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CHENG-MORDESON $\mathcal{L}$-FUZZY NORMED SPACES AND APPLICATION IN STABILITY OF FUNCTIONAL EQUATION

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Abstract. In this paper, we define and study Cheng-Mordeson $\mathcal{L}$-fuzzy normed spaces. Further, we consider the finite dimensional Cheng-Mordeson $\mathcal{L}$-fuzzy normed spaces and prove some theorems about completeness, compactness and weak convergence in these spaces. As application, we get a stability result in the setting of Cheng-Mordeson $\mathcal{L}$-fuzzy normed spaces.

1. Introduction and Preliminaries

The theory of fuzzy sets was introduced by Zadeh in 1965 [44]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive development is made in the field of fuzzy topology [2, 21, 15, 16, 18, 19, 20, 29, 39]. One of the problems in $\mathcal{L}$-fuzzy topology is to obtain an appropriate concept fuzzy normed spaces. In 1984, Katsaras [26] defined a fuzzy norm on a linear space and at the same year Wu and Fang [42] also introduced fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for fuzzy topological linear space. Some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [8, 9, 14, 28, 40, 43]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [27]. In 2003, Bag and Samanta [6] modified the definition of Cheng and Mordeson [10] by removing a regular condition.

In this paper, we define the notion of Cheng-Mordeson $\mathcal{L}$-fuzzy normed spaces using [37]. Further, we consider finite dimensional Cheng-Mordeson

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$\mathcal{L}$-fuzzy normed spaces and prove some theorems about completeness, compactness and weak convergence in these spaces.

In this paper, $\mathcal{L} = (L, \geq_L)$ is a complete lattice, i.e. a partially ordered set in which every nonempty subset admits supremum and infimum, and $0_{\mathcal{L}} = \inf L$, $1_{\mathcal{L}} = \sup L$.

Definition 1.1 (see [17])**.** 1.1 Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and let U be a non-empty set called the universe. An *$\mathcal{L}$-fuzzy set* in U is defined as a mapping $\mathcal{A}\colon U \to L$. For each u in U, $\mathcal{A}(u)$ represents the *degree* (in L) to which u is an element of $\mathcal{A}$.

Lemma 1.2 (see [12])**.** *Consider the set L^* and operation $\leq_{L^*}$ defined by*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1,\ x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 1.3 (see [4])**.** An *intuitionistic fuzzy set* $\mathcal{A}_{\zeta,\eta}$ in the universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(u, \zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ for all $u \in U$ are called the *membership degree* and the *non-membership degree*, respectively, of u in $\mathcal{A}_{\zeta,\eta}$ and, furthermore, satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

We define mapping $\wedge\colon L^2 \to L$ as

$$\wedge(x, y) = \begin{cases} x, & \text{if } x \leq_L y \\ y, & \text{if } y \leq_L x \end{cases}.$$

For example,

$$\wedge(x, y) = (\min(x_1, y_1), \max(x_2, y_2)),$$

in which $x = (x_1, x_2), y = (y_1, y_2) \in L^*$.

Definition 1.4. A *negator* on $\mathcal{L}$ is any decreasing mapping $\mathcal{N}\colon L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an *involutive negator*.

The negator N_s on $([0,1], \leq)$ defined as $N_s(x) = 1 - x$ for all $x \in [0,1]$ is called the *standard negator* on $([0,1], \leq)$. In this paper, the involutive negator $\mathcal{N}$ is fixed.

Definition 1.5. The pair $(V, \mathcal{P})$ is said to be an *Cheng-Mordeson $\mathcal{L}$-fuzzy normed space* (briefly, $CM\mathcal{L}$-fuzzy normed space) if V is vector space and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times]0, +\infty[$ satisfying the following conditions: for all $x, y \in V$ and $t, s \in]0, +\infty[$,

(a) $\mathcal{P}(x, t) = 0_{\mathcal{L}}$ for all $t \leq 0$;
(b) $\mathcal{P}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}\big(x, \frac{t}{|\alpha|}\big)$ for each $\alpha \neq 0$;
(d) $\wedge(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x + y, t + s)$;
(e) $\mathcal{P}(x, \cdot) :]0, \infty[\to L$ is continuous;

(f) $\lim_{t\to 0} \mathcal{P}(x,t) = 0_{\mathcal{L}}$ and $\lim_{t\to\infty} \mathcal{P}(x,t) = 1_{\mathcal{L}}$.

In this case $\mathcal{P}$ is called an *$\mathcal{L}$-fuzzy norm*. If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is an intuitionistic fuzzy set (see Definition 1.3), then the pair $(V, \mathcal{P}_{\mu,\nu})$ is said to be an *Cheng-Mordeson intuitionistic fuzzy normed space*.

Example 1.6. Let $(V, \|\cdot\|)$ be a normed space. We define $\wedge(a,b)$ by $\wedge(a,b) := (\min(a_1,b_1), \max(a_2,b_2))$ for all $a = (a_1,a_2), b = (b_1,b_2) \in L^*$ and let $\mathcal{P}_{\mu,\nu}$ be the intuitionistic fuzzy set on $V\times\,]0,+\infty[$ defined as follows:

$$\mathcal{P}_{\mu,\nu}(x,t) = \left(\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|}\right)$$

for all $t \in \mathbb{R}^+$. Then $(V, \mathcal{P}_{\mu,\nu})$ is a Cheng-Mordeson intuitionistic fuzzy normed space.

Definition 1.7. (1) A sequence $(x_n)_{n\in\mathbb{N}}$ in a $CM\mathcal{L}$-fuzzy normed space $(V,\mathcal{P})$ is called a *Cauchy sequence* if, for each $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$,

$$\mathcal{P}(x_n - x_m, t) >_L \mathcal{N}(\varepsilon),$$

where $\mathcal{N}$ is a negator on $\mathcal{L}$.

(2) A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be *c*onvergent to $x \in V$ in the $CM\mathcal{L}$-fuzzy normed space $(V,\mathcal{P})$, which is denoted by $x_n \stackrel{\mathcal{P}}{\to} x$ if $\mathcal{P}(x_n - x, t) \to 1_{\mathcal{L}}$, whenever $n \to +\infty$ for all $t > 0$.

(3) A $CM\mathcal{L}$-fuzzy normed space $(V,\mathcal{P})$ is said to be *complete* if and only if every Cauchy sequence in V is convergent.

Lemma 1.8 (see [37])**.** *Let $\mathcal{P}$ be a $CM\mathcal{L}$-fuzzy norm on V. Then we have the following:*

(i) $\mathcal{P}(x,t)$ *is nondecreasing with respect to t for all $x \in V$;*

(ii) $\mathcal{P}(x-y,t) = \mathcal{P}(y-x,t)$ *for all $x, y \in V$ and $t \in\,]0,+\infty[$.*

Definition 1.9. Let $(V,\mathcal{P})$ be an $CM\mathcal{L}$-fuzzy normed space and let $\mathcal{N}$ be a negator on $\mathcal{L}$. For all $t \in]0,+\infty[$, we define the *open ball* $B(x,r,t)$ with center $x \in V$ and radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ as follows:

$$B(x,r,t) = \{y \in V \mid \mathcal{P}(x-y,t) >_L \mathcal{N}(r)\}$$

and define the *unit ball* of V by

$$B(0,r,1) = \{x : \mathcal{P}(x,1) >_L \mathcal{N}(r)\}.$$

A subset $A \subseteq V$ is said to be *open* if, for each $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x,r,t) \subseteq A$. Let $\tau_{\mathcal{P}}$ denote the family of all open subsets of V. Then $\tau_{\mathcal{P}}$ is called the *t*opology induced by the $CM\mathcal{L}$-fuzzy norm $\mathcal{P}$.

Definition 1.10. Let $(V,\mathcal{P})$ be a $CM\mathcal{L}$-fuzzy normed space and let $\mathcal{N}$ be a negator on $\mathcal{L}$. A subset A of V is said to be *$\mathcal{L}F$-bounded* if there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{P}(x,t) >_L \mathcal{N}(r)$ for all $x \in A$.

Theorem 1.11. *In a $CM\mathcal{L}$-fuzzy normed space $(V, \mathcal{P})$, every compact set is closed and $\mathcal{L}F$-bounded.*

Lemma 1.12 (see [13]). *Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$-fuzzy normed space. Let $\mathcal{N}$ be a continuous negator on $\mathcal{L}$. If we define $E_{\lambda,\mathcal{P}} \colon V \to \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,\mathcal{P}}(x) = \inf\{t > 0 : \mathcal{P}(x, t) >_L \mathcal{N}(\lambda)\}$$

for all $\lambda \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ and $x \in V$. Then we have the following:

(i) *$E_{\lambda,\mathcal{P}}(\alpha x) = |\alpha| E_{\lambda,\mathcal{P}}(x)$ for all $x \in A$ and $\alpha \in \mathbb{R}$.*
(ii) *$E_{\lambda,\mathcal{P}}(x + y) \leq E_{\lambda,\mathcal{P}}(x) + E_{\lambda,\mathcal{P}}(y)$ for all $x, y \in V$.*
(iii) *A sequence $(x_n)_{n\in\mathbb{N}}$ is convergent with respect to the $CM\mathcal{L}$-fuzzy norm $\mathcal{P}$ if and only if $E_{\lambda,\mathcal{P}}(x_n - x) \to 0$. Also, the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the $CM\mathcal{L}$-fuzzy norm $\mathcal{P}$ if and only if it is a Cauchy sequence with respect to $E_{\lambda,\mathcal{P}}$.*

Lemma 1.13 (see [13]). *A subset A of $\mathbb{R}$ is $\mathcal{L}F$-bounded in $(\mathbb{R}, \mathcal{P})$ if and only if it is bounded in $\mathbb{R}$.*

Corollary 1.14 (see [13]). *If the real sequence $(\beta_n)_{n\in\mathbb{N}}$ is $\mathcal{L}F$-bounded, then it has at least one limit point.*

Definition 1.15. Let V be a vector space and let f be a real functional on V. We define

$$\tilde{V} = \{f : \mathcal{P}_0(f(x), t) \geq_L \mathcal{P}(cx, t), c \neq 0\}$$

for all $t > 0$.

Lemma 1.16 (see [38]). *If $(V, \mathcal{P})$ is a $CM\mathcal{L}$-fuzzy normed space, then we have*

(a) *the function $(x, y) \to x + y$ is continuous.*
(b) *the function $(\alpha, x) \to \alpha x$ is continuous.*

By the above lemma, a $CM\mathcal{L}$-fuzzy normed space is Hausdorff Topological Vector Space.

2. $CM\mathcal{L}$-Fuzzy Finite Dimensional Normed Spaces

Theorem 2.1. *Let $\{x_1, \cdots, x_n\}$ be a linearly independent set of vectors in vector space V and let $(V, \mathcal{P})$ be a $CM\mathcal{L}$-fuzzy normed space. Then there exist $c \neq 0$ and a $CM\mathcal{L}$-fuzzy normed space $(\mathbb{R}, \mathcal{P}_0)$ such that, for every choice of the n real scalars $\alpha_1, \cdots, \alpha_n$,*

$$\mathcal{P}(\alpha_1 x_1 + \cdots + \alpha_n x_n, t) \leq_L \mathcal{P}_0(c \sum_{j=1}^{n} |\alpha_j|, t). \tag{2.1}$$

Proof. Put $s = |\alpha_1| + \cdots + |\alpha_n|$. If $s = 0$, all α_j's must be zero and so (2.1) holds for any c. Let $s > 0$. Then (2.1) is equivalent to the inequality which we obtain from (2.1) by dividing by s and putting $\beta_j = \frac{\alpha_j}{s}$, that is,

$$\mathcal{P}(\beta_1 x_1 + \cdots + \beta_n x_n, t') \leq \mathcal{P}_0(c, t'), \quad (t' = \frac{t}{s}\sum_{j=1}^{n}|\beta_j| = 1). \tag{2.2}$$

Hence, it suffices to prove the existence of a $c \neq 0$ and $\mathcal{L}$-fuzzy norm $\mathcal{P}_0$ such that (2.2) holds. Suppose that this is not true. Then there exists a sequence $(y_m)_{m\in\mathbb{N}}$ of vectors,

$$y_m = \beta_{1,m}x_1 + \cdots + \beta_{n,m}x_n, \quad (\sum_{j=1}^{n}|\beta_{j,m}| = 1)$$

such that $\mathcal{P}(y_m, t) \to 1_{\mathcal{L}}$ as $m \to \infty$ for all $t > 0$. Since $\sum_{j=1}^{n}|\beta_{j,m}| = 1$, we have $|\beta_{j,m}| \leq 1$ and so, by Lemma 1.13, the sequence of $(\beta_{j,m})$ is $\mathcal{L}F$-bounded. By Corollary 1.14, $(\beta_{1,m})$ has a convergent subsequence. Let β_1 denote the limit of that subsequence and let $(y_{1,m})$ denote the corresponding subsequence of (y_m). By the same argument, $(y_{1,m})$ has a subsequence $(y_{2,m})$ for which the corresponding subsequence $\beta_2^{(m)}$ of real scalars convergence. Let β_2 denote the limit. Continuing this process, after n steps, we obtain a subsequence $(y_{n,m})_m$ of (y_m) such that

$$y_{n,m} = \sum_{j=1}^{n}\gamma_{j,m}x_j(\sum_{j=1}^{n}|\gamma_{j,m}| = 1)$$

and $\gamma_{j,m} \to \beta_j$ as $m \to \infty$. By Lemma 1.12 (ii), for any $\mu \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, we have

$$\begin{aligned}E_{\mu,\mathcal{P}}(y_{n,m} - \sum_{j=1}^{n}\beta_j x_j) &= E_{\mu,\mathcal{P}}(\sum_{j=1}^{n}(\gamma_{j,m} - \beta_j)x_j) \\ &\leq \sum_{j=1}^{n}|\gamma_{j,m} - \beta_j|E_{\mu,\mathcal{P}}(x_j) \to 0\end{aligned}$$

as $m \to \infty$. By Lemma 1.12 (iii), we conclude

$$\lim_{m\to\infty} y_{n,m} = \sum_{j=1}^{n}\beta_j x_j(\sum_{j=1}^{n}|\beta_j| = 1),$$

so that not all β_j can be zero. Put $y = \sum_{j=1}^{n}\beta_j x_j$. Since $\{x_1, \cdots, x_n\}$ is a linearly independent set, we have $y \neq 0$. Since $\mathcal{P}(y_m, t) \to 1_{\mathcal{L}}$ by assumption, we have $\mathcal{P}(y_{n,m}, t) \to 1_{\mathcal{L}}$. Hence it follows that

$$\mathcal{P}(y, t) = \mathcal{P}((y - y_{n,m}) + y_{n,m}, t) \geq_L \wedge(\mathcal{P}(y - y_{n,m}, t/2), \mathcal{P}(y_{n,m}, t/2)) \to 1_{\mathcal{L}}$$

and so $y = 0$, which is a contradiction. $\square$

Theorem 2.2. *Every finite dimensional subspace W of a $CM\mathcal{L}$-fuzzy normed space $(V,\mathcal{P})$ is complete. In particular, every finite dimensional $CM\mathcal{L}$-fuzzy normed space is complete.*

Proof. Let $(y_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in W such that y is its limit. Then we show that $y\in W$. Let $\dim W=n$ and $\{x_1,\cdots,x_n\}$ any linearly independent subset for Y. Then each y_m has a unique representation of the form

$$y_m=\alpha_1^{(m)}x_1+\cdots+\alpha_n^{(m)}x_n.$$

Since $(y_m)_{m\in\mathbb{N}}$ is a Cauchy sequence, for any $\varepsilon\in L\setminus\{0_\mathcal{L}\}$, there is a positive integer n_0 such that

$$\mathcal{N}(\varepsilon)<_L\mathcal{P}(y_m-y_k,t),$$

whenever $m,k>n_0$ and $t>0$. From this and the last theorem, we have

$$\begin{aligned}\mathcal{N}(\varepsilon)&<_L\mathcal{P}(y_m-y_k,t)=\mathcal{P}\Big(\sum_{j=1}^n(\alpha_j^{(m)}-\alpha_j^{(k)})x_j,t\Big)\\&\le_L\mathcal{P}_0\Big(\sum_{j=1}^n|\alpha_j^{(m)}-\alpha_j^{(k)}|c,t\Big)\le_L\mathcal{P}_0\left(1,\frac{\frac{t}{c}}{\sum_{j=1}^n|\alpha_j^{(m)}-\alpha_j^{(k)}|}\right)\\&\le_L\mathcal{P}_0\left(1,\frac{\frac{t}{c}}{|\alpha_j^{(m)}-\alpha_j^{(k)}|}\right)=\mathcal{P}_0\Big(\alpha_j^{(m)}-\alpha_j^{(k)},\frac{t}{c}\Big)\end{aligned}$$

for some $c\neq 0$ and $\mathcal{P}_0$. This shows that each of the n sequences $(\alpha_j^{(m)})_{m\in\mathbb{N}}$ where $j\in\{1,2,3,\cdots,n\}$ is a Cauchy sequence in $\mathbb{R}$. Hence these sequences converge. Let α_j denote the limit. Using these n limits $\alpha_1,\cdots,\alpha_n$, we define

$$y=\alpha_1x_1+\cdots+\alpha_nx_n.$$

Clearly, $y\in W$. Furthermore, by Lemma 1.12 (ii), for any $\mu\in L\setminus\{0_\mathcal{L},1_\mathcal{L}\}$, we have

$$E_{\mu,\mathcal{P}}(y_m-y)=E_{\mu,\mathcal{P}}\Big(\sum_{j=1}^n((\alpha_j^{(m)}-\alpha_j)x_j\Big)\le\sum_{j=1}^n|\alpha_j^{(m)}-\alpha_j|E_{\mu,\mathcal{P}}(x_j)\to 0$$

whenever $m\to\infty$. This shows that the arbitrary sequence $(y_m)_{m\in\mathbb{N}}$ is convergent in W. Hence W is complete. □

Corollary 2.3. *Every finite dimensional subspace W of a $CM\mathcal{L}$-fuzzy normed space $(V,\mathcal{P})$ is closed in V.*

Theorem 2.4. *In a finite dimensional $CM\mathcal{L}$-fuzzy normed space $(V,\mathcal{P})$, any subset $K\subset V$ is compact if and only if K is closed and $\mathcal{L}F$-bounded.*

Proof. By Theorem 1.11, compactness implies closedness and $\mathcal{L}F$-boundedness.

Conversely, let K be closed and $\mathcal{L}F$-bounded. Let $\dim V=n$ and $\{x_1,\ldots,x_n\}$ be a linearly independent set of V. We consider a sequence $(x^{(m)})_{m\in\mathbb{N}}$ in

K. Each $x^{(m)}$ has a representation by

$$x^{(m)} = \alpha_1^{(m)} x_1 + \cdots + \alpha_n^{(m)} x_n.$$

Since, K is $\mathcal{L}F$-bounded, so is $(x^{(m)})_{m\in\mathbb{N}}$ and so there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{P}(x^{(m)}, t) >_L \mathcal{N}(r)$ for all $m \in \mathbb{N}$.

On the other hand, by Theorem 2.1, there exist $c \neq 0$ and a $\mathcal{L}$-fuzzy norm $\mathcal{P}_0$ such that

$$\begin{aligned}
\mathcal{N}(r) <_L \mathcal{P}(x^{(m)}, t) &= \mathcal{P}\Big(\sum_{j=1}^{n} \alpha_j^{(m)} x_j, t\Big) \\
&\leq_L \mathcal{P}_0\Big(c \sum_{j=1}^{n} |\alpha_j^{(m)}|, t\Big) \leq_L \mathcal{P}_0\Big(1, \frac{t}{c\sum_{j=1}^{n} |\alpha_j^{(m)}|}\Big) \\
&\leq_L \mathcal{P}_0\Big(1, \frac{t}{c|\alpha_j^{(m)}|}\Big) = \mathcal{P}_0\Big(\alpha_j^{(m)}, \frac{t}{c}\Big).
\end{aligned}$$

Hence, the sequence $(\alpha_j^{(m)})_{m\in\mathbb{N}}$ for any fixed j is $\mathcal{L}F$-bounded and, by Corollary 1.14, has a limit point α_j, where $1 \leq j \leq n$. We consider that $(x^{(m)})_{m\in\mathbb{N}}$ has a subsequence $(z_m)_{m\in\mathbb{N}}$ which converges to $z = \sum_{j=1}^{n} \alpha_j x_j$. Since K is closed, $z \in K$. This shows that an arbitrary sequence $(x^{(m)})_{m\in\mathbb{N}}$ in K has a subsequence which converges in K. Hence, K is compact. □

Remark 2.5. In a $CM\mathcal{L}$-fuzzy normed space $(V, \mathcal{P})$ whenever $\mathcal{P}(x, t) >_L \mathcal{N}(r)$ for all $x \in V$, $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, we can find $t_0 \in]0, t[$ such that $\mathcal{P}(x, t_0) >_L \mathcal{N}(r)$ (see [15]).

Lemma 2.6. *Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$-fuzzy normed space and let A be a subspace of V. Define*

$$\mathcal{D}(x_1 - A, t) = \sup\{\mathcal{P}(x_1 - y, t) : y \in A\}$$

for all $x_1 \in V$ and $t > 0$. Then, for any $\varepsilon \in L \setminus \{1_{\mathcal{L}}\}$ and $x_1 \in V \setminus A$, there exists $y_1 \in A$ such that

$$\wedge(\mathcal{D}(x_1 - A, t), \varepsilon) <_L \mathcal{P}(x_1 - y_1, t) \leq_L \mathcal{D}(x_1 - A, t).$$

The proof is straightforward.

Lemma 2.7. *Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$-fuzzy normed space and let A be a subset of V. If we define*

$$p_1 = \inf\{t > 0 : \mathcal{D}(x_1 - A, t) >_L \mathcal{N}(\lambda)\}$$

and

$$p_2 = \inf\{t > 0 : \wedge(\mathcal{D}(x_1 - A, t), \varepsilon) >_L \mathcal{N}(\lambda)\},$$

in which $\varepsilon \in L \setminus \{1_{\mathcal{L}}\}$. Then there exists $\delta \in]0, t[$ such that $p_2 \geq p_1 + \delta$.

Proof. Since $\wedge(\mathcal{D}(x_1 - A, t), \varepsilon) <_L \mathcal{D}(x_1 - A, t)$, by Remark 2.5, there exists $\delta \in]0, t[$ such that $\wedge(\mathcal{D}(x_1 - A, t), \varepsilon) <_L \mathcal{D}(x_1 - A, t - \delta)$ and so

$$\begin{aligned} p_2 &= \inf\{t > 0 : \wedge(\mathcal{D}(x_1 - A, t), \varepsilon) >_L \mathcal{N}(\lambda)\} \\ &\geq \inf\{t > 0 : \mathcal{D}(x_1 - A, t - \delta) >_L \mathcal{N}(\lambda)\} \\ &= \inf\{t + \delta > 0 : \mathcal{D}(x_1 - A, t) >_L \mathcal{N}(\lambda)\} = p_1 + \delta. \end{aligned}$$

□

Lemma 2.8. *Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$-fuzzy normed space and let A be a nonempty closed subspace of V. Then $x \in A$ if and only if $\mathcal{D}(x - A, t) = 1_{\mathcal{L}}$ for all $t > 0$.*

Proof. Let $\mathcal{D}(x - A, t) = 1_{\mathcal{L}}$. By definition, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $\mathcal{P}(x - x_n, t) \to 1_{\mathcal{L}}$. Hence $x - x_n \to 0$ or equivalently $x_n \to x$ and, since A is closed, $x \in A$. The converse is trivial. □

Theorem 2.9. *Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$-fuzzy normed space and let A be a nonempty closed subspace of V. Then, for any $y \in A$, there exist $x_0 \in V \setminus A$ and $\lambda_0 \in L$ such that $x_0 \in B(0, \lambda, 1)$ and $E_{\lambda,\mathcal{P}}(x_0 - y) \geq 1$ for all $\lambda_0 <_L \lambda \leq_L 1_{\mathcal{L}}$.*

Proof. Since, A is a nonempty closed subspace of V, by Lemma 2.8, there exists $x_1 \in V \setminus A$ such that $\mathcal{D}(x_1 - A, t) <_L 1_{\mathcal{L}}$ for all $t > 0$. Let

$$\sup_{t>0} \mathcal{D}(x_1 - A, t) = \sigma.$$

Let $\lambda_0 = \mathcal{N}(\sigma)$. Then, for all $\lambda_0 <_L \lambda \leq_L 1_{\mathcal{L}}$, we have

$$\sup_{t>0} \mathcal{D}(x_1 - A, t) >_L \mathcal{N}(\lambda).$$

By the property of sup, there exists $t_0 > 0$ such that $\mathcal{D}(x_1 - A, t) >_L \mathcal{N}(\lambda)$ for all $t \geq t_0$. By Lemma 2.6, there exists $y_1 \in A$ such that

$$\wedge(\mathcal{D}(x_1 - A, t), \varepsilon) <_L \mathcal{P}(x_1 - y_1, t)$$

for all $\varepsilon \in L \setminus \{1_{\mathcal{L}}\}$ and $t \geq 0$. Taking $x_0 = \frac{x_1 - y_1}{p_2}$, by Lemma 2.7, we have

$$\begin{aligned} \mathcal{P}(x_0, 1) = \mathcal{P}\Big(\frac{x_1 - y_1}{p_2}, 1\Big) &= \mathcal{P}(x_1 - y_1, p_2) \geq_L \wedge(\mathcal{D}(x_1 - A, p_2), \varepsilon) \\ &\geq_L \wedge(\mathcal{D}(x_1 - A, p_1 + \delta), \varepsilon) >_L \wedge(\mathcal{N}(\lambda), \varepsilon). \end{aligned}$$

Since, $\varepsilon \in L \setminus \{1_{\mathcal{L}}\}$ is arbitrary, we have $\mathcal{P}(x_0, 1) >_L \mathcal{N}(\lambda)$, i.e., $x_0 \in B(0, \lambda, 1)$ for all $\lambda_0 <_L \lambda \leq_L 1_{\mathcal{L}}$. Taking $\delta_1 = \frac{\delta}{p_2}$, by Lemma 2.7, we have

$$\begin{aligned} \wedge\,(\mathcal{P}(x_0 - y, N_s(\delta_1)), \varepsilon) &= \wedge(\mathcal{P}(x_1 - (y_1 + p_2 y), p_2 N_s(\delta_1)), \varepsilon) \\ &\leq_L \wedge(\mathcal{D}(x_1 - A, p_2 - \delta), \varepsilon) \leq_L \mathcal{N}(\lambda). \end{aligned}$$

Letting $\varepsilon \to 1_{\mathcal{L}}$ and $\delta \to 0$, we have $\mathcal{P}(x_0 - y, 1) \leq_L \mathcal{N}(\lambda)$ and so

$$E_{\lambda,\mathcal{P}}(x_0 - y) \geq 1$$

for all $y \in A$ and $x_0 \in B(0, \lambda, 1)$. □

Lemma 2.10. *Let $\{x_1, \cdots, x_n\}$ be a linearly independent set of vectors in vector space V and $(V, \mathcal{P})$ be a $CM\mathcal{L}$-fuzzy normed space. Then there exists $k \neq 0$ such that, for every choice of the n real scalars $\alpha_1, \cdots, \alpha_n$,*

$$E_{\lambda,\mathcal{P}}\Big(\sum_{j=1}^{n} \alpha_j x_j\Big) \geq |k| \sum_{j=1}^{n} |\alpha_j|.$$

Proof. By Theorem 2.1, there exist $c \neq 0$ and an $\mathcal{L}$-fuzzy norm $\mathcal{P}_0$ such that

$$\mathcal{P}\Big(\sum_{j=1}^{n} \alpha_j x_j, t\Big) \leq_L \mathcal{P}_0\Big(c \sum_{j=1}^{n} |\alpha_j|, t\Big).$$

Therefore, we have

$$E_{\lambda,\mathcal{P}}\Big(\sum_{j=1}^{n} \alpha_j x_j\Big) \geq E_{\lambda,\mathcal{P}_0}\Big(c \sum_{j=1}^{n} |\alpha_j|\Big) = |c| \sum_{j=1}^{n} |\alpha_j| E_{\lambda,\mathcal{P}_0}(1).$$

Taking $k = cE_{\lambda,\mathcal{P}_0}(1)$, we have

$$E_{\lambda,\mathcal{P}}\Big(\sum_{j=1}^{n} \alpha_j x_j\Big) \geq |k| \sum_{j=1}^{n} |\alpha_j|. \qquad \square$$

Theorem 2.11. *Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$-fuzzy normed space. Then $(V, \mathcal{P})$ is finite dimensional if and only if the unit ball $B(0, \lambda, 1)$ is compact.*

Proof. Let $\dim V = n$ and $\{x_1, \cdots, x_n\}$ a basis for V. We consider any sequence $(x^{(m)})_{m \in \mathbb{N}}$ in $B(0, \lambda, 1)$. Each $x^{(m)}$ has the representation by

$$x^{(m)} = \sum_{j=1}^{n} \alpha_j^{(m)} x_j.$$

By Lemmas 2.7 and 2.10, we have

$$1 \geq E_{\lambda,\mathcal{P}}(x^{(m)}) \geq |k| \sum_{j=1}^{n} |\alpha_j^{(m)}|,$$

where $k \neq 0$. Hence the sequence $(\alpha_j^{(m)})_{m \in \mathbb{N}}$ is bounded and has a limit point α_j $(1 \leq j \leq n)$. Therefore, $(x^{(m)})_{m \in \mathbb{N}}$ has a subsequence $(x^{(m_k)})_{k \in \mathbb{N}}$ which converges to $x = \sum_{j=1}^{n} \alpha_j x_j$.

On the other hand, for any $\varepsilon \neq 0_{\mathcal{L}}$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$,

$$\mathcal{P}(x, 1 + \delta) \geq_L \wedge(\mathcal{P}(x^{(m_k)} - x, \delta), \mathcal{P}(x^{(m_k)}, 1)) \geq_L \wedge(\mathcal{N}(\varepsilon), \mathcal{N}(\lambda))$$

for all $\delta > 0$. Since $\varepsilon \neq_L 0_{\mathcal{L}}$ and $\delta > 0$ are arbitrary, it follows that

$$\mathcal{P}(x, 1) \geq_L \wedge(1_{\mathcal{L}}, \mathcal{N}(\lambda)) = \mathcal{N}(\lambda)$$

and, consequently, $x \in B(0, \lambda, 1)$. Hence, $B(0, \lambda, 1)$ is compact.

Conversely, assume that the unit balls be compact, but $(V, \mathcal{P})$ is not finite dimensional. We choose $x_1 \neq 0$ in V, for any $k_1 \in \mathbb{R}$, let $V_1 = \{k_1 x_1 : x_1 \in$

$V, k_1 \in \mathbb{R}\}$. By Theorem 2.9, for all $\lambda_{0,1} <_L \lambda \leq_L 1_{\mathcal{L}}$, there exist $x_2 \in V \setminus V_1$ and $x_2 \in B(0, \lambda, 1)$ such that $E_{\lambda,\mathcal{P}}(x_2 - x_1) \geq 1$.

In this case, x_1 and x_2 are linear independent. In fact, if x_1 and x_2 are dependent, then there exists $k_1, k_2 \in \mathbb{R}$ (we might as well assume $k_2 \neq 0$) such that $k_1x_1 + k_2x_2 = 0$ and $x_2 = \frac{-k_1}{k_2}x_1 \in V_1$, which is a contradiction.

Let $V_2 = \{k_1x_1 + k_2x_2 : x_1 \in V_1, x_2 \in V \setminus V_1, k_1, k_2 \in \mathbb{R}\}$. By Theorem 2.9, for all $\lambda_{0,2} <_L \lambda \leq_L 1_{\mathcal{L}}$, there exist $x_3 \in V \setminus V_2$ and $x_3 \in B(0, \lambda, 1)$ such that $E_{\lambda,\mathcal{P}}(x_3 - y) \geq 1$ where $y \in V_2$. In particular, if we choose $y = x_1$ and $y = x_2$, then $E_{\lambda,\mathcal{P}}(x_3 - x_1) \geq 1$ and $E_{\lambda,\mathcal{P}}(x_3 - x_2) \geq 1$. By the same way, we can choose $(x_n)_{n\in\mathbb{N}} \subset B(0, \lambda, 1)$ such that $E_{\lambda,\mathcal{P}}(x_m - x_n) \geq 1$ where $m \neq n$ for all $\lambda_{0,n-1} <_L \lambda \leq_L 1_{\mathcal{L}}$. If we put $\lambda_0 = \vee_{1\leq i\leq n-1}\lambda_{0,i}$, then the sequence $(x_n)_{n\geq 2}$ lie in $B(0, \lambda, 1)$ and $E_{\lambda,\mathcal{P}}(x_m - x_n) \geq 1$ for all $\lambda_0 <_L \lambda \leq_L 1_{\mathcal{L}}$. By Lemma 1.12, (ii), the sequence $(x_n)_{n\geq 2}$ has not any convergent subsequence in V, which is a contradiction. This completes the proof. □

Theorem 2.12. *Let $(V, \mathcal{P})$ be a finite dimensional $CM\mathcal{L}$-fuzzy normed space and let A be a closed subspace of V. Then, for all $\lambda >_L \lambda_0$, there exists $x_0 \in B(0, \lambda, 1)$ such that*

$$\inf_{y\in A} E_{\lambda,\mathcal{P}}(x_0 - y) = 1.$$

Proof. By Theorem 2.9, for any $y_n \in A$, there exist $x_n \in V \setminus A$ and $\lambda_0 \in L$ such that

$$x_n \in B(0, \lambda, 1), \quad E_{\lambda,\mathcal{P}}(x_n - y_n) \geq 1 \tag{2.3}$$

for all $\lambda >_L \lambda_0$. Since V is finite dimensional, by Theorem 2.11, $B(0, \lambda, 1)$ is compact and so there exists $x_0 \in B(0, \lambda, 1)$ such that

$$\mathcal{P}(x_{n_k} - x_0, t) \to 1_{\mathcal{L}}$$

for all $t > 0$, where $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$. Since $x_0 \in B(0, \lambda, 1)$, $E_{\lambda,\mathcal{P}}(x_0) \leq 1$. Since the null element $0 \in A$, we have

$$1 \geq E_{\lambda,\mathcal{P}}(x_0) = E_{\lambda,\mathcal{P}}(x_0 - 0) \geq \inf_{y\in A} E_{\lambda,\mathcal{P}}(x_0 - y).$$

Next, we prove that $\inf_{y\in A} E_{\lambda,\mathcal{P}}(x_0 - y) \geq 1$. By (2.1), $\mathcal{P}(x_n - y_n, 1) \leq_L \mathcal{N}(\lambda)$. Let $\mathcal{P}(x_0 - y, 1) >_L \mathcal{N}(\lambda)$ for all $y \in A$. Then, by continuity of $CM\mathcal{L}$-fuzzy norm $\mathcal{P}$ and Remark 2.5, we can find $\lambda_1 \in L$ such that, for $\delta \in]0, 1[$,

$$\mathcal{P}(x_0 - y, N_s(\delta)) >_L \mathcal{N}(\lambda_1),$$

and

$$\mathcal{N}(\lambda_1) >_L \mathcal{N}(\lambda).$$

Since $x_{n_k} \to x_0$, there exists $k_0 \in \mathbb{N}$ such that, for every $k \geq k_0$,

$$\mathcal{P}(x_{n_k} - x_0, t) >_L \mathcal{N}(\lambda_1)$$

for all $t > 0$. By triangle inequality 1.5, (d), we have

$$\begin{aligned}\mathcal{N}(\lambda) \geq_L \mathcal{P}(x_{n_k} - y_{n_k}, t) &\geq_L \wedge(\mathcal{P}(x_{n_k} - x_0, t/2), \mathcal{P}(x_0 - y_{n_k}, t/2)) \\ &\geq_L \wedge(\mathcal{N}(\lambda_1), \mathcal{N}(\lambda_1)) >_L \mathcal{N}(\lambda),\end{aligned}$$

which is a contradiction. Then, for any $y \in A$, we have $\mathcal{P}(x_0 - y, 1) \leq_L \mathcal{N}(\lambda)$, which implies $\inf_{y\in A} E_{\lambda,\mathcal{P}}(x_0 - y) \geq 1$. This completes the proof. □

Definition 2.13. A sequence $(x_m)_{m\in\mathbb{N}}$ in a $CM\mathcal{L}$-fuzzy normed space $(V, \mathcal{P})$ is said to be *weakly convergent* if there exists $x \in V$ such that, for all $f \in \tilde{V}$ and $t > 0$,

$$\mathcal{P}(f(x_m) - f(x), t) \to 1_{\mathcal{L}}.$$

This is written by

$$x_m \stackrel{W}{\to} x.$$

Theorem 2.14. *Let $(V, \mathcal{P})$ be a $CM\mathcal{L}$-fuzzy normed space and let $(x_m)_{m\in\mathbb{N}}$ be a sequence in V. Then we have the following:*

(i) *Convergence implies weak convergence with the same limit.*
(ii) *If $\dim V < \infty$, then weak convergence implies convergence.*

Proof. (i) Let $x_m \to x$. Then, for all $t > 0$, we have

$$\mathcal{P}(x_m - x, t) \to 1_{\mathcal{L}}.$$

By Definition 1.15, for every $f \in \tilde{V}$,

$$\mathcal{P}_0(f(x_m) - f(x), t) = \mathcal{P}_0(f(x_m - x), t) \geq_L \mathcal{P}(x_m - x, t/c)(c \neq 0).$$

Then $x_m \stackrel{W}{\to} x$.

(ii) Let $x_m \stackrel{W}{\to} x$ and $\dim V = n$. Let $\{x_1, \ldots, x_n\}$ be a linearly independent set of V. Then $x_m = \alpha_1^{(m)} x_1 + \cdots + \alpha_n^{(m)} x_n$ and $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$. By assumption, for all $f \in \tilde{V}$ and $t > 0$, we have

$$\mathcal{P}_0(f(x_m) - f(x), t) \to 1_{\mathcal{L}}.$$

We take in particular $f_1, \cdots, f_n$ defined by $f_j x_j = 1$ and $f_j x_i = 0$ $(i \neq j)$. Therefore, $f_j(x_m) = \alpha_j^{(m)}$ and $f_j(x) = \alpha_j$. Hence $f_j(x_m) \to f_j(x)$ implies $\alpha_j^{(m)} \to \alpha_j$. From this and Lemma 1.12 (ii), we obtain

$$E_{\mu,\mathcal{P}}(x_m - x) = E_{\mu,\mathcal{P}}\Big(\sum_{j=1}^{n} (\alpha_j^{(m)} - \alpha_j) x_j\Big) \leq \sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j| E_{\lambda,\mathcal{P}}(x_j) \to 0$$

as $m \to \infty$. This shows that $(x_m)_{m\in\mathbb{N}}$ convergence to x. □

Theorem 2.15. *A $CM\mathcal{L}$-fuzzy normed space $(V, \mathcal{P})$ is locally convex.*

Proof. It suffices to consider the family of neighborhoods of the origin, $B(0,r,t)$, with $t>0$ and $r\in L\setminus\{0_{\mathcal{L}},1_{\mathcal{L}}\}$. Let $t>0$, $r\in L\setminus\{0_{\mathcal{L}},1_{\mathcal{L}}\}$, $x,y\in B(0,r,t)$ and $\alpha\in[0,1]$. Then we have

$$\begin{aligned}\mathcal{P}(\alpha x+(1-\alpha)y,t) &\geq_L \wedge(\mathcal{P}(\alpha x,\alpha t),\mathcal{P}((1-\alpha)y,(1-\alpha)t))\\ &= \wedge(\mathcal{P}(x,t),\mathcal{P}(y,t)) >_L \mathcal{N}(r).\end{aligned}$$

Thus, $\alpha x+(1-\alpha)y$ belongs to $B(0,r,t)$ for all $\alpha\in[0,1]$. $\square$

3. Stability of Cubic Functional Equations in $\mathcal{L}$-Fuzzy Normed Spaces

The study of stability problems for functional equations is related to a question of Ulam [41] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [22]. Subsequently, the result of Hyers was generalized by T. Aoki [3] for additive mappings and by Th.M. Rassias [34] for linear mappings by considering an unbounded Cauchy difference. The paper [34] of Th.M. Rassias has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations. We refer the interested readers for more information on such problems to e.g. [5, 11, 23, 35, 36].

The functional equation

$$3f(x+3y)+f(3x-y)=15f(x+y)+15f(x-y)+80f(y) \tag{3.1}$$

is said to be the cubic functional equation since the function $f(x)=cx^3$ is its solution. Every solution of the cubic functional equation is said to be a cubic mapping. The stability problem for the cubic functional equation was proved by Jun and Kim [24] for mappings $f\colon X\to Y$, where X is a real normed space and Y is a Banach space. Later a number of mathematicians worked on the stability of some types of the cubic equation [25, 34]. In addition, Mirmostafaee, Mirzavaziri and Moslehian [33, 32], Alsina [1], Miheţ and Radu [30], Miheţ et. al. [31] and Baktash et. al. [7] investigated the stability in the settings of fuzzy, probabilistic and random normed spaces.

The aim of this note, is to provide a result on the stability of the cubic functional equation (3.1) in fuzzy normed spaces and give a better error estimation.

Now, we state our main result.

Theorem 3.1. *Let X be a linear space, $(Z,\mathcal{P}')$ be a $CM\mathcal{L}$-fuzzy normed space, $\varphi\colon X\times X\to Z$ be a function such that for some $0<\alpha<27$,*

$$\mathcal{P}'(\varphi(3x,0),t)\geq_L \mathcal{P}'(\alpha\varphi(x,0),t)\quad (x,y\in X,t>0) \tag{3.2}$$

and $\lim_{n\to\infty} \mathcal{P}'(\varphi(3^n x, 3^n y), 27^n t) = 1_L$ *for all* $x, y \in X$ *and* $t > 0$*. Let* $(Y, \mathcal{P})$ *be a complete fuzzy normed space. If* $f\colon X \to Y$ *is a mapping such that*

$$\mathcal{P}(3f(x+3y) + f(3x-y) - 15f(x+y) - 15f(x-y) - 80f(y), t) \geq_L \mathcal{P}'(\varphi(x,y), t) \tag{3.3}$$

where $x, y \in X, t > 0$*. Then there exists a unique cubic mapping* $C\colon X \to Y$ *such that*

$$\mathcal{P}(f(x) - C(x), t) \geq_L \mathcal{P}'(\varphi(x,0), (27-\alpha)t)). \tag{3.4}$$

Proof. Putting $y = 0$ in (3.3) we get

$$\mathcal{P}(\frac{f(3x)}{27} - f(x), t) \geq_L \mathcal{P}(\varphi(x,0), 27t) \quad (x \in X, t > 0). \tag{3.5}$$

Replacing x by $3^n x$ in (3.5), and using (3.2) we obtain

$$\mathcal{P}(\frac{f(3^{n+1}x)}{27^{n+1}} - \frac{f(3^n x)}{27^n}, t) \geq_L \mathcal{P}'(\varphi(3^n x, 0), 27 \times 27^n t) \geq_L \mathcal{P}'(\varphi(x,0), \frac{27 \times 27^n}{\alpha^n} t). \tag{3.6}$$

Since $\frac{f(3^n x)}{27^n} - f(x) = \sum_{k=0}^{n-1} (\frac{f(3^{k+1}x)}{27^{k+1}} - \frac{f(3^k x)}{27^k})$, by (3.6) we have

$$\mathcal{P}\left(\frac{f(3^n x)}{27^n} - f(x), t\sum_{k=0}^{n-1} \frac{\alpha^k}{27 \times 27^k}\right) \geq_L \wedge_{k=0}^{n-1} \mathcal{P}'(\varphi(x,0), t) = \mathcal{P}'(\varphi(x,0), t),$$

that is,

$$\mathcal{P}(\frac{f(3^n x)}{27^n} - f(x), t) \geq_L \mathcal{P}'\left(\varphi(x,0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{27 \times 27^k}}\right). \tag{3.7}$$

By replacing x with $3^m x$ in (3.7) we observe that:

$$\mathcal{P}(\frac{f(3^{n+m}x)}{27^{n+m}} - \frac{f(3^m x)}{27^m}, t) \geq_L \mathcal{P}'\left(\varphi(x,0), \frac{t}{\sum_{k=m}^{n+m} \frac{\alpha^k}{27 \times 27^k}}\right). \tag{3.8}$$

Then $\{\frac{f(3^n x)}{27^n}\}$ is a Cauchy sequence in $(Y, \mathcal{P})$. Since $(Y, \mathcal{P})$ is a complete $CM\mathcal{L}$-fuzzy normed space this sequence convergent to some point $C(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (3.8) to obtain

$$\mathcal{P}(\frac{f(3^n x)}{27^n} - f(x), t) \geq_L \mathcal{P}'\left(\varphi(x,0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{27 \times 27^k}}\right), \tag{3.9}$$

and so for every $\delta > 0$ we have

(3.10)
$$\begin{aligned}\mathcal{P}(C(x)-f(x),t+\delta) &\geq_L \wedge\left(\mathcal{P}(C(x)-\frac{f(3^n x)}{27^n},\delta),\mathcal{P}(\frac{f(3^n x)}{27^n}-f(x),t)\right)\\ &\geq_L \wedge\left(\mathcal{P}\left(C(x)-\frac{f(3^n x)}{27^n},\delta\right),\mathcal{P}'\left(\varphi(x,0),\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{27\times 27^k}}\right)\right).\end{aligned}$$

Taking the limit as $n \to \infty$ and using (3.10) we get

$$\mathcal{P}(C(x)-f(x),t+\delta) \geq_L \mathcal{P}'(\varphi(x,0),t(27-\alpha)). \tag{3.11}$$

Since δ was arbitrary, by taking $\delta \to 0$ in (3.11) we get

$$\mathcal{P}(C(x)-f(x),t) \geq_L \mathcal{P}'(\varphi(x,0),t(27-\alpha)).$$

Replacing x, y by $3^n x, 3^n y$ in (3.3) to get

$$\begin{aligned}\mathcal{P}\Bigg(\frac{f(3^n(x+3y))}{27^n}+\frac{f(3^n(3x-y))}{27^n}-\frac{15f(3^n(x+y))}{27^n}-\frac{15f(3^n(x-y))}{8^n}-\\ -\frac{80f(3^n(y))}{27^n},t\Bigg) \geq_L \mathcal{P}'(\varphi(3^n x,3^n y),27^n t),\end{aligned}$$

for all $x, y \in X$ and for all $t > 0$. Since $\lim_{n\to\infty}\mathcal{P}'(\varphi(3^n x,3^n y),27^n t) = 1$ we conclude that C fulfills (3.1). To Prove the uniqueness of the cubic function C, assume that there exists a cubic function $D\colon X \to Y$ which satisfies (3.4). Fix $x \in X$. Clearly $C(3^n x) = 27^n C(x)$ and $D(3^n x) = 27^n D(x)$ for all $n \in \mathbb{N}$. It follows from (3.4) that

$$\begin{aligned}\mathcal{P}(C(x)-D(x),t) &= \mathcal{P}\left(\frac{C(3^n x)}{27^n}-\frac{D(3^n x)}{27^n},t\right)\\ &\geq_L \wedge\left(\mathcal{P}\left(\frac{C(3^n x)}{27^n}-\frac{f(3^n x)}{27^n},\frac{t}{2}\right),\mathcal{P}\left(\frac{D(3^n x)}{27^n}-\frac{f(3^n x)}{27^n},\frac{t}{2}\right)\right)\\ &\geq_L \mathcal{P}'\left(\varphi(3^n x,0),27^n(27-\alpha)\frac{t}{2}\right) \geq_L \mathcal{P}'\left(\varphi(x,0),\frac{27^n(27-\alpha)\frac{t}{2}}{\alpha^n}\right).\end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{27^n(27-\alpha)t}{2\alpha^n} = \infty,$$

we get

$$\lim_{n\to\infty}\mathcal{P}'(\varphi(x,0),\frac{27^n(27-\alpha)t}{2\alpha^n}) = 1_L.$$

Therefore $\mathcal{P}(C(x)-D(x),t) = 1_L$ for all $t > 0$, whence $C(x) = D(x)$. □

Corollary 3.2. *Let X be a linear space, $L = [0, 1]$, $(Z, \mathcal{P}')$ be a $CM\mathcal{L}$-fuzzy normed space, $(Y, \mathcal{P})$ be a complete $CM\mathcal{L}$-fuzzy normed space, p, q be nonnegative real numbers and let $z_0 \in Z$. If $f\colon X \to Y$ is a mapping such that*

$$\begin{aligned}(3.12)\quad &\mathcal{P}(3f(x + 3y) + f(3x - y) - 15f(x + y) - 15f(x - y) - 80f(y), t) \\ &\qquad \geq \mathcal{P}'((\|x\|^p + \|y\|^q)z_0, t) \quad (x, y \in X, t > 0),\end{aligned}$$

$f(0) = 0$ and $p, q < 3$, then there exists a unique cubic mapping $C\colon X \to Y$ such that

$$\mathcal{P}(f(x) - C(x), t) \geq \mathcal{P}'(\|x\|^p z_0, (27 - 3^p)t)). \tag{3.13}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\varphi\colon X \times X \to Z$ be defined by $\varphi(x, y) = (\|x\|^p + \|y\|^q)z_0$. Then the corollary is followed from Theorem 3.1 by $\alpha = 3^p$. $\square$

Corollary 3.3. *Let X be a linear space, $L = [0, 1]$, $(Z, \mathcal{P}')$ be a $CM\mathcal{L}$-fuzzy normed space, $(Y, \mathcal{P})$ be a complete $CM\mathcal{L}$-fuzzy normed space and let $z_0 \in Z$. If $f\colon X \to Y$ is a mapping such that*

$$\mathcal{P}(3f(x+3y)+f(3x-y)-15f(x+y)-15f(x-y)-80f(y), t) \geq \mathcal{P}'(\varepsilon z_0, t) \tag{3.14}$$

for $x, y \in X, t > 0$ and $f(0) = 0$, then there exists a unique cubic mapping $C\colon X \to Y$ such that

$$\mathcal{P}(f(x) - C(x), t) \geq \mathcal{P}'(\varepsilon z_0, 26t). \tag{3.15}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\varphi\colon X \times X \to Z$ be defined by $\varphi(x, y) = \varepsilon z_0$. Then the corollary is followed from Theorem 3.1 by $\alpha = 1$. $\square$

References

[1] C. Alsina. On the stability of a functional equation arising in probabilistic normed spaces. In *General inequalities, 5 (Oberwolfach, 1986)*, volume 80 of *Internat. Schriftenreihe Numer. Math.*, pages 263–271. Birkhäuser, Basel, 1987.

[2] M. Amini and R. Saadati. Topics in fuzzy metric spaces. *J. Fuzzy Math.*, 11(4):765–768, 2003.

[3] T. Aoki. On the stability of the linear transformation in Banach spaces. *J. Math. Soc. Japan*, 2:64–66, 1950.

[4] K. T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1):87–96, 1986.

[5] C. Baak and M. S. Moslehian. On the stability of J^*-homomorphisms. *Nonlinear Anal.*, 63(1):42–48, 2005.

[6] T. Bag and S. K. Samanta. Finite dimensional fuzzy normed linear spaces. *J. Fuzzy Math.*, 11(3):687–705, 2003.

[7] E. Baktash, Y. J. Cho, M. Jalili, R. Saadati, and S. M. Vaezpour. On the stability of cubic mappings and quadratic mappings in random normed spaces. *J. Inequal. Appl.*, pages Art. ID 902187, 11, 2008.

[8] V. Balopoulos, A. G. Hatzimichailidis, and B. K. Papadopoulos. Distance and similarity measures for fuzzy operators. *Inform. Sci.*, 177(11):2336–2348, 2007.

[9] R. Biswas. Fuzzy inner product spaces and fuzzy norm functions. *Inf. Sci.*, 53(1-2):185–190, 1991.
[10] S. C. Cheng and J. N. Mordeson. Fuzzy linear operators and fuzzy normed linear spaces. *Bull. Calcutta Math. Soc.*, 86(5):429–436, 1994.
[11] S. Czerwik. *Functional equations and inequalities in several variables.* World Scientific Publishing Co. Inc., River Edge, NJ, 2002.
[12] G. Deschrijver and E. E. Kerre. On the relationship between some extensions of fuzzy set theory. *Fuzzy Sets and Systems*, 133(2):227–235, 2003.
[13] G. Deschrijver, D. O'Regan, R. Saadati, and S. M. Vaezpour. L-fuzzy Euclidean normed spaces and compactness. *Chaos Solitons Fractals*, 42(1):40–45, 2009.
[14] C. Felbin. Finite-dimensional fuzzy normed linear space. *Fuzzy Sets and Systems*, 48(2):239–248, 1992.
[15] A. George and P. Veeramani. On some results in fuzzy metric spaces. *Fuzzy Sets and Systems*, 64(3):395–399, 1994.
[16] A. George and P. Veeramani. On some results of analysis for fuzzy metric spaces. *Fuzzy Sets and Systems*, 90(3):365–368, 1997.
[17] J. A. Goguen. L-fuzzy sets. *J. Math. Anal. Appl.*, 18:145–174, 1967.
[18] V. Gregori and S. Romaguera. Some properties of fuzzy metric spaces. *Fuzzy Sets and Systems*, 115(3):485–489, 2000.
[19] V. Gregori and S. Romaguera. On completion of fuzzy metric spaces. *Fuzzy Sets and Systems*, 130(3):399–404, 2002. Theme: Fuzzy intervals.
[20] V. Gregori and S. Romaguera. Characterizing completable fuzzy metric spaces. *Fuzzy Sets and Systems*, 144(3):411–420, 2004.
[21] C. M. Hu. C-structure of FTS. V. Fuzzy metric spaces. *J. Fuzzy Math.*, 3(3):711–721, 1995.
[22] D. H. Hyers. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. U. S. A.*, 27:222–224, 1941.
[23] D. H. Hyers, G. Isac, and T. M. Rassias. *Stability of functional equations in several variables.* Progress in Nonlinear Differential Equations and their Applications, 34. Birkhäuser Boston Inc., Boston, MA, 1998.
[24] K.-W. Jun and H.-M. Kim. The generalized Hyers-Ulam-Rassias stability of a cubic functional equation. *J. Math. Anal. Appl.*, 274(2):267–278, 2002.
[25] K.-W. Jun, H.-M. Kim, and I.-S. Chang. On the Hyers-Ulam stability of an Euler-Lagrange type cubic functional equation. *J. Comput. Anal. Appl.*, 7(1):21–33, 2005.
[26] A. K. Katsaras. Fuzzy topological vector spaces. II. *Fuzzy Sets and Systems*, 12(2):143–154, 1984.
[27] I. Kramosil and J. Michálek. Fuzzy metrics and statistical metric spaces. *Kybernetika (Prague)*, 11(5):336–344, 1975.
[28] S. V. Krishna and K. K. M. Sarma. Separation of fuzzy normed linear spaces. *Fuzzy Sets and Systems*, 63(2):207–217, 1994.
[29] R. Lowen. *Fuzzy set theory.* Kluwer Academic Publishers, Dordrecht, 1996. Basic concepts, techniques and bibliography.
[30] D. Miheţ and V. Radu. On the stability of the additive Cauchy functional equation in random normed spaces. *J. Math. Anal. Appl.*, 343(1):567–572, 2008.
[31] D. Miheţ, R. Saadati, and S. M. Vaezpour. The stability of the quartic functional equation in random normed spaces. *Acta Appl. Math.*, 110(2):797–803, 2010.
[32] A. K. Mirmostafaee, M. Mirzavaziri, and M. S. Moslehian. Fuzzy stability of the Jensen functional equation. *Fuzzy Sets and Systems*, 159(6):730–738, 2008.
[33] A. K. Mirmostafaee and M. S. Moslehian. Fuzzy versions of Hyers-Ulam-Rassias theorem. *Fuzzy Sets and Systems*, 159(6):720–729, 2008.

[34] T. M. Rassias. On the stability of the linear mapping in Banach spaces. *Proc. Amer. Math. Soc.*, 72(2):297–300, 1978.
[35] T. M. Rassias. On the stability of functional equations and a problem of Ulam. *Acta Appl. Math.*, 62(1):23–130, 2000.
[36] T. M. Rassias, editor. *Functional equations, inequalities and applications.* Kluwer Academic Publishers, Dordrecht, 2003.
[37] R. Saadati. A note on "Some results on the IF-normed spaces". *Chaos Solitons Fractals*, 41(1):206–213, 2009.
[38] R. Saadati and S. M. Vaezpour. Some results on fuzzy Banach spaces. *J. Appl. Math. Comput.*, 17(1-2):475–484, 2005.
[39] B. Schweizer and A. Sklar. Statistical metric spaces. *Pac. J. Math.*, 10:313–334, 1960.
[40] B.-S. Shieh. Infinite fuzzy relation equations with continuous t-norms. *Inform. Sci.*, 178(8):1961–1967, 2008.
[41] S. M. Ulam. *Problems in modern mathematics.* Science Editions John Wiley & Sons, Inc., New York, 1964.
[42] C. X. Wu. Correlation properties of pseudorandom sequences. *J. Harbin Inst. Tech.*, (3):19–26, 1984.
[43] J. Z. Xiao and X. H. Zhu. Fuzzy normed space of operators and its completeness. *Fuzzy Sets and Systems*, 133(3):389–399, 2003.
[44] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.

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