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A NOTE ON FEFFERMAN–STEIN TYPE CHARACTERIZATIONS FOR CERTAIN SPACES OF ANALYTIC FUNCTIONS ON THE UNIT DISC

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Abstract. We obtain new characterizations of Bergman and Bloch spaces on the unit disc involving equivalent (quasi)-norms on these spaces. Our results are in the spirit of estimates obtained by Fefferman and Stein for Hardy spaces in $\mathbb{R}^n$.

1. Introduction

We denote by $H(\Omega)$ the space of all analytic functions in a domain $\Omega \subset \mathbb{C}$, $H^p = H^p(\mathbb{D})$ denotes the classical Hardy space on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $A^p = A^p(\mathbb{D})$ denotes the Bergman space on $\mathbb{D}$, see [3] and [6]. We set $A^p_0 = \{f \in A^p : f(0) = 0\}$. Also, $\Gamma_\alpha(\xi)$ denotes the Stolz region at $\xi \in \mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$ of aperture $\alpha > 1$. For $t > 0$ and an analytic function $f(z) = \sum_{n=0}^\infty a_n z^n$ in the unit disc the fractional derivative of f of order t is defined by $D^t f(z) = \sum_{n=0}^\infty (n+1)^t a_n z^n$, it is also analytic in $\mathbb{D}$. Area measure on $\mathbb{D}$ is denoted by dm. The Bloch space $\mathcal{B}$, defined by

$$\mathcal{B} = \left\{ f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} |f'(z)|(1-|z|) < \infty \right\}$$

is closely related to Carleson measures and corresponds to the endpoint case $p = 1$ of Q_p classes ($0 < p \le 1$), see [10]. Related spaces $\mathcal{B}_p$, $1 < p < \infty$, are defined by

$$\mathcal{B}_p(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-2} dm(z) = \|f\|^p_{\mathcal{B}_p} < \infty \right\},$$

note that $\|f\|_{\mathcal{B}}$ and $\|f\|_{\mathcal{B}_p}$ are not true norms, but $|f(0)| + \|f\|_{\mathcal{B}}$ and $|f(0)| + \|f\|_{\mathcal{B}_P}$ are norms which make respective spaces into Banach spaces.

The following result is proved by E. G. Kwon in [7]:

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Theorem 1 (see [7]). *If $0 < p < \infty$ and $0 \le \beta < p+2$, then $f \in H(\mathbb{D})$ belongs to $\mathcal{B}$ if and only if it satisfies the following condition:*

(1)
$$\sup_{a\in\mathbb{D}} \int\int_{\mathbb{D}\,\mathbb{D}} \frac{|f(z)-f(w)|^{p-\beta}}{|1-\overline{w}z|^4} |f'(z)|^\beta (1-|z|)^\beta \frac{(1-|w|)^2(1-|a|)^2}{|1-\overline{w}a|^4} dm(z)dm(w) < \infty,$$

in fact the above expression is equivalent to $\|f\|^p_{\mathcal{B}}$.

Similarly, if $1 < p < \infty$ and $0 \le \beta < p+2$, then a function $f \in H(\mathbb{D})$ belongs to $\mathcal{B}_p$ if and only if

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(w)-f(z)|^{p-\beta}}{|1-\overline{w}z|^4} |f'(z)|^\beta (1-|z|)^\beta dm(z)dm(w) < \infty, \tag{2}$$

moreover, the above expression is equivalent to $\|f\|^p_{\mathcal{B}_p}$.

We relate these estimates to the so called Fefferman–Stein type characterizations. By Fefferman–Stein characterizations we mean the following relations:

$$\|F\|_X \asymp \inf_{\omega\in S} \|\Phi(F,\omega)\|_Y, \tag{3}$$

where X and Y are (quasi)-normed subspaces of $H(\mathbb{D})$, $S = S_F$ is a certain class of measurable functions and Φ is a nonanalytic function of two variables. This idea was used to determine the predual of Q_p classes, $0 < p \le 1$, see [11, 12]. In certain cases the infimum in (3) is attained, see [2], especially section 5. Using ideas from [4] and [1] the authors there extended and used such characterizations for certain Hardy classes in $\mathbb{R}^n$. Later one of the authors provided a similar Fefferman–Stein type characterization for the analytic Hardy spaces, this is the second part of the following theorem, the first part is a classical result of N. Lusin.

Theorem 2 (see [8, 9]). *Let $0 < p, t < \infty$. Then, for $f \in H(\mathbb{D})$ we have*

$$\|f\|^p_{H^p} \asymp \int_{\mathbb{T}} \left(\int_{\Gamma_\alpha(\xi)} |D^t f|^2 (1-|z|)^{2t-2} dm(z) \right)^{p/2} d\xi. \tag{4}$$

Next, let $s > 0$ and $0 < p < 2$. Then, for $f \in H(\mathbb{D})$, we have

(5)
$$\int_{\mathbb{T}} \left(\int_{\Gamma_\alpha(\xi)} |f'(z)|^2 dm(z) \right)^{p/2} d\xi \asymp \inf_{\omega\in S_1} \left(\int_{\mathbb{D}} |f'(z)|^s (1-|z|)^{s-1} \frac{dm(z)}{\omega(z)} \right)^{p/2},$$

where

$$S_1 = \left\{ \omega \ge 0 : \| \sup_{\Gamma_\alpha(\xi)} \omega(z)(1-|z|)^{2-s} |f'(z)|^{2-s} \|_{L^{\frac{p}{2-p}}(\mathbb{T})} < 1 \right\}.$$

Here we show that similar results are true for Bloch and Bergman spaces A^p_0 in the unit disc. An interesting problem would be to obtain similar results for Q_p or other BMOA-type spaces in the unit disc.

2. Main results

In this section we state and prove the main results of this paper. They are analogous to the previously obtained results on Fefferman–Stein characterizations of Hardy spaces in $\mathbb{R}^n$ and our proofs heavily rely on the mentioned results of E. G. Kwon.

Theorem 3. *Let $1 < \alpha < 2$, $1/\alpha + 1/\alpha' = 1$ and $p \geq \alpha'$. Then for $F \in H(\mathbb{D})$ with $F(0) = 0$ we have*

$$\|F\|^p_{A^p_0} \asymp \inf_{\omega \in S_2} \left(\int_{\mathbb{D}} |F'(z)|^\alpha \omega^\alpha(z) dm(z) \right)^{1/\alpha},$$

where

$$S_2 = \left\{ \omega \geq 0 : \int_{\mathbb{D}} |F'(z)|^{(p-1)\alpha'} \omega^{-\alpha'}(z)(1-|z|)^{p\alpha'} dm(z) \leq 1 \right\}.$$

Proof. Here we use the following result from [10], Chapter 2: If $F \in H(\mathbb{D})$ and $F(0) = 0$, then

$$\int_{\mathbb{D}} |F(z)|^p dm(z) \asymp \int_{\mathbb{D}} |F(z)|^{p-\beta} |F'(z)|^\beta (1-|z|)^\beta dm(z), \tag{6}$$

where $0 < p < \infty$, $0 \leq \beta < p+2$. Taking $\beta = p$ and applying Hölder inequality we obtain

$$\|F\|^p_{A^p_0} \leq \left(\int_{\mathbb{D}} |F'(z)|^\alpha \omega^\alpha(z) dm(z) \right)^{1/\alpha} \times$$
$$\times \left(\int_{\mathbb{D}} |F'(z)|^{(p-1)\alpha'} \omega^{-\alpha'}(z)(1-|z|)^{p\alpha'} dm(z) \right)^{1/\alpha'}$$

and this gives one estimate stated in Theorem 3. It is of some interest to note that this estimate is true for all $1 < \alpha < \infty$. Now we prove the reverse estimate by choosing a special admissible test function in S_2. We can assume $\|F\|_{A^p_0} = 1$ and set

$$\tilde{\omega}(z) = \frac{|F'(z)|^{p/\alpha}(1-|z|)^{1+p/\alpha}}{|F(z)|}, \qquad z \in \mathbb{D}.$$

A straightforward calculation shows that

$$\int_{\mathbb{D}} |F'(z)|^{(p-1)\alpha'}(1-|z|)^{p\alpha'} \tilde{\omega}^{-\alpha'}(z) dm(z) = \int_{\mathbb{D}} |F'(z)|^{p-\alpha'}(1-|z|)^{p-\alpha'} |F(z)|^{\alpha'} dm(z).$$

Now (6), with $\beta = p - \alpha'$, tells us that the last expression is comparable to $\|F\|^p_{A^p_0}$ and therefore bounded by $C = C_{p,\alpha} > 0$. Hence $\omega = C^{1/\alpha'}\tilde{\omega} \in S_2$ and we have

$$\begin{aligned}\int_{\mathbb{D}} |F'(z)|^\alpha \omega^\alpha(z) dm(z) &= C \int_{\mathbb{D}} |F'(z)|^\alpha \tilde{\omega}^\alpha(z) dm(z) \\ &= C \int_{\mathbb{D}} |F'(z)|^{p+\alpha} |F(z)|^{-\alpha} (1-|z|)^{p+\alpha} dm(z).\end{aligned}$$

The last integral is, by (6) with $\beta = p + \alpha < p + 2$, bounded from above by a constant depending only on p and α and this ends the proof of Theorem 3. $\square$

To simplify formulation of our next theorem we introduce, for $a \in \mathbb{D}$,

$$d\mu_a(z,w) = \frac{(1-|w|)^2(1-|a|)^2 dm(z) dm(w)}{|1-\overline{z}w|^4 |1-\overline{w}a|^4}.$$

Theorem 4. *Let $1 < \alpha < 2$, $1/\alpha + 1/\alpha' = 1$ and $p \geq \alpha'$. Then we have, for $F \in H(\mathbb{D})$ with $F(0) = 0$*

$$\|F\|_{\mathcal{B}} \asymp \inf_{\omega \in S_3} \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} \int_{\mathbb{D}} \omega^\alpha(z,w) |F'(z)|^\alpha d\mu_a(z,w) \right)^{1/\alpha}, \tag{7}$$

where

(8)

$$S_3 = \left\{ \omega \geq 0 : \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} |F'(z)|^{\alpha'(p-1)} \omega^{-\alpha'}(z,w) (1-|z|)^{p\alpha'} d\mu_a(z,w) \leq 1 \right\}.$$

Proof. Let $F \in \mathcal{B}$, setting $\beta = p$ in (1) we obtain for arbitrary $\omega \in S_3$:

$$\begin{aligned}\|F\|_{\mathcal{B}} \leq & C \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|F'(z)|^p (1-|z|)^p}{|1-\overline{w}z|^4} \frac{(1-|w|)^2(1-|a|)^2}{|1-\overline{w}a|^4} dm(z) dm(w) \\ = & C \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} |F'(z)| \omega(z,w) |F'(z)|^{p-1} \omega^{-1}(z,w) (1-|z|)^p d\mu_a(z,w) \\ \leq & C \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} \int_{\mathbb{D}} |F'(z)|^\alpha \omega^\alpha(z,w) d\mu_a(z,w) \right)^{1/\alpha} \times \\ & \times \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} \int_{\mathbb{D}} |F'(z)|^{\alpha'(p-1)} \omega^{-\alpha'}(z,w) (1-|z|)^{p\alpha'} d\mu_a(z,w) \right)^{1/\alpha'} \\ \leq & C \sup_{a \in \mathbb{D}} \left(\int_{\mathbb{D}} \int_{\mathbb{D}} |F'(z)|^\alpha \omega^\alpha(z,w) d\mu_a(z,w) \right)^{1/\alpha}.\end{aligned}$$

Taking infimum over all $\omega \in S_3$ one obtains an estimate of $\|F\|_{\mathcal{B}}$ in terms of the right hand side in (7). Now we turn to the reverse estimate, we can assume $\|F\|_{\mathcal{B}} = 1$. Taking

$$\tilde{\omega}(z,w) = \frac{|F'(z)|^{p/\alpha} (1-|z|)^{\frac{p+\alpha}{\alpha}}}{|f(z) - f(w)|} \tag{9}$$

one obtains by an easy calculation and relation (1) with $\beta = p - \alpha'$

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}}\int_{\mathbb{D}} |F'(z)|^{\alpha'(p-1)} \omega^{-\alpha'}(z,w)(1-|z|)^{p\alpha'} d\mu_a(z,w) =$$
$$= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}}\int_{\mathbb{D}} |F'(z)|^{p-\alpha'} |f(z)-f(w)|^{\alpha'} (1-|z|)^{p-\alpha'} d\mu_a(z,w) \asymp \|F\|_{\mathcal{B}}^p.$$

As in the proof of the previous theorem this means that $\omega = C\tilde{\omega}$ is in S_3, where $C = C_{p,\alpha} > 0$. With this choice of ω we have

$$\begin{aligned}&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}}\int_{\mathbb{D}} |F'(z)|^{\alpha}\omega^{\alpha}(z,w) d\mu_a(z,w) =\\ &\quad = C^{\alpha} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}}\int_{\mathbb{D}} |F'(z)|^{p+\alpha}(1-|z|)^{p+\alpha}|f(z)-f(w)|^{-\alpha} d\mu_a(z,w)\\ &\quad \asymp \|F\|_{\mathcal{B}}^p = 1,\end{aligned}$$

where we used (1) with $\beta = p+\alpha$. This ends the proof of Theorem 4. □

This theorem has a counterpart for $\mathcal{B}_p$ spaces. It is convenient to introduce a measure $d\mu(z,w) = |1-\overline{w}z|^{-4} dm(z)dm(w)$.

Theorem 5. *Let $1 < p < \infty$, $1 < \alpha < 2$, $1/\alpha + 1/\alpha' = 1$ and $p \geq \alpha'$. Then we have, for $F \in H(\mathbb{D})$ with $F(0) = 0$:*

$$\|F\|_{\mathcal{B}_p} \asymp \inf_{\omega\in S_4} \left(\int_{\mathbb{D}}\int_{\mathbb{D}} \omega^{\alpha}(z,w)|F'(z)|^{\alpha} d\mu(z,w) \right)^{1/\alpha}, \tag{10}$$

where

$$S_4 = \left\{ \omega \geq 0 : \int_{\mathbb{D}}\int_{\mathbb{D}} |F'(z)|^{\alpha'(p-1)}\omega^{-\alpha'}(z,w)(1-|z|)^{p\alpha'} d\mu(z,w) \leq 1 \right\}.$$

Proof. The proof of this theorem parallels the proof of the previous one. Namely we use (2) with $\beta = p$ to obtain, for arbitrary $\omega \in S_4$,

$$\begin{aligned}\|F\|_{\mathcal{B}_p}^p &\asymp \int_{\mathbb{D}}\int_{\mathbb{D}} |F'(z)|\omega(z,w)|F'(z)|^{p-1}(1-|z|)^p\omega^{-1}(z,w) d\mu(z,w)\\ &\leq \left(\int_{\mathbb{D}}\int_{\mathbb{D}} |F'(z)|^{\alpha}\omega^{\alpha}(z,w) d\mu(z,w) \right)^{1/\alpha} \times\\ &\quad \times \left(\int_{\mathbb{D}}\int_{\mathbb{D}} |F'(z)|^{\alpha'(p-1)}\omega^{-\alpha'}(z,w)(1-|z|)^{\alpha' p} d\mu(z,w) \right)^{1/\alpha'}\\ &\leq \left(\int_{\mathbb{D}}\int_{\mathbb{D}} |F'(z)|^{\alpha}\omega^{\alpha}(z,w) d\mu(z,w) \right)^{1/\alpha}.\end{aligned}$$

In proving the reverse estimate one can use the same test function as in (9), the role of condition (1) is taken by condition (2). We leave details to the reader. □

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