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# APPROXIMATION PROPERTIES OF PARTIAL SUMS OF FOURIER SERIES

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Abstract. In this paper we find class of functions for which the Lebesgue estimate can be improved.

Let $C([0,2\pi])$ denote the space of continuous function $f$ with period $2\pi$. If $f \in C([0,2\pi])$ then the function

$$\omega_p(\delta, f) := \sup_x \sup_{|h| \le \delta} |\Delta_p(x; h, f)|, \omega_1(\delta, f) := \omega(\delta, f)$$

is called the modulus of continuity of the function $f$, where

$$\Delta_1(x; h, f) := f(x+h) - f(x),$$

$$\Delta_{p+1}(x; h, f) = \Delta_p(x+h; h, f) - \Delta_p(x; h, f).$$

Let $S_n(f, x)$ be the $n$th partial sums of the trigonometric Fourier series of the function $f$.

The estimation of Lebesgue (see [3], or [1]) is well-known

$$\|f - S_n(f)\| \le c\omega\left(\frac{1}{n}, f\right) \log(n+2).$$

Generalization of this estimation were studied by Chanturia [2], Oskolkov [6], Karchava [4, 5]. There arises a question: for what subclasses of $C([0,2\pi])$ the Lebesgue estimate can be improved?

We prove that the following are true

**Theorem 1.** *Let $f \in C([0,2\pi])$. Then*

$$\|f - S_n(f)\|_C \le c \sum_{k=1}^{[\log n]} \frac{\omega_k\left(\frac{1}{n}, f\right)}{2^k}.$$

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**Corollary 1.** *Let $f \in C([0,2\pi])$. Then*

$$\|f - S_n(f)\|_C \le c \max_{1 \le k \le [\log\log n]} \omega_k\left(\frac{1}{n}, f\right).$$

**Corollary 2.** *Let $f \in C([0,2\pi])$ and $\omega_k\left(\frac{1}{n}, f\right) \le M$. Then*

$$\|f - S_n(f)\|_C \to 0 \text{ as } n \to \infty.$$

**Corollary 3.** *Let $f \in C([0,2\pi])$ and $\omega_k\left(\frac{1}{n}, f\right) \le \frac{2^k}{l(k)}$, where*

$$\sum_{k=1}^{\infty} \frac{1}{l(k)} < \infty.$$

*Then*

$$\|f - S_n(f)\|_C \to 0 \text{ as } n \to \infty.$$

*Proof.* Let $T_n(x)$ be Vale-Poisson polynomial which provides best approximation of function $f$ in the space $C([0,2\pi])$ (see [3]), in particular

$$T_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t) V_n(t)\, dt, \quad \frac{1}{\pi}\int_{-\pi}^{\pi} |V_n(t)|\, dt \le 1.$$

Set

$$g(t) := f(x+t) + f(x-t) - 2f(x) - T_n(x+t) - T_n(x-t) + 2T_n(x).$$

It is evident that

$$\omega_p(\delta, T_n) \le c\omega_p(\delta, f)$$

and

$$\sum_{i=k+1}^{n} \frac{(-1)^i}{i} = \frac{(-1)^{k+1}}{2k} + P_k + Q_n, \quad P_k \le \frac{c}{k^2}, \quad Q_n \le \frac{c}{n}. \tag{1}$$

We write

$$\int_0^{\pi} g(t) D_n(t)\, dt = \int_0^{\pi} g(t)\left(D_n(t) - \frac{\sin nt}{t}\right) dt + \int_0^{\pi} g(t)\frac{\sin nt}{t} dt. \tag{2}$$

Since

$$D_n(t) - \frac{\sin nt}{t}$$

is bounded function and

$$|g(t)| \le 4E_n(f).$$

Hence

$$\left|\int_0^{\pi} g(t)\left(D_n(t) - \frac{\sin nt}{t}\right) dt\right| \le cE_n(f), \tag{3}$$

where $E_n(f)$ is best approximation of the function $f$.

On the other hand,

$$\left|\int_0^{\pi/n} g(t)\frac{\sin nt}{t}dt\right| \le \|g\|_c \int_0^{\pi/n} \frac{\sin nt}{t}dt \le cE_n(f). \tag{4}$$

Combining (2)-(4) we have

$$\int_0^{\pi} g(t) D_n(t)\,dt = \int_0^{\pi} g(t)\frac{\sin nt}{t}dt + \gamma, \tag{5}$$

where

$$|\gamma| \le cE_n(f).$$

It is evident that

$$\begin{aligned}
\left|\int_{\pi/n}^{\pi} g(t)\frac{\sin nt}{t}dt\right| &= \left|\sum_{k=1}^{n-1}\int_{\pi k/n}^{\pi(k+1)/n} g(t)\frac{\sin nt}{t}dt\right| \\
&= \left|\sum_{k=1}^{n-1}\int_0^{\pi/n} g\left(t+\frac{\pi k}{n}\right)\frac{(-1)^k \sin nt}{t+\pi k/n}dt\right| \\
&= \left|\sum_{k=1}^{n-1}\int_0^{\pi} g\left(\frac{u+\pi k}{n}\right)\frac{(-1)^k \sin u}{u+\pi k}du\right| \\
&\le \pi \max_{0\le u\le\pi}\left|\sum_{k=1}^{n-1} g\left(\frac{u+\pi k}{n}\right)\frac{(-1)^k \sin u}{u+\pi k}\right| \\
&\le \pi\left|\sum_{k=1}^{n-1} g\left(\frac{u_0+\pi k}{n}\right)\frac{(-1)^k}{u_0+\pi k}\right| \\
&\le \pi\left|\sum_{k=1}^{n-1} g\left(\frac{u_0+\pi k}{n}\right)\left(\frac{(-1)^k}{u_0+\pi k}-\frac{(-1)^k}{\pi k}\right)\right| \\
&\quad +\pi\left|\sum_{k=1}^{n-1} g\left(\frac{u_0+\pi k}{n}\right)\frac{(-1)^k}{\pi k}\right| \\
&= I + II.
\end{aligned} \tag{6}$$

where

$$\left|\sum_{k=1}^{n-1} g\left(\frac{u_0+\pi k}{n}\right)\frac{(-1)^k}{u_0+\pi k}\right| = \sup_u \left|\sum_{k=1}^{n-1} g\left(\frac{u+\pi k}{n}\right)\frac{(-1)^k}{u+\pi k}\right|.$$

It is clear that

$$I \leq \|g\|_c \sum_{k=1}^{n-1} \frac{1}{k^2} = O\left(E_n\left(f\right)\right). \tag{7}$$

Set

$$u_k := \frac{u_0 + \pi k}{n}.$$

Using Abel transformation and (1), for $II$ we obtain

$$\begin{aligned}
II &= \left| \sum_{k=1}^{n-1} g\left(u_k\right) \frac{\left(-1\right)^k}{\pi k} \right| \\
&\leq \left| \sum_{k=1}^{n-2} \left(g\left(u_{k+1}\right) - g\left(u_k\right)\right) \sum_{i=k+1}^{n-1} \frac{\left(-1\right)^i}{i} \right| + \left| g\left(u_1\right) \sum_{i=1}^{n-1} \frac{\left(-1\right)^i}{i} \right| \\
&\leq \left| \sum_{k=1}^{n-2} \left(g\left(u_{k+1}\right) - g\left(u_k\right)\right) \left( \frac{\left(-1\right)^{k+1}}{2k} + P_k + Q_n \right) \right| + \left| g\left(u_1\right) \sum_{i=1}^{n-1} \frac{\left(-1\right)^i}{i} \right| \\
&\leq \left| \sum_{k=1}^{n-2} \left(g\left(u_{k+1}\right) - g\left(u_k\right)\right) \frac{\left(-1\right)^{k+1}}{2k} \right| + O\left(E_n\left(f\right)\right) \\
&= \left| \sum_{k=1}^{n-2} \Delta_1 \left(u_k; \frac{\pi}{n}, g\right) \frac{\left(-1\right)^{k+1}}{2k} \right| + O\left(E_n\left(f\right)\right) \\
&\leq \frac{1}{2} \left| \sum_{k=1}^{n-2} \Delta_1 \left(u_k; \frac{\pi}{n}, g\right) \frac{\left(-1\right)^{k+1}}{k} \right| + \gamma_1 \\
&\leq \frac{1}{2^2} \left| \sum_{k=1}^{n-3} \Delta_2 \left(u_k; \frac{\pi}{n}, g\right) \frac{\left(-1\right)^{k+1}}{k} \right| + \gamma_1 + \gamma_2 \\
&\leq \cdots \leq \frac{1}{2^p} \left| \sum_{k=1}^{n-3} \Delta_p \left(u_k; \frac{\pi}{n}, g\right) \frac{\left(-1\right)^{k+1}}{2k} \right| + \sum_{k=1}^{p} \gamma_k
\end{aligned}$$

where

$$\gamma_k < \frac{\omega_k\left(\frac{1}{n}, f\right)}{2^k}.$$

Consequently,

$$II \leq \frac{\omega_p\left(\frac{1}{n}, f\right)}{2^p} \log n + \sum_{k=1}^{p} \frac{\omega_k\left(\frac{1}{n}, f\right)}{2^k}. \tag{8}$$

Since

$$\frac{\omega_i\left(\frac{1}{n}, f\right)}{2^i} \leq \frac{\omega_j\left(\frac{1}{n}, f\right)}{2^j} \quad (i \geq j),$$

we can write

$$\frac{\omega_p\left(\frac{1}{n}, f\right) p}{2^p} \leq \sum_{k=1}^{p} \frac{\omega_k\left(\frac{1}{n}, f\right)}{2^k}. \tag{9}$$

Set $p = [\log n]$. Then from (8) and (9) we obtain

$$II \leq \sum_{k=1}^{[\log n]} \frac{\omega_k\left(\frac{1}{n}, f\right)}{2^k}. \tag{10}$$

Combining (6), (7) and (10) complete the proof of Theorem 1. □

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