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WARPED PRODUCT SUBMANIFOLD IN GENERALIZED SASAKIAN SPACE FORM

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Abstract. In 2002, K. Matsumoto and I. Mihai established sharp inequalities for some warped product submanifolds in Sasakian space forms.

A. Olteanu, established one of these inequalities for Legendrian warped product submanifolds in generalized Sasakian space forms.

In the present paper, we generalize another inequalities for warped product submanifolds in generalized Sasakian space forms with contact structure.

1. Introduction

Let (M_1, g_1) and (M_2, g_2) be two Rimannian manifolds and f a positive differentiable function on M_1. the *warped product* of M_1 and M_2 is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$, f is called the *warped function* (see, for instance, [3] and [4]).

Let $x\colon M_1 \times_f M_2 \longrightarrow \overline{M}^m$ be an isometric immersion. We denote by h *the second fundamental form* of x. The immersion x is said to be *Mixed totally geodesic* if $h(X, Y) = 0$, for any vector fields X and Y tangent to M_1 and M_2, respectively.

In the following theorems, K. Matsumoto and I. Mihai established the sharp inequalities between the warped function of some warped product submanifolds in the Sasakian space form and the squared mean curvature, see [5].

Theorem 1.1. *Let x be a C-totally real isometric immersion of n-dimensional warped product $M_1 \times_f M_2$ into a $(2m+1)$-dimensional Sasakian space form*

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$\overline{M}(c)$ then

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c+3}{4}, \tag{1}$$

where $n_i = dimM_i$ $(i = 1, 2)$, and Δ is the Laplacian operator of M_1. Moreover, the equality case of (1) holds if and only if x is a mixed totally geodesic immersion and $n_1H_1 = n_2H_2$, where H and $H_i(i = 1, 2)$ are the mean curvature vector and partial mean curvature vectors, respectively.

Theorem 1.2. *Let $\overline{M}(c)$ be a $(2m+1)$-dimensional Sasakian space form and $M_1 \times_f M_2$ an n-dimensional warped product submanifold, such that ξ is tangent to M_1. then*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c+3}{4} - \frac{c-1}{4}, \tag{2}$$

where $n_i = dimM_i$ $(i = 1, 2)$, and Δ is the Laplacian operator of M_1. Moreover, the equality case of (2) holds if and only if x is a mixed totally geodesic immersion and $n_1H_1 = n_2H_2$, where H and $H_i(i = 1, 2)$ are the mean curvature vector and partial mean curvature vectors, respectively.

In the following theorem Olteanu established a sharp relationship between the warped function of the Legendrian warped product submanifold in the generalized Sasakian space form and the squared mean curvature (see [7]).

Theorem 1.3. *Let x be a Legendrian isometric immersion of an n-dimentional warped product $M_1 \times_f M_2$ into a $(2n+1)$-dimentional generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Then*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}\|H\|^2 + n_1 f_1, \tag{3}$$

where $n_i = dimM_i$ $(i = 1, 2)$, and Δ is the Laplacian operator of M_1. Moreover, the equality case of (3) holds if and only if x is a mixed totally geodesic immersion and $n_1H_1 = n_2H_2$, where H and $H_i(i = 1, 2)$ are the mean curvature vector and partial mean curvature vectors, respectively.

In this paper we are going to generalize another inequalities, by establishing the sharp relationships between the warped function of the warped product submanifolds in the generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ with contact structure and the squared mean curvature such that structure vector field of $\overline{M}(f_1, f_2, f_3)$ is tangent to these submanifolds.

2. Preliminaries

In this section, we recall some definitions and basic formulas which we will use later.

A $(2n + 1)$-dimensional Riemannian manifold $(\overline{M}, g)$ is said to be *almost contact metric* if there exist on $\overline{M}$ a (1,1)-tensor field ϕ, a vector field ξ(is called the structure vector field) and a 1-form η such that $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields X, Y on $\overline{M}$. Also, it can be simply proved that in an almost contact metric manifold we have $\phi\xi = 0$ and $\eta \circ \phi = 0$ (see for instance [1]). We denote an almost contact metric manifold by $(\overline{M}, \phi, \xi, \eta, g)$.

If in an almost contact manifold $(\overline{M}, \phi, \xi, \eta, g)$,

$$2\Phi(X, Y) = d\eta(X, Y),$$

where $\Phi(X, Y) = g(Y, \phi X)$, then $(\overline{M}, \phi, \xi, \eta, g)$ is called the *contact metric manifold*. A contact metric manifold is called the *K-contact metric manifold* if the structure vector field be a *killing vector field*, it is easy to see that in K-contact metric manifold, we have

$$\nabla_X \xi = \phi X,$$

in which $X \in \tau(\overline{M})$.

If in an almost contact metric manifold $(\overline{M}, \phi, \xi, \eta, g)$,

$$\Big(\nabla_X \phi\Big)(Y) = \eta(Y)X - g(X, Y)\xi,$$

then we call $(\overline{M}, \phi, \xi, \eta, g)$ is the *Sasakian manifold*. It is easy to see that a Sasakian manifold is contact metric manifold.

Let $(\overline{M}, \phi, \xi, \eta, g)$ be an almost contact manifold. If $\pi_p \subset T_p\overline{M}$ is generated by $\{X, \phi X\}$ where $0 \neq X \in T_p\overline{M}$ is normal to ξ_p, is called the ϕ-section of $\overline{M}$ at p and $K(\pi_p)$ is ϕ-sectional curvature. If in a Sasakian manifold, there exist $c \in \Re$ such that for any $p \in \overline{M}$, $K(\pi_p) = c$ then we call $\overline{M}$ is the *Sasakian space form* and denote it by $\overline{M}(c)$. In [6] we see that in a Sasakian space form $\overline{M}(c)$, the curvature tensor is

$$\begin{aligned}\overline{R}(X, Y)Z = {} & \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\ & + \frac{c-1}{4}\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ & + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ & - g(Y, Z)\eta(X)\xi\}.\end{aligned}$$

Almost contact manifolds are said to be Generalized Sasakian space form if

$$\begin{aligned}\overline{R}(X, Y)Z = {} & f_1\{g(Y, Z)X - g(X, Z)Y\} \\ & + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ & - g(Y, Z)\eta(X)\xi\},\end{aligned} \tag{4}$$

where f_1, f_2, f_3 are differentiable functions on $\overline{M}$ and we denote this kind of manifold by $\overline{M}(f_1, f_2, f_3)$. It is clear that a Sasakian space form is a generalized Sasakian space form, but the converse is not necessarily true.

Let M^n be a submanifold of $\overline{M}^{2m+1}$ and h is the second fundamental form of M and $\overline{R}$ and R are the curvature tensors of $\overline{M}$ and M respectively. The Gauss equation is given by

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g\Big(h(X,W),h(Y,Z)\Big) - g\Big(h(X,Z),h(Y,W)\Big), \tag{5}$$

for any vector fields X, Y, Z, W on M.

Let

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i),$$

be the mean curvature vector field of M, in which $\{e_1,\dots,e_{2m+1}\}$ is a local orthonormal frame for $\overline{M}$ such the $e_1,\dots,e_n$ are tangent to M. Thus,

$$n^2\|H\|^2 = \sum_{i,j=1}^{n} g\Big(h(e_i,e_i),h(e_j,e_j)\Big). \tag{6}$$

As, it is known, M is said to be minimal if H vanishes identically. Also, we set

$$h_{ij}^r = g\Big(h(e_i,e_j),e_r\Big),\ i,j\in\{1,\dots,n\},\ r\in\{n+1,\dots,2m+1\}, \tag{7}$$

the coefficients of the second fundamental form h with respect to $\{e_1,\dots,e_n,\dots,e_{2m+1}\}$, and

$$\|h\|^2 = \sum_{i,j=1}^{n} g\Big(h(e_i,e_j),h(e_i,e_j)\Big). \tag{8}$$

Now, by (6) and (8), the Gauss equation can be rewritten as follows:

$$\sum_{1\le i,j\le n} \overline{R}_m(e_j,e_i,e_i,e_j) = R - n^2\|H\|^2 + \|h\|^2. \tag{9}$$

in which R is the scalar curvature of M.

Let M^n be a Riemannian manifold and $\{e_1,\dots,e_n\}$ be a local orthonormal frame of M. For a differentiable function f on M, the Laplacian Δf of f is defined by

$$\Delta f = \sum_{j=1}^{n}\Big((\nabla_{e_j}e_j)f - e_j(e_jf)\Big). \tag{10}$$

We recall the following result of B. Y. Chen for later use.

Lemma 2.1 ([2]). *Let $n\ge 2$ and $a_1,\dots,a_n$ and b are real numbers such that*

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^{n} a_i^2 + b\Big).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = \ldots = a_n.$$

3. Warped product submanifolds tangent to the structure vector field

In this section, we investigate warped product submanifold $M = M_1 \times_f M_2$ tangent to the structure vector field ξ in a generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ with contact structure.

We distinguish the following three cases:

(a) ξ tangent to M_1;
(b) ξ tangent to M_2;
(c) $\xi = \xi_1 + \xi_2$ such that ξ_1 and ξ_2 are nonzero at any point of M and tangent to M_1 and M_2 respectively.

Theorem 3.1. *Let $\overline{M}(f_1, f_2, f_3)$ be a $(2m+1)$-dimensional generalized Sasakian space form with contact structure and $M_1 \times_f M_2$ an n-dimensional warped product submanifold of $\overline{M}$.*

a. If ξ is tangent to M_1, then

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1 f_1 - f_3 \tag{11}$$

b. If ξ is tangent to M_2, then

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1 f_1 - \frac{n_1}{n_2} f_3, \tag{12}$$

c. If $\xi = \xi_1 + \xi_2$ such that ξ_1 and ξ_2 are nonzero at any point of $M_1 \times_f M_2$ and tangent to M_1 and M_2 respectively, then

$$\begin{aligned} n_2 \frac{\Delta f}{f} \leq & \Big(n_2 - g(\xi_1, \xi_1)\Big)\Big(n_1 - g(\xi_2, \xi_2)\Big) f_1 \\ & - \bigg(n_2 \Big(g(\xi_1, \xi_1)\Big)^2 + n_1 \Big(g(\xi_2, \xi_2)\Big)^2 + 3g(\xi_1, \xi_1) g(\xi_2, \xi_2) - 1\bigg) f_3 \\ & + 3(n-2)\Theta(f_2) + \frac{n^2}{4}\|H\|^2 \end{aligned} \tag{13}$$

in which for any $x \in M_1 \times M_2$

$$\Theta(f_2)(x) := \begin{cases} f_2(x), & f_2(x) > 0, \\ 0, & f_2(x) \leq 0, \end{cases}$$

and $n_i = \dim M_i (i = 1, 2)$ and Δ is the Laplacian operator of M_1.

d. The equality in (11) and (12) hold if and only if $M_1 \times_f M_2$ is a mixed totally geodesic submanifold of $\overline{M}$ and $n_1 H_1 = n_2 H_2$, where H and $H_i (i = 1, 2)$ are the mean curvature vector and partial mean curvature vectors, respectively.

Proof. a. Let $M_1 \times_f M_2$ be a warped product submanifold of a generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$.

Since $M_1 \times_f M_2$ is a warped product, it is easily seen that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z, \tag{14}$$

for any vector fields X and Z tangent to M_1 and M_2, respectively (see [8]). If X and Z are orthonormal vector fields, then the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\Big((\nabla_X X)f - X^2 f\Big). \tag{15}$$

We choose a local orthonormal frame $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m+1}\}$ such that $e_1, \dots, e_{n_1} = \xi$ are tangent to M_1, $e_{n_1+1}, \dots, e_n$ are tangent to M_2 and e_{n+1} is parallel to H. Then using (14), we have

$$\frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s), \tag{16}$$

for each $s \in \{n_1 + 1, \dots, n\}$. From (4) and (9) we have

$$n^2\|H\|^2 = 2\tau - n(n-1)f_1 + 2(n-1)f_3 - 3f_2P + \|h\|^2, \tag{17}$$

where $2\tau = R$, that is

$$\tau = \sum_{1 \le j < i \le n} K(e_j \wedge e_i)$$

and

$$P := \sum_{1 \le i,j \le n} \Big(g(e_j, \phi e_i)\Big)^2 = \sum_{1 \le i,j \le n_1} \Big(g(e_j, \phi e_i)\Big)^2,$$

because

$$2g(e_i, \phi e_j) = d\eta(e_j, e_i) = \bigg(e_j\Big(\eta(e_i)\Big) - e_i\Big(\eta(e_j)\Big) - \eta([e_j, e_i])\bigg).$$

Therefore, if $i, j \in \{n_1 + 1, \dots, n\}$ or $i \in \{1, \dots, n_1\}$ and $j \in \{n_1 + 1, \dots, n\}$ then $g(e_i, \phi e_j) = 0$. now set

$$\delta := 2\tau - n(n-1)f_1 - 3f_2P + 2(n-1)f_3 - \frac{n^2}{2}\|H\|^2, \tag{18}$$

then (17) can be rewritten as

$$n^2\|H\|^2 = 2(\delta + \|h\|^2). \tag{19}$$

With respect to the above orthonormal frame, (19) takes the following form:

$$\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 = 2\Big(\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{1 \le i \ne j \le n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\Big).$$

If we put $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$, then the above equation becomes

$$\Big(\sum_{i=1}^{3} a_i\Big)^2 = 2\Big(\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1\le i\ne j\le n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \sum_{2\le i\ne j\le n_1} h_{ii}^{n+1} h_{jj}^{n+1} - \sum_{n_1+1\le i\ne j\le n} h_{ii}^{n+1} h_{jj}^{n+1}\Big).$$

Thus, a_1, a_2, a_3 satisfy the Lemma 2.1 (for $n = 3$), i.e.,

$$\Big(\sum_{i=1}^{3} a_i\Big)^2 = 2\Big(b + \sum_{i=1}^{3} a_i^2\Big),$$

with

$$b = \delta + \sum_{1\le i\ne j\le n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \sum_{2\le i\ne j\le n_1} h_{ii}^{n+1} h_{jj}^{n+1} - \sum_{n_1+1\le i\ne j\le n} h_{ii}^{n+1} h_{jj}^{n+1}.$$

Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3$. In the case under consideration, this means

$$\sum_{1\le j<i\le n_1} h_{jj}^{n+1} h_{ii}^{n+1} + \sum_{n_1+1\le j<i\le n} h_{jj}^{n+1} h_{ii}^{n+1} \ge \frac{\delta}{2} + \sum_{1\le j<i\le n} (h_{ji}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ji}^r)^2. \tag{20}$$

Equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{j=n_1+1}^{n} h_{jj}^{n+1}. \tag{21}$$

From (16) and the Gauss equation, we have

$$\begin{aligned} n_2\frac{\Delta f}{f} &= \tau - \sum_{1\le j<i\le n_1} K(e_j\wedge e_i) - \sum_{n_1+1\le j<i\le n} K(e_j\wedge e_i) \\ &= \tau - \frac{n_1(n_1-1)}{2}f_1 - \frac{3}{2}f_2\sum_{1\le i,j\le n_1}\Big(g(e_j,\phi e_i)\Big)^2 + (n_1-1)f_3 \\ &\quad - \frac{n_2(n_2-1)}{2}f_1 - \underbrace{\frac{3}{2}f_2\sum_{n_1+1\le i,j\le n}\Big(g(e_j,\phi e_i)\Big)^2}_{0} \\ &\quad - \sum_{r=n+1}^{2m+1}\sum_{1\le j<i\le n_1}\Big(h^r_{ii}h^r_{jj} - (h^r_{ji})^2\Big) \\ &\quad - \sum_{r=n+1}^{2m+1}\sum_{n_1+1\le j<i\le n}\Big(h^r_{ii}h^r_{jj} - (h^r_{ji})^2\Big). \end{aligned} \tag{22}$$

From (20) and (22), we obtain

$$\begin{aligned} n_2\frac{\Delta f}{f} &\le \tau - \frac{n(n-1)}{2}f_1 + n_1n_2f_1 - \frac{3}{2}Pf_2 + (n_1-1)f_3 - \frac{\delta}{2} \\ &\quad - \sum_{\substack{1\le j\le n_1\\ n_1+1\le i\le n}}(h^{n+1}_{ji})^2 - \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h^r_{ji})^2 \\ &\quad + \sum_{r=n+2}^{2m+1}\sum_{1\le j<i\le n_1}\Big((h^r_{ji})^2 - h^r_{ii}h^r_{jj}\Big) \\ &\quad + \sum_{r=n+2}^{2m+1}\sum_{n_1+1\le j<i\le n}\Big((h^r_{ji})^2 - h^r_{ii}h^r_{jj}\Big), \end{aligned}$$

Therefore,

$$\begin{aligned} n_2\frac{\Delta f}{f} &\le \tau - \frac{n(n-1)}{2}f_1 + n_1n_2f_1 - \frac{3}{2}Pf_2 + (n_1-1)f_3 - \frac{\delta}{2} \\ &\quad - \sum_{r=n+1}^{2m+1}\sum_{j=1}^{n_1}\sum_{i=n_1+1}^{n}(h^r_{ji})^2 - \frac{1}{2}\sum_{r=n+2}^{2m+1}(\sum_{j=1}^{n_1}h^r_{jj})^2 \\ &\quad - \frac{1}{2}\sum_{r=n+2}^{2m+1}(\sum_{j=n_1+1}^{n}h^r_{jj})^2 \\ &\le \tau - \frac{n(n-1)}{2}f_1 + n_1n_2f_1 - \frac{3}{2}Pf_2 + (n_1-1)f_3 - \frac{\delta}{2} \\ &= \frac{n^2}{4}\|H\|^2 + n_1n_2f_1 - n_2f_3. \end{aligned} \tag{23}$$

Which implies the inequality (11).

We see that the equality sign of (23) holds if and only if

$$h^r_{ji} = 0 \text{ for } 1 \le j \le n_1,\ n_1 + 1 \le i \le n,\ n + 1 \le r \le 2n + 1, \tag{24}$$

and

$$\sum_{i=1}^{n_1} h^r_{ii} = \sum_{j=n_1+1} h^r_{jj} = 0, \quad n + 2 \le r \le 2m + 1. \tag{25}$$

Obviously (24) is equivalent to the mixed totally geodesic of the warped product $M_1 \times_f M_2$ and (21) and (25) implies $n_1 H_1 = n_2 H_2$.

The converse statement is straightforward.

b. We choose a local orthonormal frame $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m+1}\}$ such that $e_1, \dots, e_{n_1}$ are tangent to M_1, $e_{n_1+1}, \dots, e_n = \xi$ are tangent to M_2 and e_{n+1} is parallel to H.

Use the similar computation in part (a) to get (17), in which

$$P := \sum_{1 \le i,j \le n} \Big(g(e_j, \phi e_i)\Big)^2 = \sum_{n_1+1 \le i,j \le n-1} \Big(g(e_j, \phi e_i)\Big)^2,$$

Using (18), we have (19). Then with the same method in proof of part (a) and using Gauss equation, we have

$$\begin{aligned} n_2 \frac{\Delta f}{f} &= \tau - \sum_{1 \le j < i \le n_1} K(e_j \wedge e_i) - \sum_{n_1+1 \le j < i \le n} K(e_j \wedge e_i), \\ &= \tau - \frac{n_1(n_1 - 1)}{2} f_1 - \underbrace{\frac{3}{2} f_2 \sum_{1 \le i,j \le n_1} \Big(g(e_j, \phi e_i)\Big)^2}_{0} + (n_2 - 1) f_3 \\ &\quad - \frac{n_2(n_2 - 1)}{2} f_1 - \frac{3}{2} f_2 \sum_{n_1+1 \le i,j \le n} \Big(g(e_j, \phi e_i)\Big)^2 \\ &\quad - \sum_{r=n+1}^{2m+1} \sum_{1 \le j < i \le n_1} \Big(h^r_{ii} h^r_{jj} - (h^r_{ji})^2\Big) \\ &\quad - \sum_{r=n+1}^{2m+1} \sum_{n_1+1 \le j < i \le n} \Big(h^r_{ii} h^r_{jj} - (h^r_{ji})^2\Big). \end{aligned} \tag{26}$$

Applying lemma 2.1 and doing similar computations as in the proof of part (a), (26) leads to

$$n_2 \frac{\Delta f}{f} \le \tau - \frac{n(n-1)}{2} f_1 + n_1 n_2 f_1 + (n_2 - 1) f_3 - \frac{3}{2} f_2 P - \frac{\delta}{2}. \tag{27}$$

Using (18), the inequality (27) becomes

$$n_2 \frac{\Delta f}{f} \le \frac{n^2}{4} \|H\|^2 + n_1 n_2 f_1 - n_1 f_3,$$

i.e. the inequality is proved.

It can be proved, just similar to part (a) that the equality holds in the above relation if and only if $M_1 \times_f M_2$ is a mixed totally geodesic submanifold of $\overline{M}$ and $n_1 H_1 = n_2 H_2$. Therefore (d) is proved.

c. We choose a local normal frame $\{e_1, \ldots, e_{2m+1}\}$ such that $e_1, \ldots, e_{n_1} = \xi_1$ tangent to M_1, $e_{n_1+1}, \ldots, e_n = \xi_2$ tangent to M_2 and e_{n+1} parallel to H and for any $i \in \{1, \ldots, 2m+1\} - \{n_1, n\}$, $\|e_i\| = 1$.

From (9) and Gauss equation, we have

$$\begin{aligned} n^2\|H\|^2 = 2\tau + \|h\|^2 &- \Big(n^2 - 3n + 2 - 2g(\xi_1,\xi_1)g(\xi_2,\xi_2)\Big)f_1 \\ &+2\Big((n-2) + (5-2n)g(\xi_1,\xi_1)g(\xi_2,\xi_2)\Big)f_3 \\ &-3(2P_1 + P)f_2. \end{aligned} \tag{28}$$

in which

$$P := \sum_{\substack{1\le i,j\le n \\ i,j\ne n_1,n}} \Big(g(e_j,\phi e_i)\Big)^2 = \sum_{1\le i,j\le n_1-1} \Big(g(e_j,\phi e_i)\Big)^2 + \sum_{n_1+1\le i,j\le n} \Big(g(e_j,\phi e_i)\Big)^2$$

and

$$P_1 := 2\sum_{\substack{j=1 \\ j\ne n_1,n}}^{n} \Big(g(e_j,\phi\xi_1)\Big)^2 = 2\sum_{\substack{j=1 \\ j\ne n_1,n}}^{n} \Big(g(e_j,\phi\xi_2)\Big)^2.$$

We denote

$$\begin{aligned} \delta := 2\tau &- \Big(n^2 - 3n + 2 - 2g(\xi_1,\xi_1)g(\xi_2,\xi_2)\Big)f_1 \\ &+2\Big((n-2) + (5-2n)g(\xi_1,\xi_1)g(\xi_2,\xi_2)\Big)f_3 \\ &-3(2P_1 + P)f_2 - \frac{n^2}{2}\|H\|^2, \end{aligned} \tag{29}$$

Then (28) can be written as (19). We use the same method as in the proof of part (a). Using again the Gauss equation and (16), we have

$$\begin{aligned} n_2\frac{\Delta f}{f} &= \tau - \sum_{1\le j<i\le n_1} K(e_j\wedge e_i) - \sum_{n_1+1\le j<i\le n} K(e_j\wedge e_i) \\ &= \tau - \Big(n_1 g(\xi_1,\xi_1) + n_2 g(\xi_2,\xi_2) + \frac{n^2-3n+4}{2} - n_1n_2 - 1\Big)f_1 \\ &\quad + \Big(n_1\Big(g(\xi_1,\xi_1)\Big)^2 + n_2\Big(g(\xi_2,\xi_2)\Big)^2 + 2g(\xi_1,\xi_1)g(\xi_2,\xi_2) - 1\Big)f_3 \\ &\quad -3\Big(\sum_{j=1}^{n_1-1}\Big(g(e_j,\xi_1)\Big)^2 + \sum_{j=n_1+1}^{n-1}\Big(g(e_j,\xi_2)\Big)^2 + \frac{1}{2}P\Big)f_2 \\ &\quad - \sum_{r=n+1}^{2m+1}\sum_{1\le j<i\le n_1}\Big(h_{ii}^r h_{jj}^r - (h_{ji}^r)^2\Big) - \sum_{r=n+1}^{2m+1}\sum_{n_1+1\le j<i\le n}\Big(h_{ii}^r h_{jj}^r - (h_{ji}^r)^2\Big). \end{aligned}$$

(30)

Applying lemma 2.1 and doing the similar computations as in the proof of part (a), (30) leads to

$$
\begin{aligned}
n_2\frac{\Delta f}{f} \leq \tau - \Big(n_1 g(\xi_1,\xi_1) + n_2 g(\xi_2,\xi_2) + \frac{n^2-3n+4}{2} - n_1n_2 - 1\Big) f_1 \\
+\Big(n_1\Big(g(\xi_1,\xi_1)\Big)^2 + n_2\Big(g(\xi_2,\xi_2)\Big)^2 + 2g(\xi_1,\xi_1)g(\xi_2,\xi_2) - 1\Big) f_3 \\
-3\Big(\sum_{j=1}^{n_1-1}\Big(g(e_j,\xi_1)\Big)^2 + \sum_{j=n_1+1}^{n-1}\Big(g(e_j,\xi_2)\Big)^2 + \frac{1}{2}P\Big) f_2 - \frac{\delta}{2}.
\end{aligned}
$$

Using (28), the above inequality becomes

$$
\begin{aligned}
n_2\frac{\Delta f}{f} \leq \Big(n_1n_2 - n_1 g(\xi_1,\xi_1) - n_2 g(\xi_2,\xi_2) + g(\xi_1,\xi_1)g(\xi_2,\xi_2)\Big) f_1 + \frac{3}{2}P_1 f_2 \\
-\Big(n_2\Big(g(\xi_1,\xi_1)\Big)^2 + n_1\Big(g(\xi_2,\xi_2)\Big)^2 + 3g(\xi_1,\xi_1)g(\xi_2,\xi_2) - 1\Big) f_3 \\
+\frac{n^2}{4}\|H\|^2,
\end{aligned}
\tag{31}
$$

since

$$
g(\phi\xi_1,\phi\xi_1) = g(\xi_1,\xi_1) - \Big(\eta(\xi_1)\Big)^2 \leq 1.
$$

On the other hand for $j \in \{1,\dots,n_1-1,n_1+1,\dots,n-1\}$ we have

$$
\begin{aligned}
0 \leq g(e_j-\phi\xi_1, e_j-\phi\xi_1) = g(e_j,e_j) + g(\phi\xi_1,\phi\xi_1) - 2g(e_j,\phi\xi_1) \\
\Rightarrow g(e_j,\phi\xi_1) \leq \frac{1}{2}\Big(g(e_j,e_j) + g(\phi\xi_1,\phi\xi_1)\Big) \leq 1.
\end{aligned}
$$

From the last inequality we have $0 \leq P_1 \leq 2(n-2)$. using (31) and $\Theta(f_2)$ to get (13). ☐

Corollary 3.1. *Let $\overline{M}(f_1,f_2,f_3)$ be a $(2m+1)$-dimensional generalized Sasakian space form with contact structure and $M_1\times_f M_2$ an n-dimensional minimal warped product submanifold, such that f is a harmonic function.*

a. If ξ is tangent to M_1, then

$$f_3 \leq n_1 f_1, \tag{32}$$

b. If ξ is tangent to M_2, then

$$f_3 \leq n_2 f_1, \tag{33}$$

c. If $\xi = \xi_1+\xi_2$ such that ξ_1 and ξ_2 are nonzero at any point of $M_1\times_f M_2$ and tangent to M_1 and M_2 respectively, then

$$f_3 \leq \frac{f_1}{A}\Big(g(\xi_1,\xi_1) - n_2\Big)\Big(g(\xi_2,\xi_2) - n_1\Big) + \frac{3}{A}(n-2)\Theta(f_2) \tag{34}$$

where $n_i = \dim M_i$ *(i=1,2),* Δ *is Laplacian operator of* M_1 *and*

$$A := \left(n_2\left(g(\xi_1,\xi_1)\right)^2 + n_1\left(g(\xi_2,\xi_2)\right)^2 + 3g(\xi_1,\xi_1)g(\xi_2,\xi_2) - 1\right).$$

d. The equality case of (32) and (33) hold if and only if $M_1 \times_f M_2$ *is a minimal mixed totally geodesic submanifold and* $n_1 H_1 = n_2 H_2$*, where* $H_i (i = 1, 2)$*, is partial mean curvature vector.*

$A \neq 0$ in c because $0 < g(\xi_1,\xi_1) < 1$ and $0 < g(\xi_2,\xi_2) < 1$ and

$$A = (n_2 - 1)\left(g(\xi_1,\xi_1)\right)^2 + (n_1 - 1)\left(g(\xi_2,\xi_2)\right)^2 + g(\xi_1,\xi_1)g(\xi_2,\xi_2).$$

Proof. $H = 0$ over $M_1 \times_f M_2$, and $\Delta f = 0$ over M_1, hence this corollary follows from Theorem 3.1. □

Corollary 3.2. *Let* $\overline{M}(f_1, f_2, f_3)$ *be a* $(2m + 1)$*-dimensional generalized Sasakian space form with contact structure and* $M_1 \times_f M_2$ *an* n*-dimensional minimal warped product submanifold, such that* f *is an eigenfunction of Laplacian on* M_1 *with the corresponding eigenvalue* $\lambda > 0$*. If*

a. ξ *is tangent to* M_1*, then*

$$f_3 < n_1 f_1, \tag{35}$$

b. ξ *is tangent to* M_2*, then*

$$f_3 < n_2 f_1, \tag{36}$$

c. $\xi = \xi_1 + \xi_2$ *such that* ξ_1 *and* ξ_2 *are nonzero at any point of* $M_1 \times_f M_2$ *and tangent to* M_1 *and* M_2 *respectively, then*

$$f_3 < \frac{f_1}{A}\left(g(\xi_1,\xi_1) - n_2\right)\left(g(\xi_2,\xi_2) - n_1\right) + \frac{3}{A}(n-2)\Theta(f_2), \tag{37}$$

where $n_i = \dim M_i$ *(i=1,2),* Δ *is Laplacian operator of* M_1 *and*

$$A := \left(n_2\left(g(\xi_1,\xi_1)\right)^2 + n_1\left(g(\xi_2,\xi_2)\right)^2 + 3g(\xi_1,\xi_1)g(\xi_2,\xi_2) - 1\right).$$

Proof. If f is an eigenfunction of Laplacian on M_1 with eigenvalue $\lambda > 0$ then

$$\frac{\Delta f}{f} = \frac{\lambda f}{f} = \lambda > 0,$$

therefore from Theorem 3.1, this corollary is proved. □

Corollary 3.3. *Let* $\overline{M}(f_1, f_2, f_3)$ *be a* $(2m + 1)$*-dimensional generalized Sasakian space form with contact structure and* $M_1 \times_f M_2$ *an* n*-dimensional warped product submanifold and* $\xi = \xi_1 + \xi_2$ *such that* ξ_1 *and* ξ_2 *are nonzero at any point of* $M_1 \times_f M_2$ *and tangent to* M_1 *and* M_2 *respectively,*

a. If $g(\xi_2,\xi_2) \to 0$*, then*

$$\frac{\Delta f}{f} \leq \frac{n_2 - 1}{n_2}(n_1 f_1 - f_3) + \frac{n^2}{4n_2}\|H\|^2, \tag{38}$$

b. If $g(\xi_1, \xi_1) \to 0$, *then*

$$\frac{\Delta f}{f} \leq \frac{n_1 - 1}{n_2}(n_2 f_1 - f_3) + \frac{n^2}{4n_2}\|H\|^2, \tag{39}$$

The equality case of (38) and (39) hold if and only if $M_1 \times_f M_2$ *is a mixed totally geodesic submanifold and* $n_1 H_1 = n_2 H_2$, *in which* $H_i (i = 1, 2)$, *is partial mean curvature vector.*

Proof. By inequality (31), the proof is evident. □

Corollary 3.4. *Let* $\overline{M}(f_1, f_2, f_3)$ *be a* $(2m + 1)$*-dimensional generalized Sasakian space form with contact structure and* $M_1 \times_f M_2$ *an* n*-dimensional minimal warped product submanifold and* $\xi = \xi_1 + \xi_2$ *such that* ξ_1 *and* ξ_2 *are nonzero at any point of* $M_1 \times_f M_2$ *and tangent to* M_1 *and* M_2 *respectively,*

a. If $g(\xi_2, \xi_2) \to 0$, *then*

$$\frac{\Delta f}{f} \leq \frac{n_2 - 1}{n_2}(n_1 f_1 - f_3), \tag{40}$$

b. If $g(\xi_1, \xi_1) \to 0$, *then*

$$\frac{\Delta f}{f} \leq \frac{n_1 - 1}{n_2}(n_2 f_1 - f_3). \tag{41}$$

The equality case of (40) and (41) hold if and only if $M_1 \times_f M_2$ *is a mixed totally geodesic submanifold and* $n_1 H_1 = n_2 H_2$, *in which* $H_i (i = 1, 2)$, *is partial mean curvature vector.*

Remark 3.1. In part (a) of Corollary 3.3, if $f_3 \leq n_1 f_1$ then

$$\frac{n_2 - 1}{n_2}(n_1 f_1 - f_3) \leq (n_1 f_1 - f_3),$$

therefore (38) reduces to (11), and if $f_3 \leq n_2 f_1$ then

$$\frac{n_1 - 1}{n_2}(n_2 f_1 - f_3) \leq \frac{n_1}{n_2}(n_2 f_1 - f_3),$$

therefore (39) reduces to (12).

Theorem 3.2. *Let* $\overline{M}(f_1, f_2, f_3)$ *be a* $(2m + 1)$*-dimensional generalized Sasakian space form with* K*-contact metric structure and* $M_1 \times_f M_2$ *an* n*-dimensional warped product submanifold. If* ξ *is tangent to* M_2 *then*

$$f_3 \leq \frac{n^2}{4n_1}\|H\|^2 + n_2 f_1, \tag{42}$$

the equality case hold if and only if $M_1 \times_f M_2$ *is a mixed totally geodesic submanifold and* $n_1 H_1 = n_2 H_2$, *where* $H_i (i = 1, 2)$, *is partial mean curvature vector.*

Proof. For any $X \in \tau(M_1 \times_f M_2)$ tangent to M_1, over $M_1 \times_f M_2$ we have

$$\nabla_X \xi = \frac{Xf}{f}\xi.$$

on the other hand by K-contactnes we have

$$0 = g(\phi X, \xi) = g(\frac{Xf}{f}\xi, \xi) = \frac{Xf}{f}$$

Thus $Xf = 0$, therefore f is constant and $\Delta f = 0$. From (12) and (44), inequality (42) is proved.

The proof of the last part of theorem is similar to Theorem 3.1. □

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