Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
28 (2012), 47–58
www.emis.de/journals
ISSN 1786-0091

A NEW PARANORMED SEQUENCE SPACE AND SOME MATRIX TRANSFORMATIONS

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ABSTRACT. In this paper, we introduce the space $r^q(u,p)$. We proved its completeness property and has shown its linear isomorphism to $l(p)$. Also, investigations have been made for computing its α-, β- and γ- duals. Furthermore, we constructed the basis of $r^q(u,p)$. Finally, we characterize the classes $(r^q(u,p) : l_\infty)$, $(r^q(u,p) : c)$ and $(r^q(u,p) : c_0)$ of infinite matrices.

1. Preliminaries, Background and Notations

A sequence space is defined to be a linear space of real or complex sequences. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers respectively. Let ω denote the space of all sequences (real or complex); l_∞ and c respectively, denotes the space of all bounded sequences, the space of convergent sequences. A linear Topological space X over the field of real numbers $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $h\colon X \to \mathbb{R}$ such that $h(\theta) = 0$,$h(-x) = h(x)$ and scalar multiplication is continuous, that is, $|\alpha_n - \alpha| \to 0$ and $h(x_n - x) \to 0$ imply $h(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha's$ in $\mathbb{R}$ and $x's$ in X , where θ is a zero vector in the linear space X. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H$and $M = \max\{1, H\}$. Then, the linear spaces $l(p)$ and $l_\infty(p)$ were defined by Maddox [7] (see also [11, 12, 16]) as follows:

$$l(p) = \{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\}, \text{ with } 0 < p_k \leq H < \infty,$$
$$l_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\},$$

2010 *Mathematics Subject Classification.* 46A45; 40C05.

Key words and phrases. Sequence space of non-absolute type; paranormed sequence space; α-, β- and γ-duals; matrix transformations.

which are complete spaces paranormed by

$$g_1(x) = \left[\sum_k |x_k|^{p_k}\right]^{1/M} \text{ and } g_2(x) = \sup_k |x_k|^{p_k/M} \text{ iff } \inf p_k > 0.$$

We shell assume throughout that $p_k^{-1} + (p_k{}')^{-1}$ provided $1 < \inf p_k \leq H < \infty$ and we denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

For the sequence space X and Y, define the set

$$S(X : Y) = \{z = (z_k) \in \omega : xz = (x_k z_k) \in Y \ \forall\ x \in X\}. \tag{1}$$

With the notation of (1) , the α-, β- and γ- duals of a sequence space X, which are respectively denoted by X^α , X^β and X^γ and are defined by

$$X^\alpha = S(X : l_1),\ X^\beta = S(X : cs) \text{ and } X^\gamma = S(X : bs).$$

If a sequence space X paranormed by h contains a sequence (b_n) with the property that for every $x \in X$ there is a unique sequence of scalars (α_n) such that

$$\lim_n h(x - \sum_{k=0}^{n} \alpha_k b_k) = 0$$

then (b_n) is called a Schauder basis (or briefly basis) for X. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum \alpha_k b_k$.

Let X and Y be two subsets of ω. Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, the matrix A defines the $A-$transformation from X *into* Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$,the A-transform of x exists and is in Y; where $(Ax)_n = \sum_k a_{nk} x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(X : Y)$,we denote the class of all such matrices. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A = \{x = (x_k) : x = (x_k) \in \omega\}. \tag{2}$$

Let (q_k) be a sequence of positive numbers and let us write,

$$Q_n = \sum_{k=0}^{n} q_k$$

for $n \in \mathbb{N}$. Then the matrix $R^q = (r^q_{nk})$ of the Riesz mean (R, q_n) is given by

$$r^q_{nk} = \begin{cases} \frac{q_k}{Q_n} & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

The Riesz mean (R, q_n) is regular if and only if $Q_n \to \infty$ as $n \to \infty$ (see, Petersen [14, p. 10]).

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors (see,[2, 3, 4, 10, 11, 13]). In the present paper, following (see, [2, 3, 4]), we introduce Riesz sequence space $r^q(u,p)$, and prove that the space $r^q(u,p)$ is a complete paranormed linear space and show it is linearly isomorphic to the space $l(p)$. We also compute α-, β- and γ-duals of the space $r^q(u,p)$. Furthermore, we construct the basis for the space $r^q(u,p)$. in the final section of the present paper we characterize the classes $(r^q(u,p) : l_\infty)$, $(r^q(u,p) : c)$ and $(r^q(u,p) : c_0)$ of infinite matrices.

2. The Riesz Sequence space $r^q(u,p)$ of non-absolute type

In the present section, we introduce Riesz sequence space $r^q(u,p)$, prove that the space $r^q(u,p)$ is a complete paranormed linear space and show it is linearly isomorphic to the space $l(p)$. We also compute α-, β- and γ-duals of the space $r^q(u,p)$. Finally, we give basis for the space $r^q(u,p)$, where $u = (u_k)$ is a sequence such that $u_k \neq 0$ for all $k \in \mathbb{N}$.

We define the Riesz sequence space $r^q(u,p)$ as the set of all sequences such that R^q transform of it is in the space $l(p)$, that is,

$$r^q(u,p) = \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \right|^{p_k} \right\} < \infty, \ (0 < p_k \leq H < \infty).$$

In the case $(u_k) = e = (1, 1, \ldots)$, the sequence spaces $r^q(u,p)$ reduces to $r^q(p)$, introduced by Altay and Başar [1].

With the notation of (2) that

$$r^q(u,p) = \{l(p)\}_{R^q}.$$

Define the sequence $y = (y_k)$, which will be used, by the R^q-transform of a sequence $x = (x_k)$, i.e.,

$$y_k = \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j. \tag{3}$$

Now, we begin with the following theorem which is essential in the text.

Theorem 2.1. *$r^q(u,p)$ is a complete linear metric space paranormed by g defined*

$$g(x) = \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \right|^{p_k} \right]^{\frac{1}{M}} \quad \textit{with } 0 < p_k \leq H < \infty.$$

Proof. The linearity of $r^q(u,p)$ with respect to the co-ordinate wise addition and scalar multiplication follows from the inequalities which are satisfied for $z, x \in r^q(u,p)$ (see[9, p. 30])

$$\left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k} u_j q_j (z_j + x_j)\right|^{p_k}\right]^{\frac{1}{M}} \tag{4}$$
$$\leq \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k} u_j q_j z_j\right|^{p_k}\right]^{\frac{1}{M}} + \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k} u_j q_j x_j\right|^{p_k}\right]^{\frac{1}{M}}$$

and for any $\alpha \in \mathbb{R}$ (see [8])

$$|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}. \tag{5}$$

It is clear that, $g(\theta) = 0$ and $g(x) = g(-x)$ for all $x \in r^q(u,p)$. Again the inequality (4) and (5), yield the subadditivity of g and

$$g(\alpha x) \leq \max\{1, |\alpha|\} g(x).$$

Let $\{x^n\}$ be any sequence of points of the space $r^q(u,p)$ such that $g(x^n - x) \to 0$ and (α_n) is a sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality,

$$g(x^n) \leq g(x) + g(x^n - x)$$

holds by subadditivity of g, $\{g(x^n)\}$ is bounded and we thus have

$$g(\alpha_n x^n - \alpha x) = \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k} u_j q_j (\alpha_n x_j^n - \alpha x_j)\right|^{p_k}\right]^{\frac{1}{M}}$$
$$\leq |\alpha_n - \alpha|\, g(x^n) + |\alpha|\, g(x^n - x)$$

which tends to zero as $n \to \infty$. That is to say that the scalar multiplication is continuous. Hence, g is paranorm on the space $r^q(u,p)$.

It remains to prove the completeness of the space $r^q(u,p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $r^q(u,p)$, where $x^i = \{x_0^i, x_2^i, \ldots\}$. Then, for a given $\epsilon > 0$ there exists a positive integer $n_0(\epsilon)$ such that

$$g(x^i - x^j) < \epsilon \tag{6}$$

for all $i, j \geq n_0(\epsilon)$. Using definition of g and for each fixed $k \in \mathbb{N}$ that

$$\left|(R^q x^i)_k - (R^q x^j)_k\right| \leq \left[\sum_k \left|(R^q x^i)_k - (R^q x^j)_k\right|^{p_k}\right]^{\frac{1}{M}} < \epsilon$$

for $i, j \geq n_0(\epsilon)$, which leads us to the fact that $\{(R^q x^0)_k, (R^q x^1)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say, $(R^q x^i)_k \to ((R^q x)_k$ as $i \to \infty$. Using these infinitely many limits $(R^q x)_0, (R^q x)_1, \ldots$, we define the sequence $\{(R^q x)_0, (R^q x)_1, \ldots\}$. From (6) for each $m \in \mathbb{N}$ and $i, j \geq n_0(\epsilon)$.

$$\sum_{k=0}^{m} \left|(R^q x^i)_k - (R^q x^j)_k\right|^{p_k} \leq g(x^i - x^j)^M < \epsilon^M. \tag{7}$$

Take any $i, j \geq n_0(\epsilon)$. First, let $j \to \infty$ in (7) and then $m \to \infty$, we obtain

$$g(x^i - x) \leq \epsilon.$$

Finally, taking $\epsilon = 1$ in (7) and letting $i \geq n_0(1)$ we have by Minkowski's inequality for each $m \in \mathbb{N}$ such that

$$\left[\sum_{k=0}^{m} |(R^q x)_k|^{p_k}\right]^{\frac{1}{M}} \leq g(x^i - x) + g(x^i) \leq 1 + g(x^i),$$

which implies that $x \in r^q(u,p)$. Since $g(x - x^i) \leq \epsilon$ for all $i \geq n_0(\epsilon)$, it follows that $x^i \to x$ as $i \to \infty$, hence we have shown that $r^q(u,p)$ is complete. $\square$

Note that one can easily see the absolute property does not hold on the space $r^q(u,p)$, that is $g(x) \neq g(|x|)$ for at least one sequence in the space $r^q(u,p)$ and this says that $r^q(u,p)$ is a sequence space of non-absolute type.

Theorem 2.2. *The Riesz sequence space $r^q(u,p)$ of non-absolute type is linearly isomorphic to the space $l(p)$, where $0 < p_k \leq H < \infty$.*

Proof. To prove the theorem, we should show the existence of a linear bijection between the spaces $r^q(u,p)$ and $l(p)$, where $0 < p_k \leq H < \infty$. With the notation of (3), define the transformation T from $r^q(u,p)$ to $l(p)$ by $x \to y = Tx$. The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y \in l(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \frac{1}{u_k q_k}\{Q_k y_k - Q_{k-1} y_{k-1}\} \text{ for } k \in \mathbb{N}.$$

Then

$$\begin{aligned} g(x) = \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k} u_j q_j x_j\right|^{p_k}\right]^{\frac{1}{M}} &= \left[\sum_k \left|\sum_{j=0}^{k} \delta_{kj} y_j\right|^{p_k}\right]^{\frac{1}{M}} \\ &= \left[\sum_k |y_k|^{p_k}\right]^{\frac{1}{M}} = g_1(y) < \infty. \end{aligned}$$

Thus, we have $x \in r^q(u,p)$. Consequently, T is surjective and is paranorm preserving. Hence, T is a linear bijection and this says us that the spaces $r^q(u,p)$ and $l(p)$ are linearly isomorphic. $\square$

First we state some lemmas which are needed in proving the theorems.

Lemma 2.3 ([5, Theorem 5.10]). *(i) Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (l(p) : l_1)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in N} a_{nk} B^{-1} \right|^{p_k'} < \infty.$$

(ii) Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (l(p) : l_1)$ if and only if

$$\sup_{N\in\mathcal{F}} \sup_k \left| \sum_{n\in N} a_{nk} B^{-1} \right|^{p_k} < \infty.$$

Lemma 2.4 ([6, Theorem 6]). *(i) Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (l(p) : l_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_n \sum_k |a_{nk} B^{-1}|^{p_k'} < \infty. \tag{8}$$

(ii) Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (l(p) : l_\infty)$ if and only if

$$\sup_{n,k} |a_{nk}|^{p_k} < \infty. \tag{9}$$

Lemma 2.5 ([6, Theorem 1]). *Let $0 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (l(p) : c)$ if and only if (8) and (9) hold and*

$$\lim_n a_{nk} = \beta_k \ for\ k \in \mathbb{N}$$

also holds.

Theorem 2.6. *Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Define the sets $D_1(u,p)$ and $D_2(u,p)$, as follows*

$$D_1(u,p) = \bigcup_{B>1} \{a = (a_k) \in \omega : \sup_{n\in F} \sum_k \left| \sum_{n\in N} (-1)^{n-k} \frac{a_n}{u_n q_n} Q_k B^{-1} \right|^{p_k'} < \infty\}$$

and

$$D_2(u,p) = \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sum_k \left| \triangle \left(\frac{a_k}{u_k q_k} \right) Q_k B^{-1} \right|^{p_k'} < \infty \ and \ \left\{ \left(\frac{a_k}{u_k q_k} Q_k B^{-1} \right)^{p_k'} \right\} \in l_\infty \right\}.$$

Then

$$[r^q(u,p)]^\alpha = D_1(u,p) \ and \ [r^q(u,p)]^\beta = [r^q(u,p)]^\gamma = D_2(u,p).$$

Proof. Let us take any $a = (a_k) \in \omega$. We can easily derive with (3) that

$$a_n x_n = \sum_{i=n-1}^{n} (-1)^{n-i} \frac{a_n}{u_n q_n} Q_i y_i = (Cy)_n \tag{10}$$

for $n \in \mathbb{N}$ where, $C = (c_{nk})$ is defined as

$$c_{nk} = \begin{cases} (-1)^{n-k} \frac{a_n}{u_n q_n} Q_k, & \text{if } n-1 \le k \le n, \\ 0, & \text{if } 0 \le k < n-1 \text{ or } k > n, \end{cases}$$

where $n, k \in \mathbb{N}$. Thus we observe by combining (10) with (i) of Lemma 2.4 that $ax = (a_n x_n) \in l_1$ whenever $x = (x_n) \in r^q(u,p)$ if and only if $Cy \in l_1$ whenever $y \in l(p)$. This gives the result that $[r^q(u,p)]^\alpha = D_1(u,p)$. Consider the equation,

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n-1} \triangle\left(\frac{a_k}{u_k q_k}\right) Q_k y_k + \frac{a_n}{u_n q_n} Q_n y_n = (Dy)_n \text{ for } n \in \mathbb{N}, \tag{11}$$

where, $D = (d_{nk})$ is defined as

$$d_{nk} = \begin{cases} \triangle\left(\frac{a_k}{u_k q_k}\right) Q_k, & \text{if } 0 \le k \le n-1, \\ \frac{a_n}{u_n q_n} Q_k, & \text{if } k = n, \\ 0, & \text{if } k > n, \end{cases}$$

where $n, k \in \mathbb{N}$. Thus we deduce from Lemma 2.5 with (11) that $ax = (a_n x_n) \in cs$ whenever $x = (x_n) \in r^q(u,p)$ if and only if $Dy \in c$ whenever $y \in l(p)$. Therefore, we derive from (8) that

$$\sum_k \left| \triangle\left(\frac{a_k}{u_k q_k}\right) Q_k B^{-1} \right|^{p_k'} < \infty \;\; and \;\; \sup_{k \in \mathbb{N}} \left| \frac{a_k}{u_k q_k} Q_k B^{-1} \right|^{p_k'} < \infty \tag{12}$$

which shows that that $[r^q(u,p)]^\beta = D_2(u,p)$.

As this, from Lemma 2.4 together with (11) that $ax = (a_k x_k) \in bs$ whenever $x = (x_n) \in r^q(u,p)$ if and only if $Dy \in l_\infty$ whenever $y = (y_k) \in l(p)$. Therefore, we again obtain the condition (12) which means that $[r^q(u,p)]^\gamma = D_2(u,p)$. $\square$

Theorem 2.7. *Let Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Define the sets $D_3(u,p)$ and $D_4(u,p)$, as follows*

$$D_3(u,p) = \left\{ a = (a_k) \in \omega : \sup_{N \in F} \sup_k \left| \sum_n (-1)^{n-k} \frac{a_n}{u_n q_n} Q_k B^{-1} \right|^{p_k} < \infty \right\}$$

and

$$D_4(u,p)$$
$$= \left\{ a = (a_k) \in \omega : \sup_k \left| \triangle\left(\frac{a_k}{u_k q_k}\right) Q_k \right|^{p_k} < \infty \text{ and } \sup_k \left| \frac{a_k}{u_k q_k} Q_k \right|^{p_k} < \infty \right\}.$$

Then $[r^q(u,p)]^\alpha = D_3(u,p)$ and $[r^q(u,p)]^\beta = [r^q(u,p)]^\beta = D_4(u,p)$.

Proof. This is obtained by proceeding as in the proof of Theorem 2.6, above by using second parts of Lemmas 2.3, 2.4 and 2.5 instead of the first parts. So, we omit the details. ☐

Theorem 2.8. *Define the sequence $b^{(k)}(q) = b_n^{(k)}(q)$ of the elements of the space $r^q(u,p)$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(q) = \begin{cases} (-1)^{n-k}\frac{Q_k}{u_n q_n}, & \text{if } k \leq n \leq k+1, \\ 0, & \text{if } \ 0 \leq n < k \text{ or } n > k+1. \end{cases}$$

Then, the sequence $b^{(k)}(q)$ is a basis for the space $r^q(u,p)$ and any $x \in r^q(u,p)$ has a unique representation of

$$x = \sum_k \lambda_k(q) b^{(k)}(q) \tag{13}$$

where, $\lambda_k(q) = (R^q x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \leq H < \infty$.

Proof. It is clear that $b^{(k)}(q) \subset r^q(u,p)$, since

$$R^q b^{(k)}(q) = e^{(k)} \in l_p \ for\ k \in \mathbb{N} \text{ and } 0 < p_k \leq H < \infty. \tag{14}$$

Let $x \in r^q(u,p)$ be given. For every non-negative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) b^{(k)}(q). \tag{15}$$

Applying R^q to (15) with (14) we obtain

$$R^q x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) R^q b^{(k)}(q) = \sum_{k=0}^{m} (R^q x)_k e^{(k)}$$

and

$$\left(R^q(x - x^{[m]})\right)_i = \begin{cases} 0, & \text{if } 0 \leq i \leq m, \\ (R^q x)_i\,, & \text{if } i > m, \end{cases}$$

where $i, m \in \mathbb{N}$. Given $\varepsilon > 0$, there exists an integer m_0 such that

$$\left(\sum_{i=m}^{\infty} |(R^q x)_i|^{p_k}\right)^{\frac{1}{M}} < \frac{\varepsilon}{2},$$

for all $m \geq m_0$. Hence

$$g\left(x - x^{[m]}\right) = \left(\sum_{i=m}^{\infty} |(R^q x)_i|^{p_k}\right)^{\frac{1}{M}} \leq \left(\sum_{i=m_0}^{\infty} |(R^q x)_i|^{p_k}\right)^{\frac{1}{M}} < \frac{\varepsilon}{2} < \varepsilon,$$

for all $m \geq m_0$, which proves that $x \in r^q(u,p)$ is represented as (13).

Let us show the uniqueness of the representation for $x \in r^q(u,p)$ given by (13). Suppose, on the contrary; that there exists a representation $x =$

$\sum_k \mu_k(q)b^k(q)$. Since the linear transformation T from $r^q(u,p)$ to $l(p)$ used in the Theorem 2.2 is continuous we have

$$(R^q x)_n = \sum_k \mu_k(q)\left(R^q b^k(q)\right)_n = \sum_k \mu_k(q)e_n^{(k)} = \mu_n(q)$$

for $n \in \mathbb{N}$, which contradicts the fact that $(R^q x)_n = \lambda_n(q)$ for all $n \in \mathbb{N}$. Hence, the representation (13) is unique. $\square$

3. Matrix Mappings on the space $r^q(u,p)$

In this section, we characterize the matrix mappings from the space $r^q(u,p)$ to the spaces l_∞, c and c_0.

Theorem 3.1. *(i) Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p) : l_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$C(B) = \sup_n \sum_k \left|\triangle\left(\frac{a_{nk}}{u_k q_k}\right) Q_k B^{-1}\right|^{p_k'} < \infty \tag{16}$$

$$\left\{\left(\frac{a_{nk}}{u_k q_k} Q_k B^{-1}\right)^{p_k}\right\} \in l_\infty \text{ for } n \in \mathbb{N}. \tag{17}$$

(ii) Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p) : l_\infty)$ if and only if

$$\sup_n \left|\triangle\left(\frac{a_{nk}}{u_k q_k}\right) Q_k\right|^{p_k} < \infty. \tag{18}$$

Proof. We only prove the part (i) and (ii) may be proved in a similar fashion. So, let $A \in (r^q(u,p) : l_\infty)$ and $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then Ax exists for $x \in r^q(u,p)$ and implies that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{r^q(u,p)\}^\beta$ for each $n \in \mathbb{N}$. Now the necessities of (16) and (17) hold and $x \in r^q(u,p)$. In this situation, since $\{a_{nk}\}_{k\in\mathbb{N}} \in \{r^q(u,p)\}^\beta$ for every fixed $n \in \mathbb{N}$, the A-transform of x exists. Consider the following equality obtained by using the relation (3) that

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m-1} \triangle\left(\frac{a_{nk}}{u_k q_k}\right) Q_k y_k + \frac{a_{nk}}{u_m q_m} Q_m y_m \text{ for } m, n \in \mathbb{N}. \tag{19}$$

Taking into account the assumptions we derive from (19) as $m \to \infty$ that

$$\sum_k a_{nk}x_k = \sum_k \triangle\left(\frac{a_{nk}}{u_k q_k}\right) Q_k y_k \text{ for } n \in \mathbb{N}. \tag{20}$$

Now, by combining (20) and the inequality which holds for any $B > 0$ and any complex numbers a, b

$$|ab| \leq B\left(\left|aB^{-1}\right|^{p'} + |b|^p\right)$$

with $p^{-1} + (p')^{-1} = 1$ (see [6]), one can easily see that

$$\sup_{n\in\mathbb{N}}\left|\sum_k a_{nk}x_k\right| \leq \sup_{n\in\mathbb{N}}\left|\sum_k \triangle\left(\frac{a_{nk}}{u_kq_k}\right)Q_k\right||y_k|$$
$$\leq B\left[C(B)+g_1^B(y)\right]<\infty. \quad \square$$

Theorem 3.2. *Let $0<p_k\leq H<\infty$ for every $k\in\mathbb{N}$. Then $A\in(r^q(u,p):c)$ if and only if (16),(17) and (18) hold and there is a sequence (α_k) of scalars such that*

$$\lim_n \triangle\left(\frac{a_{nk}-\alpha_k}{u_kq_k}\right)Q_k=0, forevery\ k\in\mathbb{N}. \tag{21}$$

Proof. Let $A\in(r^q(u,p):c)$ and $1<p_k\leq H<\infty$ for every $k\in\mathbb{N}$. Since $c\subset l_\infty$, the necessities of (16) and (17) are trivial by Theorem 3.1 (above).

Because of $A-$transform of every $x\in r^q(u,p)$ exists and is in c by hypothesis, $Ax^{(k)}$ is also in c for every $k\in\mathbb{N}$, which shows that $\left\{\triangle\left(\frac{a_{nk}}{u_kq_k}\right)Q_k\right\}_{n\in\mathbb{N}}\in c$ where $x^{(k)}=\left\{x_n^{(k)}(q)\right\}$ is defined by

$$x_n^{(k)}(q)=\begin{cases}(-1)^{n-k}\frac{Q_k}{u_nq_n}, & \text{if } k\leq n\leq k+1,\\ 0, & \text{if } 0\leq n<k \text{ or } n>k+1,\end{cases}$$

and is in the space $r^q(u,p)$ for every $k\in\mathbb{N}$. This proves necessity of (21).

Conversely, suppose that (16), (17) and (21) hold, and $x\in r^q(u,p)$. Then $\{a_{nk}\}\in\{r^q(u,p)\}^\beta$ for each $n\in\mathbb{N}$, which implies that Ax exists. We observe for every $m,n\in\mathbb{N}$ that

$$\sum_{k=0}^m\left|\triangle\left(\frac{a_{nk}}{u_kq_k}\right)Q_kB^{-1}\right|^{p_k}\leq \sup_n\sum_k\left|\triangle\left(\frac{a_{nk}}{u_kq_k}\right)Q_kB^{-1}\right|^{p_k}$$

which gives the fact by letting $m,n\to\infty$ with (21) and (16) that

$$\sum_k\left|\triangle\left(\frac{\alpha_k}{u_kq_k}\right)Q_kB^{-1}\right|^{p_k}<\infty. \tag{22}$$

Also, we have from (17) by letting $n\to\infty$ that $\left\{\left(\frac{a_{nk}}{u_kq_k}Q_kB^{-1}\right)^{p_k}\right\}\in l_\infty$ which leads us together (22) that $(\alpha_k)\in D_2(u,p)$. That is to say that the series $\sum_k\alpha_kx_k$ converges for every $x\in r^q(u,p)$.

Let us now consider the equality obtained from (20) with $a_{nk}-\alpha_k$ instead of a_{nk}:

$$\sum_k(a_{nk}-\alpha_k)x_k=\sum_k\triangle\left(\frac{a_{nk}-\alpha_k}{u_kq_k}\right)Q_ky_k\ for\ n\in\mathbb{N}. \tag{23}$$

Thus, we have at this stage from Lemma 2.6 with $\beta_k=0$ for all $k\in\mathbb{N}$ that the matrix $\left\{\triangle\left(\frac{a_{nk}-\alpha_k}{q_k}\right)Q_k\right\}_{n,k\in\mathbb{N}}$ belong to a class of $(l(p):c_0)$. Thus, we see by (23) that

$$\lim_n \sum_k (a_{nk} - \alpha_k)x_k = 0. \tag{24}$$

Combining (24) with the above results one can see that $Ax \in c$, which is what we wished to prove. □

Now, if we take $\alpha_k = 0$ for each $k \in \mathbb{N}$, we have the following corollary:

Corollary 1. *Let $0 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (r^q(u,p) : c_0)$ if and only if (16), (17) and (18)hold and (20) also holds with $\alpha_k = 0$ for each $k \in \mathbb{N}$.*

References

[1] B. Altay and F. Başar. On the paranormed Riesz sequence spaces of non-absolute type. *Southeast Asian Bull. Math.*, 26(5):701–715, 2003.

[2] F. Başar. Infinite matrices and Cesàro sequence spaces of non-absolute type. *Math. J. Ibaraki Univ.*, 31:1–12, 1999.

[3] F. Başar and B. Altay. Matrix mappings on the space $bs(p)$ and its α-, β- and γ-duals. *Aligarh Bull. Math.*, 21(1):79–91, 2002.

[4] B. Choudhary and S. K. Mishra. On Köthe-Toeplitz duals of certain sequence spaces and their matrix transformations. *Indian J. Pure Appl. Math.*, 24(5):291–301, 1993.

[5] K.-G. Grosse-Erdmann. Matrix transformations between the sequence spaces of Maddox. *J. Math. Anal. Appl.*, 180(1):223–238, 1993.

[6] C. G. Lascarides and I. J. Maddox. Matrix transformations between some classes of sequences. *Proc. Cambridge Philos. Soc.*, 68:99–104, 1970.

[7] I. J. Maddox. Spaces of strongly summable sequences. *Quart. J. Math. Oxford Ser. (2)*, 18:345–355, 1967.

[8] I. J. Maddox. Paranormed sequence spaces generated by infinite matrices. *Proc. Cambridge Philos. Soc.*, 64:335–340, 1968.

[9] I. J. Maddox. *Elements of functional analysis*. Cambridge University Press, Cambridge, second edition, 1988.

[10] E. Malkowsky. *Matrix transformtion between spaces of absolutely and strongly summable sequences*. Habilitationsschrift, Giessen, 1988.

[11] Mursaleen. Matrix transformations between some new sequence spaces. *Houston J. Math.*, 9(4):505–509, 1983.

[12] H. Nakano. Modulared sequence spaces. *Proc. Japan Acad.*, 27:508–512, 1951.

[13] P. N. Ng and P. Y. Lee. Cesàro sequence spaces of non-absolute type. *Comment. Math. Prace Mat.*, 20(2):429–433, 1977/78.

[14] G. M. Petersen. *Regular matrix transformations*. McGraw-Hill Publishing Co., Ltd., London-New York-Toronto, Ont., 1966.

[15] S. Simons. The sequence spaces $l(p_\nu)$ and $m(p_\nu)$. *Proc. London Math. Soc. (3)*, 15:422–436, 1965.

[16] A. Wilansky. *Summability through functional analysis*, volume 85 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1984. Notas de Matemática [Mathematical Notes], 91.

Received November 11, 2011.

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