)$, let x_{n_i} and x_{m_k} be subsequence of $\{x_n\}$ such that $x_{n_i}\rightharpoonup p_1$ and $x_{m_k}\rightharpoonup p_2$, respectively. Obviously, $p_1,\ p_2\in F(T)\cap EP(F)$. Since $\lim_n\|x_n-p\|$ exists, by Opical's condition, if $p_1\ne p_2$, we obtain that

$$\begin{aligned}\limsup_i\|x_{n_i}-p_1\| &< \limsup_i\|x_{n_i}-p_2\| = \limsup_k\|x_{m_k}-p_2\| \qquad (3.24)\\ &< \limsup_k\|x_{m_k}-p_1\| = \limsup_i\|x_{n_i}-p_1\|,\end{aligned}$$

contradictory, hence $\omega_w(x_n)$ is a single-point set. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let K be a closed convex subset of a real Hilbert space H, $f\colon H\to H$ is a contraction mapping with contraction constant $\alpha\in(0,1)$, $T\colon K\to K$ is a nonexpansive mapping, F is a bifunction of $K\times K$ onto $\boldsymbol{R}$ satisfying (A1) − (A4). $G\colon K\to K$ is a $L-$Lipschitzian mapping. $F(T)\cap EP(F)\ne\emptyset$, let $\{x_n\}$ and $\{u_n\}$ be defined by Algorithm 1.2, then $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $q\in F(T)\cap EP(F)$ if and only if $\liminf_n d(x_n, F(T)\cap EP(F))=0$.*

Proof. $\forall p \in F(T) \cap EP(F)$, by Theorem 3.2 we have that $\lim_n \|x_n - p\|$ exists, $\lim_n \|u_n - x_n\| = 0$ and $\{x_n\}, \{u_n\}$ are bounded.

It is obvious that if $x_n \to p$ and $u_n \to p$, then $\liminf_n d(x_n, F(T) \cap EP(F)) = 0$. Conversely, it follows from (3.17) that

$$\|x_{n+1} - p\| \leq \|x_n - p\| + \lambda_{n+1} M', \tag{3.25}$$

where M' is a constant such that

$$\mu L \|x_n - p\| + \mu \|G(p)\| \leq M', \quad n \geq 1.$$

Thus, we have

$$d(x_{n+1}, F(T) \cap EP(F)) \leq d(x_n, F(T) \cap EP(F)) + \lambda_{n+1} M'. \tag{3.26}$$

Further by Lemma 2.8 we obtain that $\lim_n d(x_{n+1}, F(T) \cap EP(F))$ exists. Moreover,

$$\lim_n d(x_{n+1}, F(T) \cap EP(F)) = \liminf_n d(x_{n+1}, F(T) \cap EP(F)) = 0. \tag{3.27}$$

Next, we prove $\{x_n\}$ is a *Cauchy* sequence. Let $N > 1$ be a nonnegative integer. Suppose that $n > m > N$, it follows from (3.25) that

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\ &\leq \|x_{n-1} - p\| + \|x_{m-1} - p\| + M'(\lambda_n + \lambda_m) \\ &\leq \|x_N - p\| + \|x_N - p\| + 2M' \Sigma_{i=N+1}^n \lambda_i \\ &= 2\|x_N - p\| + 2M' \Sigma_{i=N+1}^n \lambda_i, \end{aligned} \tag{3.28}$$

this implies that

$$\|x_n - x_m\| \leq 2d(x_N, F(T) \cap EP(F)) + 2M' \Sigma_{i=N+1}^n \lambda_i. \tag{3.29}$$

The inequality (3.29) shows that $\{x_n\}$ is a *Cauchy* sequence. Therefore, there exists $q \in H$ such that $\{x_n\}$ converge strongly to q. Since $\lim_n \|x_n - Tx_n\| = 0$, we have $q \in F(T)$. Since $\lim_n \|x_n - u_n\| = 0$, $\{u_n\}$ also converge strongly to q. Again from Lemma 2.5, we have that $q \in EP(F)$. Consequently, $q \in F(T) \cap EP(F)$. This completes the proof of Theorem 3.3. ☐

References

[1] E. Blum and W. Oettli. From optimization and variational inequalities to equilibrium problems. *Math. Student*, 63(1-4):123–145, 1994.

[2] P. L. Combettes and S. A. Hirstoaga. Equilibrium programming in Hilbert spaces. *J. Nonlinear Convex Anal.*, 6(1):117–136, 2005.

[3] K. Goebel and W. A. Kirk. *Topics in metric fixed point theory*, volume 28 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[4] M. O. Osilike and S. C. Aniagbosor. Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings. *Math. Comput. Modelling*, 32(10):1181–1191, 2000.

[5] T. Suzuki. Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces. *Fixed Point Theory Appl.*, (1):103–123, 2005.

[6] A. Tada and W. Takahashi. Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem. *J. Optim. Theory Appl.*, 133(3):359–370, 2007.

[7] L. Wang. An iteration method for nonexpansive mappings in Hilbert spaces. *Fixed Point Theory Appl.*, pages Art. ID 28619, 8, 2007.
[8] H.-K. Xu. Iterative algorithms for nonlinear operators. *J. London Math. Soc. (2)*, 66(1):240–256, 2002.
[9] I. Yamada. The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings. In *Inherently parallel algorithms in feasibility and optimization and their applications (Haifa, 2000)*, volume 8 of *Stud. Comput. Math.*, pages 473–504. North-Holland, Amsterdam, 2001.

Received December 22, 2009.

Zhenhua He,
Department of Mathematics,
Honghe University,
Mengzi, Yunnan 661100,
China.
E-mail address: zhenhuahe@126.com

Feng Gu,
Department of Mathematics,
Hangzhou Normal University,
Zhejiang, 310036,
China.
E-mail address: gufeng99@sohu.com