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SIMPLE PROOFS OF THREE HOMOGENEOUS CYCLIC INEQUALITIES OF THREE VARIABLES OF DEGREE THREE AND FOUR

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Abstract. In this short note, the authors give simple proofs of three homogeneous cyclic inequalities of three variables of degree three and four which were proved quite complicated in [1].

1. Introduction and Main Results

Throughout this paper, we will customarily use the cyclic sum and product symbols, that is:

$$\sum f(x) = f(x) + f(y) + f(z),$$

$$\sum f(y,z) = f(x,y) + f(y,z) + f(z,x)$$

and

$$\prod f(x) = f(x)f(y)f(z).$$

T. Ando obtained the following results in [1].

Theorem 1.1. *For non-negative real numbers x, y, z and any real number t, the following inequality holds:*

$$\begin{aligned} t^2 \sum x^3 + (t^4 - 2t) \sum x^2 y \geq (2t^3 - 1) \sum xy^2 \\ + (3t^4 - 6t^3 + 3t^2 - 6t + 3) \prod x. \end{aligned} \tag{1.1}$$

The equality holds if and only if $x = y = z$ or $x : y : z = t : 0 : 1$ or any cyclic permutation thereof.

Theorem 1.2. (1) *For any real numbers x, y, z and t such that $|t| \geq 2$, the following inequalities hold:*

$$\sum x^4 + \frac{t^2+8}{4} \sum y^2 z^2 + \frac{t(t-2)}{2} \sum x \cdot \prod x \geq t \sum xy(x^2 + y^2). \tag{1.2}$$

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Here, the equality holds if and only if $2\sum x^2 = t\sum yz$.
(2) *For any non-negative real numbers* x, y, z *and* t, *the following inequality holds:*

$$\begin{aligned} 2t^3\sum x^4 \geq &t^2(3-t^4)\sum x^3y + (3t^4-1)\sum xy^3 \\ &+ (t^6-3t^4+2t^3-3t^2+1)\sum x\cdot\prod x. \end{aligned} \tag{1.3}$$

The equality holds if and only if $x = y = z$ *or* $x : y : z = t : 0 : 1$ *or any cyclic permutation thereof.*

In the next two sections, the authors will prove inequalities (1.1)–(1.3) with simple methods. And we will also point out that inequality (1.2) holds for any real number t.

2. The Proof of Theorem 1.1

Proof. (i) While $x = y = z$, the inequality (1.1) holds obviously.

(ii) While

$$(x-y)^2 + (y-z)^2 + (z-x)^2 \neq 0, \tag{2.1}$$

we set

$$m = \sum x^2y - 3\prod x,$$

$$n = \sum xy^2 - 3\prod x,$$

$$p = \sum x^3 - 3\prod x.$$

It's easy to find that $m, n, p > 0$ by $AM - GM$ inequality and (2.1). Inequality (1.1) is equivalent to

$$mt^4 - 2nt^3 + pt^2 - 2mt + n \geq 0$$

or

$$mt^2\left(t - \frac{n}{m}\right)^2 + n\left(1 - \frac{m}{n}t\right)^2 + \left[p - \left(\frac{n^2}{m} + \frac{m^2}{n}\right)\right]t^2 \geq 0. \tag{2.2}$$

For

$$p - \left(\frac{n^2}{m} + \frac{m^2}{n}\right) = \frac{xyz[\sum(y-z)^2]^3}{8mn} \geq 0, \tag{2.3}$$

inequality (2.2) holds, i.e. inequality (1.1) holds.
From (i) and (ii), we complete the proof of inequality (1.1). □

3. The Proof of Theorem 1.2

3.1. The Proof of Inequality (1.2).

Proof. Inequality (1.2) is equivalent to

$$\frac{1}{4}[2(x^2+y^2+z^2)-t(xy+yz+zx)]^2 \geq 0. \tag{3.1}$$

Inequality (3.1) holds obviously, hence, inequality (1.2) holds. □

Remark 1. It's easy to find that inequality (3.1) holds for any real numbers x, y, z, t.

3.2. The Proof of Inequality (1.3).

In order to prove inequality (1.3), we require the following lemma.

Lemma 1. *If $x, y, z \geq 0$, then*

$$\begin{aligned} &\sum x^7 - \sum (y+z)x^6 - \sum (y^2+z^2)x^5 + \sum (y^3+z^3)x^4 \\ &\quad + \sum yz(y+z)x^4 + \prod x \cdot \sum x^4 - \prod x \cdot \sum y^2z^2 \\ &\quad - \prod x^2 \cdot \sum x \geq 0. \end{aligned} \tag{3.2}$$

Proof. For inequality (3.2) is symmetric with x, y, z, there is no harm in supposing $x \leq y \leq z$, and take $y = x+a, z = x+a+b (a, b \geq 0)$, then inequality (3.2) is equivalent to

$$\begin{aligned} &4(b^2+ab+a^2)^2x^3 + 2(2a+b)(b^2+ab+a^2)(4b^2+ab+a^2)x^2 \\ &\quad + (19b^2a^4+39b^4a^2+5b^6+23b^5a+33a^3b^3+3ba^5+a^6)x \\ &\quad + b^2(2a+b)(b^4+4ab^3+6a^2b^2+4a^3b+2a^4) \geq 0. \end{aligned} \tag{3.3}$$

For $x \geq 0$ and $a, b \geq 0$, inequality (3.3) holds true. Hence, the proof of inequality (3.2) is complete. □

Proof. (i) While $x = y = z$, the inequality (1.3) holds obviously.

(ii) While $(x-y)^2+(y-z)^2+(z-x)^2 \neq 0$, then we set

$$u = \sum x^3y - \prod x \sum x,$$

$$v = \sum xy^3 - \prod x \sum x,$$

$$w = \sum x^4 - \prod x \sum x.$$

It's easy to find that $u, v, w > 0$ by sequence inequality and (2.1). Inequality (1.3) is equivalent to

$$ut^6 - 3vt^4 + 2wt^3 - 3ut^2 + v \geq 0$$

or

$$\frac{(t\sqrt{u}+2\sqrt{v})(t\sqrt{u}-\sqrt{v})^2t^3}{\sqrt{u}}+\frac{(\sqrt{v}+2t\sqrt{u})(\sqrt{v}-t\sqrt{u})^2}{\sqrt{v}} \\ +2\left[w-\left(v\sqrt{\frac{v}{u}}+u\sqrt{\frac{u}{v}}\right)\right]t^3\geq 0. \tag{3.4}$$

And by Lemma 1, we have

$$\begin{aligned} & uvw^2-(u^2+v^2)^2 \\ =&\frac{1}{4}xyz(x+y+z)^2\left[\sum(y-z)^2\right]^2\cdot[\sum x^7-\sum(y+z)x^6 \\ & -\sum(y^2+z^2)x^5+\sum(y^3+z^3)x^4+\sum yz(y+z)x^4 \\ & +\prod x\cdot\sum x^4-\prod x\cdot\sum y^2z^2-\prod x^2\cdot\sum x]\geq 0, \end{aligned} \tag{3.5}$$

hence,

$$\begin{aligned} w-\left(v\sqrt{\frac{v}{u}}+u\sqrt{\frac{u}{v}}\right) &= \frac{\sqrt{uv}w-(u^2+v^2)}{\sqrt{uv}} \\ &= \frac{uvw^2-(u^2+v^2)^2}{\sqrt{uv}[\sqrt{uv}w+(u^2+v^2)]}\geq 0. \end{aligned}$$

For $t\geq 0$, inequality (3.4) holds obviously, i.e. inequality (1.3) holds. From (i) and (ii), the proof of inequality (1.3) is complete. □

References

[1] T. Ando, *Some Homogeneous Cyclic Inequalities of Three Variables Of Degree Three and Four*, Aust. J. Math. Anal. Appl. **7** (2010), no. 2, 1–14.

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