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ALMOST EVERYWHERE CONVERGENCE OF CONE-LIKE RESTRICTED DOUBLE FEJÉR MEANS ON COMPACT TOTALLY DISCONNECTED GROUPS

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Dedicated to the memory of Professor Árpád Varecza on the occasion of his 70th birthday.

Abstract. In the present paper we prove the a.e. convergence of Fejér means of integrable functions with respect to the two-dimensional representative product systems on a bounded compact totally disconnected group provided that the set of indices is in a cone-like set.

1. Introduction

Now, we give a brief introduction, for more details see [3]. Moreover, see the book of Hewitt and Ross [9] and Schipp, Wade, Simon and Pál [11].

Denote by $\mathbb{N}, \mathbb{P}$ the set of natural numbers and the set of positive integers, respectively. Let $m := (m_k, k \in \mathbb{N})$ be a sequence of positive integers such that $m_k \geq 2$ and G_k a finite group with order m_k, $(k \in \mathbb{N})$. Suppose that each group has discrete topology and normalized Haar measure μ_k. Let G_m be the compact group formed by the complete direct product of the groups G_k with the product of the topologies, operations and measures (μ). Thus, each $x \in G_m$ is a sequence $x := (x_0, x_1, \ldots)$, where $x_k \in G_k$, $(k \in \mathbb{N})$. We call this sequence the expansion of x. The compact totally disconnected (simply CTD) group G_m is called a bounded group if the sequence m is bounded. All over this paper the boundedness of the group G_m is supposed. Set $m_* := \max\{m_k : k \in \mathbb{N}\}$. A neighborhood base of the topology can be given in the following way:

$$I_0(x) := G_m \quad I_n(x) := \{y \in G_m : y_k = x_k \text{ for } 0 \leq k < n\},$$

where $x \in G_m, n \in \mathbb{P}$. Let $M_0 := 1$ and $M_{k+1} := m_k M_k$, $k \in \mathbb{N}$, every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbb{N}$. This allows us to say that the sequence $(n_0, n_1, \ldots)$ is the expansion of n with respect to the number system m. We often use the following notations:

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$|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$, $n_{(k)} := \sum_{j=0}^{k-1} n_k M_k$, and $n^{(k)} := \sum_k^\infty n_k M_k$, where $|n|$ is called the order of n.

Denote by Σ_k the dual object of the finite group G_k ($k \in \mathbb{N}$). Thus, each $\sigma \in \Sigma_k$ is a set of continuous irreducible unitary representations of G_k which are equivalent to some fixed representation $U^{(\sigma)}$. Let d_σ be the dimension of its representation space and let $\{\xi_1, \xi_2, \ldots, \xi_{d_\sigma}\}$ be a fixed, but arbitrary orthonormal basis in the representation space. The functions $u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j\rangle$ ($i, j \in \{1, \ldots, d_\sigma\}, x \in G_k$) are called the coordinate functions for $U^{(\sigma)}$ and the basis $\{\xi_1, \xi_2, \ldots, \xi_{d_\sigma}\}$. Let $\{\varphi_k^s : 0 \leq s < m_k\}$ be a system of all normalized coordinate functions of the group G_k. We suppose that $\varphi_k^0 \equiv 1$. Thus, for every $0 \leq s < m_k$ there exists a $\sigma \in \Sigma_k$, $i, j \in \{1, \ldots, d_\sigma\}$ such that $\varphi_s(x) = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x)$ for $x \in G_k$. Let ψ be the product system of the functions φ_k^s, that is

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \quad (x \in G_m),$$

where n is of the form $n = \sum_{k=0}^\infty n_k M_k$ and $x = (x_0, x_1, \ldots)$. The system ψ is orthonormal and complete on $L^2(G_m)$ [7]. If the group G_k is the discrete cyclic group of order m_k for each $k \in \mathbb{N}$, then the system ψ is the well-known Vilenkin system and G_m is a Vilenkin group[1]. As special cases we have the Walsh system and the Walsh group, too [11]. The system ψ is called representative product system of the CTD group.

Let us consider the Dirichlet and Fejér kernel functions

$$D_n(y, x) := \sum_{k=0}^{n-1} \psi_k(y)\overline{\psi_k}(x), \quad K_n(y, x) := \frac{1}{n} \sum_{k=1}^{n} D_k(y, x)$$

and $D_0 = K_0 := 0$. The M_nth Dirichlet kernel has got a closed form [7]

$$D_{M_n}(y, x) = \begin{cases} M_n, & \text{for } y \in I_n(x) \\ 0, & \text{otherwise.} \end{cases}$$

Recently, the behavior of the Dirichlet kernel was discussed by Toledo [14, 12]. The Fourier coefficients, the partial sums of Fourier series and the Fejér means are defined in the usual way for $f \in L^1(G_m)$. It is known that

$$\sigma_n f(y) = \int_{G_m} f(x) K_n(y, x) d\mu(x).$$

Denote $G_m \times G_{\tilde{m}}$ the two-dimensional compact totally disconnected (CTD) group. Define the two-dimensional Dirichlet and Fejér kernel functions as the Kronecker product of the one-dimensional functions

$$D_n(y, x) := D_{n_1}(y^1, x^1) D_{n_2}(y^2, x^2), \quad K_n(y, x) := K_{n_1}(y^1, x^1) K_{n_2}(y^2, x^2),$$

where $y := (y^1, y^2), x := (x^1, x^2) \in G_m \times G_{\tilde{m}}$ and $n = (n_1, n_2) \in \mathbb{N}^2$. The following is well-known

$$\sigma_n f(y) = \int_{G_m \times G_{\tilde{m}}} f(u^1, u^2) K_{n_1}(y^1, u^1) K_{n_2}(y^2, u^2) d\mu(u^1, u^2)$$

In the present paper we also suppose that $m = \tilde{m}$ and we write simply $G_m^2 = G_m \times G_{\tilde{m}}$, although we know that $G_m \neq G_{\tilde{m}}$ may be happened. During the proofs C and c denote constants which may depend only on m_* and could vary at different occurrences.

In [7] Gát and Toledo proved the fact that the Fejér means of the Fourier series with respect to representative product systems on bounded groups converge to the function in L^p -norm $(1 \leq p < \infty)$, although we can not state the same for the Fourier series in general. In 2009 they extended this statement to Cesàro means of order α where $0 < \alpha < 1$ [8]. On the other hand the behavior of the partial sums worse than in the commutative (Vilenkin, Walsh) case. Let G_m be the complete product of $\mathcal{S}_3$. If $1 < p < \infty$ and $p \neq 2$, then there exists an $f \in L^p(G_m)$ such that $S_n f$ does not converge to the function f in L^p-norm [13].

The almost everywhere convergence of the one-dimensional Fejér means was proved by Gát in [3]. Our paper deal with the a.e. convergence of the two-dimensional Fejér means provided that the set of indices is in a cone-like set. We note that until the present time there was not any result given in dimension 2 for the Fourier series on CTD groups.

For double Walsh-Fourier series, Móricz, Schipp and Wade [10] proved that $\sigma_n f$ converge to f a.e. in the Pringsheim sense (that is, no restriction on the indices other than $\min(n_1, n_2) \to \infty$) for all functions $f \in L \log^+ L$. For double Walsh system Gát [4] and Weisz [15] proved that the Fejér means of an integrable function converge almost everywhere to the function itself provided that the indices satisfy the inequality $\beta^{-1} \leq n_1/n_2 \leq \beta$ with some fixed $\beta > 1$. Recently, a common generalization of the results of Gát, Weisz and the result of Móricz, Schipp, Wade (with respect to Walsh system) was given in the same direction and way as Gát did in [5] with respect to the trigonometric system. A necessary and sufficient condition for cone-like sets was given by the author and Gát in order to preserve the convergence property for Walsh system [6].

In the present paper we prove the a.e. convergence of Fejér means of integrable functions with respect to two-dimensional representative product systems of a bounded CTD group provided that the set of indices is in a cone-like set.

2. Almost everywhere convergence of double Fejér means with cone resriction

The following Lemmas are the basis of our proof:

Lemma 2.1 (Gát [3]). *Let $A, t, s, n \in \mathbb{N}$, and $x \in I_{t+1}(u)$, $u \in G_m$. Then*

$$\int_{I_t(u)\setminus I_{t+1}(u)} \sup_{M_A \le n^{(s)} < M_{A+1}} |K_{n^{(s)},M_s}(y,x)| d\mu(y) \le c M_{\min(s,t)} \sqrt{\frac{M_A}{M_{\max(s,t)}}}.$$

Lemma 2.2. *Let $A, t \in \mathbb{N}$, and $x \in I_{t+1}(u)$, $u \in G_m$. Then*

$$\int_{I_t(u)\setminus I_{t+1}(u)} \sup_{n \ge M_A} |K_n(y,x)| d\mu(y) \le c \sqrt{\frac{M_t}{M_A}}.$$

Proof. Set $x \in I_{t+1}(u)$, $u \in G_m$. By Lemma 2.1 and the method of Gát we write that

$$\begin{aligned}
\int_{I_t(u)\setminus I_{t+1}(u)} \sup_{n \ge M_A} |K_n(y,x)| d\mu(y) &\le \sum_{B=A}^{\infty} \int_{I_t(u)\setminus I_{t+1}(u)} \sup_{|n|=B} |K_n(y,x)| d\mu(y) \\
&\le \sum_{B=A}^{\infty} \frac{1}{M_B} \int_{I_t(u)\setminus I_{t+1}(u)} \sup_{|n|=B} \sum_{s=0}^{B} \sum_{j=0}^{m_s-2} |K_{n^{(s+1)}+jM_s,M_s}(y,x)| d\mu(y) \\
&\le \sum_{B=A}^{\infty} \frac{1}{M_B} \sum_{s=0}^{B} \int_{I_t(u)\setminus I_{t+1}(u)} \sup_{|n|=B} |K_{n^{(s)},M_s}(y,x)| d\mu(y) \\
&\le c \sum_{B=A}^{\infty} \frac{1}{M_B} \left(\sum_{s=0}^{t} M_s \sqrt{\frac{M_B}{M_t}} + \sum_{s=t+1}^{B} M_t \sqrt{\frac{M_B}{M_s}} \right) \\
&\le c \sum_{B=A}^{\infty} \frac{1}{M_B} M_t \sqrt{\frac{M_B}{M_t}} \le c \sqrt{\frac{M_t}{M_A}}.
\end{aligned}$$

□

We define the maximal operator $\sigma^{\#}$ by

$$\sigma^{\#} f := \sup_{\substack{n \in \mathbb{P}^2 \\ \beta^{-1} \le n_1/n_2 \le \beta}} |\sigma_n f|,$$

where $\beta > 1$ is a fixed parameter. For this maximal operator we have the following theorem:

Theorem 2.3. *The operator $\sigma^{\#}$ is of weak type $(1,1)$.*

By standard argument we get that

Theorem 2.4. *Let $f \in L^1(G_m^2)$ and $\beta \ge 1$ be a fixed parameter. Then the relation*

$$\lim_{\substack{\wedge n \to \infty \\ \beta^{-1} \le n_1/n_2 \le \beta}} \sigma_n f = f \qquad a.e.$$

holds.

By the help of Lemma 2.2, the well-known Calderon-Zygmund decomposition Lemma and the method of Gát and Blahota in [2] (with necessary changes) we have the proof of Theorem 2.3. On the other hand, Theorem 2.3 could be reach as a corollary of our main theorem in the following section.

At last, we note that Theorem 2.4 is unknown until the present time.

3. Pointwise convergence of cone-like restricted two-dimensional Fejér means

Now, we define the cone-like sets. Let $\alpha : [1, +\infty) \to [1, +\infty)$ be a strictly monotone increasing continuous function with property $\lim_{+\infty} \alpha = +\infty$, $\alpha(1) = 1$, and $\beta : [1, +\infty) \to [1, +\infty)$ be a monotone increasing function with property $\beta(1) > 1$.

Define the cone-like restriction sets of $\mathbb{N}^2$ as follows [5, 6]:

$$\mathbb{N}_{\alpha,\beta,1} := \left\{ n \in \mathbb{N}^2 : \frac{\alpha(n_1)}{\beta(n_1)} \leq n_2 \leq \alpha(n_1)\beta(n_1) \right\},$$
$$\mathbb{N}_{\alpha,\beta,2} := \left\{ n \in \mathbb{N}^2 : \frac{\alpha^{-1}(n_2)}{\beta(n_2)} \leq n_1 \leq \alpha^{-1}(n_2)\beta(n_2) \right\}.$$

For $\alpha(x) := x$, $\beta(x) := \beta$ ($\beta \in (1, \infty)$) we get the restriction set $\mathbb{N}_{\alpha,\beta,1} = \mathbb{N}_{\alpha,\beta,2} = \left\{ n \in \mathbb{N}^2 : \frac{1}{\beta} \leq \frac{n_2}{n_1} \leq \beta \right\}$ used in [2, 4, 15].

Let $\beta(x) = \beta$ be a constant function. It is natural that $\mathbb{N}_{\alpha,\beta_1,1} \subset \mathbb{N}_{\alpha,\beta_2,1}$ and $\mathbb{N}_{\alpha,\beta_1,2} \subset \mathbb{N}_{\alpha,\beta_2,2}$ for any $\beta_1 \leq \beta_2$.

For $i = 1, 2$ set

$$\mathbb{N}_{\alpha,i} := \{\mathbb{N}_{\alpha,\beta,i} : \beta > 1\}.$$

For a fixed $i \in \{1, 2\}$, we say that $\mathbb{N}_{\alpha,i}$ is weaker than $\mathbb{N}_{\alpha,3-i}$, if for all $L \in \mathbb{N}_{\alpha,i}$, there exists an $\tilde{L} \in \mathbb{N}_{\alpha,3-i}$ such that $L \subset \tilde{L}$. This will be denoted by $\mathbb{N}_{\alpha,i} \prec \mathbb{N}_{\alpha,3-i}$. If $\mathbb{N}_{\alpha,1} \prec \mathbb{N}_{\alpha,2}$ and $\mathbb{N}_{\alpha,2} \prec \mathbb{N}_{\alpha,1}$, then we say that $\mathbb{N}_{\alpha,1}$ and $\mathbb{N}_{\alpha,2}$ are equivalent and denote this by $\mathbb{N}_{\alpha,1} \sim \mathbb{N}_{\alpha,2}$.

We say that the function α is a cone-like restriction function (CRF), if $\mathbb{N}_{\alpha,1} \sim \mathbb{N}_{\alpha,2}$. Set $\mathbb{N}_\alpha := \mathbb{N}_{\alpha,1} \cup \mathbb{N}_{\alpha,2}$. We say that the cone-like set $L \in \mathbb{N}_\alpha$ is based on the function α.

The properties of a CRF is characterized in the following statement [5]: Function α is a CRF if and only if there exist $\zeta, \gamma_1, \gamma_2 > 1$ such that the inequality

$$\gamma_1\alpha(x) \leq \alpha(\zeta x) \leq \gamma_2\alpha(x) \tag{3.1}$$

holds for each $x \geq 1$.

In other words, the condition $\gamma_1\alpha(x) \leq \alpha(\zeta x) \leq \gamma_2\alpha(x)$ is very natural, since it is necessary and sufficient in order to have that for all restriction set $L \in \mathbb{N}_{\alpha,1}$ there exists a restriction set $\tilde{L} \in \mathbb{N}_{\alpha,2}$ such that $L \subset \tilde{L}$, and in the same way backwards also.

We define the maximal operator σ_L^* by

$$\sigma_L^* := \sup_{n \in L} |\sigma_n f|.$$

For the maximal operator σ_L^* we prove the following theorem:

Theorem 3.1. *Let α be CRF, $L \in \mathbb{N}_\alpha$. Then the operator σ_L^* is of weak type $(1,1)$.*

By standard argument we get that

Theorem 3.2. *Let α be CRF, $L \in \mathbb{N}_\alpha$. Then for any $f \in L^1(G_m^2)$ the relation*

$$\lim_{\substack{\wedge n\to\infty \\ n\in L}} \sigma_n f = f \qquad a.e.$$

holds.

We immediately have Theorem 2.4 in the previous section as a corollary. To prove Theorem 3.1 we need the following decomposition Lemma of Calderon and Zygmund type proved in [5].

Lemma 3.3. *Let the function $\varphi_j : [1,+\infty) \to [1,+\infty)$ be monotone increasing and continuous with property $\lim_{+\infty} \varphi_j = +\infty$ $(j = 1,2)$. Set $\phi_j := \lfloor \varphi_j \rfloor$ $(j = 1,2)$ (where $\lfloor x \rfloor$ denotes the lower integer part of x).*

Let $f \in L^1$ and $\lambda > \|f\|_1$. Then there exists a sequence of integrable functions (f_i) such that

$$f = \sum_{i=0}^{\infty} f_i,$$

where $\|f_0\|_\infty \le C\lambda$, $\|f_0\|_1 \le C\|f\|_1$ and $\operatorname{supp} f_i \subset I_{k^{i,1}}(x_1^i) \times I_{k^{i,2}}(x_2^i) =: J_1^i \times J_2^i$ $((x_1^i, x_2^i) \in G_m^2)$ with measures

$$\mu(I_{k^{i,1}}(x_1^i)) = 1/M_{\phi_1(s_i)} \text{ and } \mu(I_{k^{i,2}}(x_2^i)) = 1/M_{\phi_2(s_i)}$$

for some $s_i \ge 1$. Moreover, $\int_{G_m^2} f_i = 0$ $(i \ge 1)$, the sets $J_1^i \times J_2^i$ are disjoint, and with the definition $F := \bigcup_{i=1}^{\infty} (I_{k^{i,1}}(x_1^i) \times I_{k^{i,2}}(x_2^i))$ we have $\mu(F) \le C\|f\|_1/\lambda$.

Proof of Theorem 3: During the proof of Theorem 3.1 we follow the method of the author and Gát in [6]. But, we have to make necessary changes, because we have got another structure than in the Walsh case.

Let $L \in \mathbb{N}_\alpha$. Without loss of generality, we suppose that $L = \mathbb{N}_{\alpha,\beta,1}$ for some $\beta > 1$. First, we choose functions $\phi_1(s) := |s|$ (that is, $\phi_1(s)$ is the order of s and $M_{|s|} \le s < M_{|s|+1}$) and $\phi_2(s) := |\alpha(s)|$, where α is CRF (we note that the continuous functions φ_1, φ_2 can be constructed). We apply Lemma 3.3 for the functions $\phi_1(s), \phi_2(s)$.

Set $f \in L^1(G_m^2)$ and $\operatorname{supp} f \subset J_1 \times J_2$ with measure $\mu(J_i) = \frac{1}{M_{\phi_i(s)}}$ for some $s \ge 1$ $(i = 1,2)$. Set $k^j := \phi_j(s)$, that is $J_i = I_{k^i}(x^i)$ for $j = 1,2$.

By the help of Lemma 2.2 we prove the inequality

$$\int_{I_{k^1}(x^1)\times I_{k^2}(x^2)} \sup_{n\in L} |\sigma_n f| \le c\|f\|_1. \tag{3.2}$$

We decompose the set $\overline{I_{k^1}(x^1)\times I_{k^2}(x^2)}$ as the following union:

$$\left(\overline{I_{k^1}(x^1)}\times\overline{I_{k^2}(x^2)}\right)\cup\left(I_{k^1}(x^1)\times\overline{I_{k^2}(x^2)}\right)\cup\left(\overline{I_{k^1}(x^1)}\times I_{k^2}(x^2)\right).$$

We set $\delta:=\zeta^{\log_{\gamma_1}(2\beta m_*)+1}$. $n_1\le M_{\phi_1(s)}/\delta$ yields that

$$\begin{aligned}n_2&\le\beta\alpha(n_1)\le\beta\alpha(M_{\phi_1(s)}\zeta^{-\log_{\gamma_1}(2\beta m_*)-1})\\&\le\beta\frac{1}{\gamma_1^{\log_{\gamma_1}2\beta m_*+1}}\alpha(M_{\phi_1(s)})\le\frac{\alpha(s)}{2m_*}<M_{\phi_2(s)}.\end{aligned}$$

Moreover, $\zeta,\gamma_1,\gamma_2>1$ gives $n_1<M_{k^1}$ and $n_2<M_{k^2}$. In this case the (k,l)-th Fourier coefficients are zeros for $k\le n_1$ and $l\le n_2$. More exactly,

$$\hat f(k,l)=\int_{G_m^2}f(\overline{\psi}_k\times\overline{\psi}_l)=\int_{I_{k^1}(x^1)\times I_{k^2}(x^2)}f(\overline{\psi}_k\times\overline{\psi}_l)=(\overline{\psi}_k\times\overline{\psi}_l)\int_{I_{k^1}(x^1)\times I_{k^2}(x^2)}f=0.$$

This yields that $\sigma_nf=0$. That is, we have to suppose that $n_1>M_{\phi_1(s)}/\delta\ge M_{k^1-c_*}$. From this we write that

$$\begin{aligned}n_2&\ge\frac{\alpha(n_1)}{\beta}\ge\frac{\alpha(M_{\phi_1(s)}m_*/\delta m_*)}{\beta}\ge\frac{1}{\beta\gamma_2^{\log_\zeta m_*+\log_{\gamma_1}2\beta m_*+1}}\alpha(M_{\phi_1(s)}m_*)\\&\ge\frac{\alpha(s)}{\delta'}\ge\frac{M_{\phi_2}(s)}{\delta'}\ge M_{k^2-c^*}.\end{aligned}$$

Now, we discuss the integral $\int_{\overline{I_{k^1}(x^1)}\times\overline{I_{k^2}(x^2)}}\sup_{n\in L}|\sigma_nf|$.

Now, we decompose the sets $\overline{I_{k^i}(x^i)}$ $(i=1,2)$ in the usual way. Using the notation

$$J^{a,b}:=(I_a(x^1)\backslash I_{a+1}(x^1))\times(I_b(x^2)\backslash I_{b+1}(x^2))$$

for $a=0,1,\dots,k^1-1,\ b=0,1,\dots,k^2-1$ we have that

$$\overline{I_{k^1}(x^1)}\times\overline{I_{k^2}(x^2)}=\bigcup_{a=0}^{k^1-1}\bigcup_{b=0}^{k^2-1}J^{a,b}.$$

By Lemma 2.2 (with $u^1\in I_{k^1}(x^1)=J^1$) and theorem of Fubini we get that

$$\begin{aligned}&\int_{J^{a,b}}\sup_{n\in L}\left|\int_{J^1\times J^2}f(u^1,u^2)K_{n_1}(y^1,u^1)K_{n_2}(y^2,u^2)d\mu(u^1,u^2)\right|d\mu(y^1,y^2)\\&\le\int_{J^1\times J^2}|f(u^1,u^2)|\int_{J^{a,b}}\sup_{n\in L}|K_{n_1}(y^1,u^1)K_{n_2}(y^2,u^2)|d\mu(y^1,y^2)d\mu(u^1,u^2)\\&\le\int_{J^1\times J^2}|f(u^1,u^2)|\int_{J^{a,b}}\sup_{n_1\ge M_{k^1-c_*}}|K_{n_1}(y^1,u^1)|\times\\&\quad\times\sup_{n_2\ge M_{k^2-c^*}}|K_{n_2}(y^2,u^2)|d\mu(y^1,y^2)d\mu(u^1,u^2)\\&\le c\|f\|_1\sqrt{\frac{M_aM_b}{M_{k^1}M_{k^2}}}.\end{aligned}$$

Moreover, we write that

$$\int_{\overline{I_{k^1}(x^1)}\times\overline{I_{k^2}(x^2)}}\sup_{n\in L}|\sigma_n f| \le \sum_{a=0}^{k^1-1}\sum_{b=0}^{k^2-1}\int_{J^{a,b}}\sup_{n\in L}|\sigma_n f|$$
$$\le c\|f\|_1 \sum_{a=0}^{k^1-1}\sum_{b=0}^{k^2-1}\sqrt{\frac{M_a M_b}{M_{k^1}M_{k^2}}} \le c\|f\|_1.$$

We discuss the integral $\int_{I_{k^1}(x^1)\times\overline{I_{k^2}(x^2)}}\sup_{n\in L}|\sigma_n f|$.

For $r \ge k^1$ and a fixed $x^1 \in G_m$ we set an $\epsilon := (x^1_0, \dots, x^1_{k^1-1}, \epsilon_{k^1}, \dots, \epsilon_r, x^1_{r+1}, \dots)$, where $\epsilon_i \in G_i$ $(i = k^1, \dots, r)$.

Then

$$I_{k^1}(x^1) = \bigcup_{\substack{\epsilon_i\in G_i\\ i=k^1,\dots,r}} I_{r+1}(\epsilon).$$

For each $a = k^1, \dots, r$, $b = 0, 1, \dots, k^2 - 1$ and an arbitrary ϵ we define the sets $J^{a,b}_\epsilon$ and J^b_ϵ by

$$J^{a,b}_\epsilon := (I_a(\epsilon)\backslash I_{a+1}(\epsilon)) \times \left(I_b(x^2)\backslash I_{b+1}(x^2)\right)$$

and

$$J^b_\epsilon := I_{r+1}(\epsilon) \times \left(I_b(x^2)\backslash I_{b+1}(x^2)\right).$$

Then we have the following disjoint decomposition of the set $I_{k^1}(x^1)\times\overline{I_{k^2}(x^2)}$:

$$I_{k^1}(x^1)\times\overline{I_{k^2}(x^2)} = \left(\bigcup_{a=k^1}^{r}\bigcup_{b=0}^{k^2-1} J^{a,b}_\epsilon\right)\bigcup\left(\bigcup_{b=0}^{k^2-1} J^b_\epsilon\right).$$

We introduce the following abbreviation:

$$S^L_r := \sup_{\substack{M_{r-c}\le n_1\le M_{r+c}\\ n\in L}}$$

It is easy to see that, $c\alpha(M_r) \le n_2 \le C\alpha(M_r)$ for $n \in L$ and $M_r \le n_1 \le cM_r$. Theorem of Fubini and the decomposition given above yields that

$$
\begin{aligned}
&\int_{I_{k^1}(x^1)\times\overline{I_{k^2}(x^2)}} \sup_{n\in L} |\sigma_n f(y^1,y^2)| d\mu(y^1,y^2) \le \\
&\le \sum_{r=k^1}^{\infty} \int_{I_{k^1}(x^1)\times\overline{I_{k^2}(x^2)}} S_r^L \Big| \int_{I_{k^1}(x^1)\times I_{k^2}(x^2)} f(u^1,u^2) K_{n_1}(y^1,u^1) K_{n_2}(y^2,u^2) d\mu(u^1,u^2)\Big| d\mu(y^1,y^2) \\
&\le \sum_{r=k^1}^{\infty} \sum_{\epsilon} \int_{I_{k^1}(x^1)\times\overline{I_{k^2}(x^2)}} S_r^L \Big| \int_{I_{r+1}(\epsilon)\times I_{k^2}(x^2)} f(u^1,u^2) K_{n_1}(y^1,u^1) K_{n_2}(y^2,u^2) d\mu(u^1,u^2)\Big| d\mu(y^1,y^2) \\
&\le \sum_{r=k^1}^{\infty} \sum_{\epsilon} \sum_{a=k^1}^{r} \sum_{b=0}^{k^2-1} \int_{J_\epsilon^{a,b}} S_r^L \Big| \int_{I_{r+1}(\epsilon)\times I_{k^2}(x^2)} f(u^1,u^2) K_{n_1}(y^1,u^1) K_{n_2}(y^2,u^2) d\mu(u^1,u^2)\Big| d\mu(y^1,y^2) \\
&\quad + \sum_{r=k^1}^{\infty} \sum_{\epsilon} \sum_{b=0}^{k^2-1} \int_{J_\epsilon^{b}} S_r^L \Big| \int_{I_{r+1}(\epsilon)\times I_{k^2}(x^2)} f(u^1,u^2) K_{n_1}(y^1,u^1) K_{n_2}(y^2,u^2) d\mu(u^1,u^2)\Big| d\mu(y^1,y^2) \\
&\le \sum_{r=k^1}^{\infty} \sum_{\epsilon} \sum_{a=k^1}^{r} \sum_{b=0}^{k^2-1} \int_{I_{r+1}(\epsilon)\times I_{k^2}(x^2)} |f(u^1,u^2)| \int_{J_\epsilon^{a,b}} \sup_{M_{r-c}\le n_1} |K_{n_1}(y^1,u^1)| \times \\
&\quad \times \sup_{c\alpha(M_r)\le n_2} |K_{n_2}(y^2,u^2)| d\mu(y^1,y^2) d\mu(u^1,u^2) \\
&\quad + \sum_{r=k^1}^{\infty} \sum_{\epsilon} \sum_{b=0}^{k^2-1} \int_{I_{r+1}(\epsilon)\times I_{k^2}(x^2)} |f(u^1,u^2)| \int_{J_\epsilon^{b}} \sup_{n_1\le M_{r+c}} |K_{n_1}(y^1,u^1)| \times \\
&\quad \times \sup_{c\alpha(M_r)\le n_2} |K_{n_2}(y^2,u^2)| d\mu(y^1,y^2) d\mu(u^1,u^2) \\
&=: I + II.
\end{aligned}
$$

We discuss I. We use Lemma 2.2 and $u^1 \in I_{r+1}(\epsilon)$, $u^2 \in I_{k^2}(x^2)$. Thus,

$$
I \le c\|f\|_1 \sum_{r=k^1}^{\infty} \sum_{a=k^1}^{r} \sum_{b=0}^{k^2-1} \sqrt{\frac{M_a M_b}{M_r \alpha(M_r)}} \le c\|f\|_1 \sum_{r=k^1}^{\infty} \sqrt{\frac{M_{k^2}}{\alpha(M_r)}}.
$$

From Lemma 2.2 (with $u^2 \in I_{k^2}(x^2)$) and the fact that $|K_n| \le cn$ (see [3]) we get that

$$
\begin{aligned}
II &\le c \sum_{r=k^1}^{\infty} \sum_{\epsilon} \sum_{b=0}^{k^2-1} \int_{I_{r+1}(\epsilon)\times I_{k^2}(x^2)} |f(u^1,u^2)| M_r \frac{1}{M_r} \sqrt{\frac{M_b}{\alpha(M_r)}} d\mu(u^1,u^2) \\
&\le c\|f\|_1 \sum_{r=k^1}^{\infty} \sqrt{\frac{M_{k^2}}{\alpha(M_r)}}.
\end{aligned}
$$

Now, we show that $\sum_{r=k^1}^{\infty}\sqrt{\frac{M_{k^2}}{\alpha(M_r)}} \leq c$. Since, α is strictly monotone increasing we have that $\alpha(M_r) \geq \alpha(M_{k^1}2^{r-k^1})$. We write for an arbitrary A (we will give more details about A later)

$$\sum_{r=k^1}^{\infty}\sqrt{\frac{1}{\alpha(M_r)}} \leq \sum_{j=0}^{A-1}\sum_{i=0}^{\infty}\sqrt{\frac{1}{\alpha(M_{k^1}2^{Ai+j})}}.$$

Now, we choose A so big such that the inequality

$$\sqrt{\alpha(M_{k^1}2^{Ai+j+A})} \geq \sqrt{\gamma_1^{A\log_\zeta 2}\alpha(M_{k^1}2^{Ai+j})} \geq 2\sqrt{\alpha(M_{k^1}2^{Ai+j})}$$

holds. (We could choose such an A because $\gamma_1, \zeta > 1$.) From this

$$\sum_{r=k^1}^{\infty}\sqrt{\frac{1}{\alpha(M_r)}} \leq c\sum_{j=0}^{A-1}\sqrt{\frac{1}{\alpha(M_{k^1}2^j)}} \leq c\sqrt{\frac{1}{\alpha(M_{k^1})}}.$$

$M_{k^2} \leq \alpha(s)$ and $\alpha(M_{k^1}) = \alpha(M_{|s|}) \geq \alpha(s/m_*) \geq c\alpha(s)$ yields that

$$\sum_{r=k^1}^{\infty}\sqrt{\frac{M_{k^2}}{\alpha(M_r)}} \leq c\sqrt{\frac{M_{k^2}}{\alpha(M_{k^1})}} \leq c.$$

The discussion of the integral $\int_{\overline{I_{k^1}(x^1)\times I_{k^2}(x^2)}} \sup_{n\in L}|\sigma_n f|$ follows.

Using the substitutions $t = \alpha(s)$ and $s = \alpha^{-1}(t)$, we write that

$$\overline{I_{|s|}(x^1)\times I_{|\alpha(s)|}(x^2)} = \overline{I_{|\alpha^{-1}(t)|}(x^1)\times I_{|t|}(x^2)}.$$

That is

$$\overline{I_{\phi_1(s)}(x^1)\times I_{\phi_2(s)}(x^2)} = \overline{I_{\tilde{\phi}_2(t)}(x^1)\times I_{\tilde{\phi}_1(t)}(x^2)}.$$

If α is CRF, then α^{-1} is CRF, too (for more details see [6]). By this

$$\int_{\overline{I_{\phi_1(s)}(x^1)\times I_{\phi_2(s)}(x^2)}} \sup_{n\in L}|\sigma_n f| \leq \int_{\overline{I_{\tilde{\phi}_2(t)}(x^1)\times I_{\tilde{\phi}_1(t)}(x^2)}} \sup_{n\in \tilde{L}}|\sigma_n f|.$$

The discussion above gives that

$$\int_{\overline{I_{k^1}(x^1)\times I_{k^2}(x^2)}} \sup_{n\in L}|\sigma_n f| \leq c\|f\|_1.$$

The fact that $\int_{G_m}|K_n(y,x)|d\mu(y) \leq c$ for all $n\in\mathbb{N}, x\in G_m$ (see [3]) implies that the operator σ_L^* is of type (∞,∞). This, inequality (3.2) and Lemma 3.3 give by standard argument our theorem. □

By the interpolation lemma of Marcinkiewicz [11] and the fact that the operator σ_L^* is sublinear we immediately have the following corollary.

Corollary 3.4. *Let α be CRF and $L \in \mathbb{N}_\alpha$. Then the operator σ_L^* is of type (p,p) for all $1 < p \leq \infty$.*

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