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SOME HARDY SPACE ESTIMATES FOR MULTILINEAR SINGULAR INTEGRAL OPERATOR

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Abstract. In this paper, we establish the boundedness for some multilinear singular integral operators on Hardy and Herz type Hardy spaces. The operators include Calderon-Zygmund singular integral operators.

1. Introduction

Let $b \in BMO(R^n)$ and T be the Calderon-Zygmund operator. The commutator $[b,T]$ generated by b and T is defined by $[b,T]f(x) = b(x)Tf(x) - T(bf)(x)$. By a classical result of Coifman, Rochberg and Weiss(see [6]), we know that the commutator $[b,T]$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. However, it was observed that $[b,T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ and from $L^1(R^n)$ to $L^{1,\infty}(R^n)$ for $p \le 1$. But, if $H^p(R^n)$ is replaced by a suitable atomic space $H^p_b(R^n)$(see [1, 14]), then $[b,T]$ is bounded from $H^p_b(R^n)$ to $L^p(R^n)$ for $p \in (n/(n+1), 1]$. In recent years, the theory of Herz space and Herz type Hardy space, as a local version of Lebesgue space and Hardy space, have been developed (see [8, 9, 11, 12]). The main purpose of this paper is to establish the boundedness properties of some multilinear operators related to certain non-convolution type singular integral operators on Hardy and Herz type Hardy spaces. The operators include Calderón-Zygmund singular integral operators.

2. Notations and Theorems

In this paper, we study the singular integral operators as following. Let $T : S \to S'$ be a linear operator and there exists a locally integrable function

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$K(x,y)$ on $R^n \times R^n \setminus \{(x,y) \in R^n \times R^n : x = y\}$ such that

$$Tf(x) = \int_{R^n} K(x,y)f(y)dy$$

for every bounded and compactly supported function f, where K satisfies: for fixed $\varepsilon > 0$ and $n > \delta \geq 0$,

$$|K(x,y)| \leq C|x-y|^{-n+\delta}$$

and

$$|K(y,x) - K(z,x)| + |K(x,y) - K(x,z)| \leq C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon+\delta}$$

if $2|y-z| \leq |x-z|$. Let m_i be positive integers$(i = 1, \ldots, l)$, $m_1 + \cdots + m_l = m$ and A_i be some functions on $R^n (i = 1, \ldots, l)$. The multilinear operator related to T is defined by

$$T^A(f)(x) = \int_{R^n} \frac{\prod_{i=1}^{l} R_{m_i+1}(A_i; x, y)}{|x-y|^m} K(x,y)f(y)dy,$$

where

$$R_{m_i+1}(A_i; x, y) = A_i(x) - \sum_{|\beta| \leq m_i} \frac{1}{\beta!} D^{\beta} A_i(y)(x-y)^{\beta}.$$

Note that when $m = 0$, T^A is just the multilinear commutator of T and A (see [16]). While when $m > 0$, T^A is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3, 5, 4, 13]). In [7], the weighted $L^p (p > 1)$-boundedness of the multilinear operator related to some singular integral operator are obtained. The main purpose of this paper is to study the boundedness of the multilinear singular integral operator T^A on some Hardy and Herz-Hardy spaces.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [10])

$$f^{\#}(x) \approx \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^{\infty}(R^n)$ and $||f||_{BMO} = ||f^{\#}||_{L^{\infty}}$.

Definition 1. Let A_i be some function on R^n and m_i be positive integers$(i = 1, \ldots, l)$, $m_1 + \cdots + m_l = m$ and $0 < p \leq 1$. A bounded measurable function a on R^n is said to be a $(p, D^m A)$ atom if

i) $\operatorname{supp} a \subset Q = Q(x_0, r)$,

ii) $||a||_{L^\infty} \leq |Q|^{-1/p}$,

iii) $\int_{R^n} a(y)dy = \int_{R^n} a(y) \prod_{\nu=1}^{k} D^\beta A_\nu(y)dy = 0$ for $|\beta| = m_i, i = 1, \ldots, l$ and $k = 1, \ldots, l$;

A tempered distribution f is said to belong to $H^p_{D^m A}(R^n)$, if, in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),$$

where a_j's are $(p, D^m A)$ atoms, $\lambda_j \in C$ and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover, $||f||_{H^p_{D^m A}} \approx \left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p}$.

Definition 2. Let $0 < p, q < \infty$, $\alpha \in R$. For $k \in Z$, define $B_k = \{x \in R^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.

The homogeneous Herz space is defined by

$$\dot{K}^{\alpha,p}_q(R^n) = \{f \in L^q_{loc}(R^n \setminus \{0\}) : ||f||_{\dot{K}^{\alpha,p}_q} < \infty\}, \tag{1}$$

where

$$||f||_{\dot{K}^{\alpha,p}_q} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q}\right]^{1/p}.$$

The nonhomogeneous Herz space is defined by

$$K^{\alpha,p}_q(R^n) = \{f \in L^q_{loc}(R^n) : ||f||_{K^{\alpha,p}_q} < \infty\}, \tag{2}$$

where

$$||f||_{K^{\alpha,p}_q} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q} + ||f\chi_{B_0}||^p_{L^q}\right]^{1/p}.$$

Definition 3. Let A_i be a function on R^n and m_i be positive integers ($i = 1, \ldots, l$), $m_1 + \cdots + m_l = m$, $\alpha \in R$, $0 < p < \infty$, $1 < q \leq \infty$. A function $a(x)$ on R^n is called a central $(\alpha, q, D^m A)$-atom (or a central $(a, q, D^m A)$-atom of restrict type), if

1) $\operatorname{supp} a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),

2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,

3) $\int_{R^n} a(x)dx = \int_{R^n} a(y) \prod_{i=1}^{k} D^\beta A_i(y)dy = 0$ for $|\beta| = m_i, i = 1, \ldots, l$ and $k = 1, \ldots, l$;

A tempered distribution f is said to belong to $H\dot{K}^{\alpha,p}_{q,D^m A}(R^n)$ (or $HK^{\alpha,p}_{q,D^m A}(R^n)$), if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=1}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, where a_j is a central $(\alpha, q, D^m A)$-atom (or a central $(\alpha, q, D^m A)$-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$(or $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$), moreover, $||f||_{H\dot{K}^{\alpha,p}_{q,D^m A}}$(or $||f||_{HK^{\alpha,p}_{q,D^m A}}$)$\approx \left(\sum_j |\lambda_j|^p\right)^{1/p}$.

Now, we can state our results as following.

Theorem 1. *Let* $\max(n/(n+1), n/(n+\varepsilon-\delta)) < q \le 1$, $1/q = 1/p - \delta/n$, $D^\beta A_i \in BMO(R^n)$ *for all* β *with* $|\beta| = m_i$ *and* $i = 1, \ldots, l$. *Suppose that* T^A *is bounded from* $L^s(R^n)$ *to* $L^r(R^n)$ *for any* $1 < s < n/\delta$ *and* $1/r = 1/s - \delta/n$. *Then* T^A *is bounded from* $H^p_{D^m A}(R^n)$ *to* $L^q(R^n)$.

Theorem 2. *Let* $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = \delta/n$, $n(1-1/q_1) \le \alpha < \min(n(1-1/q_1)+1, n(1-1/q_1)+\varepsilon)$ *and* $D^\beta A_i \in BMO(R^n)$ *for all* β *with* $|\beta| = m_i$ *and* $i = 1, \ldots, l$. *Suppose that* T^A *is bounded from* $L^s(R^n)$ *to* $L^r(R^n)$ *for any* $1 < s < n/\delta$ *and* $1/r = 1/s - \delta/n$. *Then* T^A *is bounded from* $H\dot{K}^{\alpha,p}_{q_1,D^m A}(R^n)$ *to* $\dot{K}^{\alpha,p}_{q_2}(R^n)$.

Remark 1. Theorem 2 is also hold for nonhomogeneous Herz and Herz type Hardy space.

3. Proofs of Theorems

To prove the theorems, we need the following lemma.

Lemma 1 (see [4]). *Let* A *be a function on* R^n *and* $D^\beta A \in L^q(R^n)$ *for* $|\beta| = m$ *and some* $q > n$. *Then*

$$|R_m(A; x, y)| \le C|x-y|^m \sum_{|\beta|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\beta A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ *is the cube centered at* x *and having side length* $5\sqrt{n}|x-y|$.

Proof of Theorem 1: It suffices to prove that there exists a constant $C > 0$ such that for every $(p, D^m A)$ atom a,

$$||T^A(a)||_{L^q} \le C.$$

Let a be a $(p, D^m A)$ atom supported on a cube $Q = Q(x_0, d)$. We write

$$\int_{R^n} |T^A(a)(x)|^q dx = \int_{2Q} |T^A(a)(x)|^q dx + \int_{(2Q)^c} |T^A(a)(x)|^q dx = I + II.$$

For I, taking $r, s > 1$ with $q < s < n/\delta$ and $1/r = 1/s - \delta/n$, by Holder's inequality and the (L^s, L^r)-boundedness of T^A, we get

$$I \le C||T^A(a)||^q_{L^r} |Q(x_0, 2d)|^{1-q/r} \le C||a||^q_{L^s} |Q|^{1-q/r} \le C|Q|^{-q/p+q/s+1-q/r} \le C.$$

To obtain the estimate of II, we need to estimate $T^A(a)(x)$ for $x \in (2Q)^c$. Without loss of generality, we may assume $l = 2$. Let $\tilde{A}_i(x) = A_i(x) - \sum_{|\beta|=m_i} \frac{1}{\beta!}(D^\beta A_i)_Q x^\beta$. Then $R_{m_i}(A_i; x, y) = R_{m_i}(\tilde{A}_i; x, y)$ and $D^\beta \tilde{A}_i = D^\beta A_i -$

$(D^\beta A_i)_Q$ for $|\beta| = m_i$. We write, by the vanishing moment of a,

$$T^A(a)(x) = \int_{R^n}\left[\frac{K(x,y)}{|x-y|^m} - \frac{K(x,x_0)}{|x-x_0|^m}\right] R_{m_1}(\tilde{A}_1;x,y)R_{m_2}(\tilde{A}_2;x,y)a(y)dy$$
$$+\int_{R^n}\frac{K(x,x_0)}{|x-x_0|^m}[R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;x,x_0)]R_{m_2}(\tilde{A}_2;x,y)a(y)dy$$
$$+\int_{R^n}\frac{K(x,x_0)}{|x-x_0|^m}[R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x,x_0)]R_{m_1}(\tilde{A}_1;x,x_0)a(y)dy$$
$$-\sum_{|\beta_2|=m_2}\frac{1}{\beta_2!}\int_{R^n}\left[\frac{K(x,y)(x-y)^{\beta_2}}{|x-y|^m} - \frac{K(x,x_0)(x-x_0)^{\beta_2}}{|x-x_0|^m}\right]\times$$
$$\times R_{m_1}(\tilde{A}_1;x,y)D^{\beta_2}\tilde{A}_2(y)a(y)dy$$
$$-\sum_{|\beta_2|=m_2}\frac{1}{\beta_2!}\int_{R^n}\frac{K(x,x_0)(x-x_0)^{\beta_2}}{|x-x_0|^m}[R_{m_1}(\tilde{A}_1;x,y) - R_{m_1}(\tilde{A}_1;x,x_0)]\times$$
$$\times D^{\beta_2}\tilde{A}_2(y)a(y)dy$$
$$-\sum_{|\beta_1|=m_1}\frac{1}{\beta_1!}\int_{R^n}\left[\frac{K(x,y)(x-y)^{\beta_1}}{|x-y|^m} - \frac{K(x,x_0)(x-x_0)^{\beta_1}}{|x-x_0|^m}\right]\times$$
$$\times R_{m_2}(\tilde{A}_2;x,y)D^{\beta_1}\tilde{A}_1(y)a(y)dy$$
$$-\sum_{|\beta_1|=m_1}\frac{1}{\beta_2!}\int_{R^n}\frac{K(x,x_0)(x-x_0)^{\beta_1}}{|x-x_0|^m}[R_{m_1}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x,x_0)]\times$$
$$\times D^{\beta_1}\tilde{A}_1(y)a(y)dy$$
$$+\sum_{|\beta_1|=m_1,|\beta_2|=m_2}\frac{1}{\beta_1!\beta_2!}\int_{R^n}\left[\frac{K(x,y)(x-y)^{\beta_1+\beta_2}}{|x-y|^m} - \frac{K(x,x_0)(x-x_0)^{\beta_1+\beta_2}}{|x-x_0|^m}\right]\times$$
$$\times D^{\beta_1}\tilde{A}_1(y)D^{\beta_2}\tilde{A}_2(y)a(y)dy$$
$$= II_1(x) + II_2(x) + II_3(x) + II_4(x) + II_5(x) + II_6(x) + II_7(x) + II_8(x).$$

By Lemma and the following inequality (see [17])

$$|b_{Q_1} - b_{Q_2}| \le C\log(|Q_2|/|Q_1|)||b||_{BMO} \quad \text{for } Q_1 \subset Q_2,$$

we know that, for $y \in Q$ and $x \in 2^{k+1}Q \setminus 2^kQ$,

$$|R_{m_i}(\tilde{A}_i;x,y)| \le C|x-y|^{m_i}\sum_{|\beta|=m_i}(||D^\beta A_i||_{BMO} + |(D^\beta A_i)_{Q(x,y)} - (D^\beta A_i)_Q|)$$
$$\le Ck|x-y|^{m_i}\sum_{|\beta|=m_i}||D^\beta A_i||_{BMO}.$$

Note that $|x-y| \sim |x-x_0|$ for $y \in Q$ and $x \in R^n \setminus 2Q$, we obtain, by the condition on K,

$|II_1(x)| \le$

$$\leq C\int_{R^n}\left(\frac{|y-x_0|}{|x-x_0|^{m+n+1-\delta}}+\frac{|y-x_0|^{\varepsilon}}{|x-x_0|^{m+n+\varepsilon-\delta}}\right)\prod_{i=1}^{2}|R_{m_i}(\tilde{A}_i;x,y)||a(y)|dy$$

$$\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\int_Q k^2\left(\frac{|y-x_0|}{|x-x_0|^{n+1-\delta}}+\frac{|y-x_0|^{\varepsilon}}{|x-x_0|^{n+\varepsilon-\delta}}\right)|a(y)|dy$$

$$\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k^2\left(\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right).$$

For $II_2(x)$, by the formula (see [5]):

$$R_{m_i}(\tilde{A}_i;x,y)-R_{m_i}(\tilde{A}_i;x,x_0)=\sum_{|\gamma|<m_i}\frac{1}{\gamma!}R_{m_i-|\gamma|}(D^{\gamma}\tilde{A}_i;x,x_0)(x-y)^{\gamma}$$

and Lemma, we have

$$|R_{m_i}(\tilde{A}_i;x,y)-R_{m_i}(\tilde{A}_i;x,x_0)|\leq C\sum_{|\gamma|<m_i}\sum_{|\beta|=m_i}|x-x_0|^{m_i-|\gamma|}|x-y|^{|\gamma|}||D^{\beta}A_i||_{BMO},$$

thus

$$|II_2(x)|\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\int_Q k\frac{|y-x_0|}{|x-x_0|^{n+1-\delta}}|a(y)|dy$$

$$\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}.$$

Similarly,

$$|II_3(x)|\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}.$$

For $II_4(x)$, similar to the proof of $II_1(x)$ and $II_2(x)$, we get

$$|II_4(x)|\leq C\sum_{|\beta_1|=m_1}||D^{\beta_1}A_1||_{BMO}\sum_{|\beta_2|=m_2}k\left(\frac{|Q|^{1/n-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right)\times$$

$$\times\int_Q|D^{\beta_2}A_2(y)-(D^{\beta_2}A_2)_Q|dy$$

$$\leq C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k\left(\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right).$$

Similarly,

$$|II_5(x)| \le C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}.$$

$$|II_6(x)| \le C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k\left(\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right).$$

$$|II_7(x)| \le C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)k\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}.$$

For $II_8(x)$, taking $1 < r_1, r_2 < \infty$ such that $1/r_1 + 1/r_2 = 1$, then, by Holder's inequality,

$$\begin{aligned}
&|II_8(x)| \le \\
&\le C\sum_{|\beta_1|=m_1,|\beta_2|=m_2}\int_{R^n}\left|\frac{K(x,y)(x-y)^{\beta_1+\beta_2}}{|x-y|^m}-\frac{K(x,x_0)(x-x_0)^{\beta_1+\beta_2}}{|x-x_0|^m}\right|\times \\
&\quad\times|D^{\beta_1}\tilde{A}_1(y)||D^{\beta_2}\tilde{A}_2(y)||a(y)|dy \\
&\le C\sum_{|\beta_1|=m_1}\left(\int_Q|D^{\alpha_1}A_1(y)-(D^{\beta_1}A_1)_Q|^{r_1}dy\right)^{1/r_1} \\
&\quad\times\sum_{|\beta_2|=m_2}\left(\int_Q|D^{\alpha_2}A_2(y)-(D^{\beta_2}A_2)_Q|^{r_2}dy\right)^{1/r_2}\left(\frac{|Q|^{1/n-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right) \\
&\le C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\left(\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right).
\end{aligned}$$

Thus, recall that $\max(n/(n+1), n/(n+\varepsilon-\delta)) < q \le 1$,

$$\begin{aligned}
II &\le \sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|T^A(a)(x)|^q dx \\
&\le C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}k^{2q}\times \\
&\quad\times\left(\frac{|Q|^{1/n+1-1/p}}{|x-x_0|^{n+1-\delta}}+\frac{|Q|^{\varepsilon/n+1-1/p}}{|x-x_0|^{n+\varepsilon-\delta}}\right)^q dx \\
&\le C\left[\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\right]^q\sum_{k=1}^{\infty}k^{2q}[2^{knq(1/p-1/n-1)}+2^{knq(1/p-\varepsilon/n-1)}]
\end{aligned}$$

$$\le C\left[\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\right]^q.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2: Without loss of generality, we may assume $l=2$. Let $f\in H\dot{K}^{\alpha,p}_{q_1,D^mA}(R^n)$ and $f(x)=\sum_{j=-\infty}^{\infty}\lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 3. We write

$$\begin{aligned}||T^A(f)||^p_{\dot{K}^{\alpha,p}_{q_2}} &\le \sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|||T^A(a_j)\chi_k||_{L^{q_2}}\right)^p\\ &+\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||T^A(a_j)\chi_k||_{L^{q_2}}\right)^p\\ &= J+JJ.\end{aligned}$$

For JJ, by the (L^{q_1},L^{q_2})-boundedness of T^A, we get

$$\begin{aligned}JJ &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|||a_j||_{L^{q_1}}\right)^p\\ &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|2^{-j\alpha}\right)^p\\ &\le \begin{cases} C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^p2^{-j\alpha p}\right), & 0<p\le 1\\ C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^p2^{-j\alpha p/2}\right)\left(\sum_{j=k-2}^{\infty}2^{-j\alpha p'/2}\right)^{p/p'}, & p>1\end{cases}\\ &\le \begin{cases} C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p}\right), & 0<p\le 1\\ C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p/2}\right)\left(\sum_{k=-\infty}^{j+2}2^{(k-j)\alpha p'/2}\right)^{p/p'}, & p>1\end{cases}\\ &\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\le C||f||^p_{H\dot{K}^{\alpha,p}_{q_1,D^mA}}.\end{aligned}$$

For J, similar to the proof of Theorem 1, we get, for $x\in C_k$, $j\le k-3$,

$$\begin{aligned}&|T^A(a_j)(x)|\le\\ &\le C\prod_{i=1}^{2}\left(\sum_{|\beta_i|=m_i}||D^{\beta_i}A_i||_{BMO}\right)\left(\frac{|B_j|^{1/n}}{|x|^{n+1-\delta}}+\frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta}}\right)\int_{R^n}|a_j(y)|dy\\ &+C\sum_{|\beta_1|=m_1}||D^{\beta_1}A_1||_{BMO}\left(\frac{|B_j|^{1/n}}{|x|^{n+1-\delta}}+\frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta}}\right)\sum_{|\beta_2|=m_2}\int_{R^n}|a_j(y)||D^{\beta_2}\tilde{A}_2(y)|dy\end{aligned}$$

$$+C\sum_{|\beta_2|=m_2}||D^{\beta_2}A_2||_{BMO}\left(\frac{|B_j|^{1/n}}{|x|^{n+1-\delta}}+\frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta}}\right)\sum_{|\beta_1|=m_1}\int_{R^n}|a_j(y)||D^{\beta_1}\tilde{A}_1(y)|dy$$

$$+C\left(\frac{|B_j|^{1/n}}{|x|^{n+1-\delta}}+\frac{|B_j|^{\varepsilon/n}}{|x|^{n+\varepsilon-\delta}}\right)\sum_{|\beta_1|=m_1,|\beta_2|=m_2}\int_{R^n}|a_j(y)||D^{\beta_1}\tilde{A}_1(y)||D^{\beta_2}\tilde{A}_2(y)|dy$$

$$\le C\left(\frac{2^{j(1+n(1-1/q_1)-\alpha)}}{|x|^{n+1-\delta}}+\frac{2^{j(\varepsilon+n(1-1/q_1)-\alpha)}}{|x|^{n+\varepsilon-\delta}}\right).$$

To be simply, denote $W(j,k)=2^{(j-k)(1+n(1-1/q_1)-\alpha)}+2^{(j-k)(\varepsilon+n(1-1/q_1)-\alpha)}$, then

$$J\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p\left[\frac{2^{j(1+n(1-1/q_1)-\alpha)}}{2^{k(n+1-\delta)}}+\frac{2^{j(\varepsilon+n(1-1/q_1)-\alpha)}}{2^{k(n+\varepsilon-\delta)}}\right]^p\right)2^{knp/q_2}$$

$$\le\begin{cases}C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}W(j,k)^p, & 0<p\le 1\\ C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\left[\sum_{k=j+3}^{\infty}W(j,k)^{p/2}\right]\left[\sum_{k=j+3}^{\infty}W(j,k)^{p'/2}\right]^{p/p'}, & p>1\end{cases}$$

$$\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\le C||f||^p_{H\dot{K}^{\alpha,p}_{q_1,D^mA}}.$$

These yield the desired result and finish the proof of Theorem 2. □

4. Examples

In this section we shall apply Theorem 1 and 2 of the paper to the Calderón-Zygmund singular integral operator.

Let T be the Calderón-Zygmund operator (see [2, 10, 15, 18], the multilinear operator related to T is defined by

$$T^A(f)(x)=\int_{R^n}\frac{\prod_{i=1}^{l}R_{m_i+1}(A_i;x,y)}{|x-y|^m}K(x,y)f(y)dy.$$

In particular, the multilinear commutator related to T is (see [11])

$$T^A(f)(x)=\int_{R^n}\left[\prod_{i=1}^{l}(A_i(x)-A_i(y))\right]K(x,y)f(y)dy.$$

Then it is easily to see that T satisfies the conditions in Theorem 1 and 2.

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