Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
28 (2012), 59–68
www.emis.de/journals
ISSN 1786-0091

RICCI SOLITONS IN LORENTZIAN α-SASAKIAN MANIFOLDS

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Abstract. We study Ricci solitons in Lorentzian α-Sasakian manifolds. It is shown that a symmetric parallel second order covariant tensor in a Lorentzian α-Sasakian manifold is a constant multiple of the metric tensor. Using this it is shown that if $\mathcal{L}_V g + 2S$ is parallel, V is a given vector field then (g, V) is Ricci soliton. Further, by virtue of this result Ricci solitons for $(2n+1)$-dimensional Lorentzian α-Sasakian manifolds are obtained. Next, Ricci solitons for 3-dimensional Lorentzian α-Sasakian manifold whose scalar curvature is constant are obtained.

1. Introduction

Ricci flow is an excellent tool for simplifying the structure of a manifold and smooth out the topology of that manifold to make it look more symmetric. It is defined for Riemannian manifolds of any dimension. It is a process which deforms the metric of a Riemannian manifold analogous to the diffusion of heat there by smoothing out the regularity in the metric. It is given by

$$\frac{\partial g}{\partial t} = -2 \operatorname{Ric} g.$$

For example, if $ds^2 = e^{2p(x,y)}(dx^2 + dy^2)$, then to compute the Ricci tensor and Laplace-Beltrami operator for two dimensional Riemannian manifold we use the differential forms method of Elie Cartan. We obtain an expression for the Ricci flow:

$$\frac{\partial p}{\partial t} = \triangle p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}.$$

2010 *Mathematics Subject Classification.* 53C15, 53C20, 53C21, 53C25, 53D10.

Key words and phrases. Ricci soliton, Lorentzian metric, Sasakian manifold, Einstein.

The authors express their thanks to Department of Science and Technology, Government of India for providing financial assistance under major research project No.SR/S4/MS:482/07.

This is manifestly analogous to the best known of all diffusion equations, the heat equation that is,

$$\frac{\partial T}{\partial t} = \triangle T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2},$$

where now $\triangle = D_x^2 + D_y^2$ is the usual Laplacian on the Euclidean plane.

Let $X(t)$ be a time dependent family of smooth vector fields on M generated by a family of diffeomorphisms $\{\phi_t : t \in R\}$ that is one parameter group of transformations, then the relation between $f\colon M \to R$ and $\{\phi_t : t \in R\}$ is

$$X(\phi_t(p))f = \frac{df \circ \phi_t}{dt}(p).$$

Let $\sigma(t)$ be a smooth function of time. Since $\phi_t\colon M \to M$ is a diffeomorphism and $g(t)$ is a Riemannian metric on M (codomain) then by definition of pull back $\phi_t^* g(t)$ is a metric on M (domain).

Set $\tilde{g}(t) = \sigma(t)\phi_t^*(g(t))$ then we have [21]

$$\frac{\partial \tilde{g}}{\partial t} = \sigma^{'}(t)\phi_t^*(g(t)) + \sigma(t)\phi_t^*\frac{\partial g}{\partial t} + \sigma(t)\phi_t^*(L_X g). \tag{1.1}$$

Suppose we have a metric g_0, a vector field Y and $\lambda \in R$ (all independent of time) such that

$$\mathcal{L}_Y g_0 + 2\,\mathrm{Ric}\, g_0 + 2\lambda g_0 = 0. \tag{1.2}$$

If we choose $g(t) = g_0$, $\sigma(t) = 1 - 2\lambda t$ and $X(t) = \frac{1}{\sigma(t)}Y$ which gives a family of diffeomorphisms ϕ_t with ϕ_0 identity then using (1.2) in (1.1) $\tilde{g}$ defined above is a Ricci flow with $g(0) = g_0$ that is

$$\frac{\partial \tilde{g}}{\partial t} = -2\,\mathrm{Ric}\,\tilde{g}. \tag{1.3}$$

Hence $\mathcal{L}_X g_0 + 2\,\mathrm{Ric}\, g_0 + 2\lambda g_0 = 0$ is a solution of the Ricci flow and is known as Ricci soliton.

Hereafter, we use the notation S instead of Ric for Ricci tensor.

Thus a Ricci soliton on a Riemannian manifold is defined by

$$\mathcal{L}_X g + 2S + 2\lambda g = 0. \tag{1.4}$$

It is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

An η-Ricci soliton introduced in the paper [3] as a data (g, V, λ, μ) :

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0. \tag{1.5}$$

1.1. **Example (Hamilton Cigar Soliton).** Let $M = R^2$ and $\phi_t\colon R^2 \to R^2$ defined by $\phi_t(x,y) = (e^{-2t}x, e^{-2t}y)$ forms a family of one parameter group of diffeomorphisms. The vector field X generated by $\{\phi_t\}$ is $X = -2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$. The metric g_0 is obtained as $g_0 = \frac{dx^2+dy^2}{1+x^2+y^2}$, $\tilde{g}(t) = \phi_t^*(g_0) = \frac{dx^2+dy^2}{e^{4t}+x^2+y^2}$, $\mathrm{Ric}\, g_0 = \frac{2}{1+x^2+y^2}g_0$, $\mathcal{L}_X g_0 = \frac{4}{1+x^2+y^2}g_0$. Using (1.4) we have $\lambda = 0$. Hence this Ricci soliton is steady and is called cigar soliton because it is a asymptotic to a flat cylinder at infinity.

In 1923, Eisenhart [7] proved that if a positive definite Riemannian manifold (M, g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible. In 1925, Levy [12] has obtained the necessary and sufficient conditions for the existence of such tensors. Recently Sharma [9] and [19] has generalized Levy's result by showing that a second order parallel(not necessarily symmetric and non singular) tensor on an n-dimensional $(n > 2)$ space of constant curvature is a constant multiple of the metric tensor. Sharma has also proved in [16] that on a Sasakian manifold there is no nonzero parallel 2-form. In 1964, Y. Wong [23] proved that the existence of linear connections w.r.t which given tensor fields are parallel or recurrent. Also the parallelism of h is involved and appears in his paper as the theory of totally geodesic maps, and $\nabla h = 0$ is equivalent with the fact that $I\colon (M, g) \to (M, h)$ is a totally geodesic map. In 2007, Lovejoy Das [5] in his paper proved that a second order symmetric parallel tensor on an α-K-contact $(\alpha \in R_0)$ manifold is a constant multiple of the associated metric tensor and he also proved that there is no nonzero skew symmetric second order parallel tensor on an α-Sasakian manifold.

Constantin Calin and Mircea Crasmareanu [2] have extended the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds. They have studied the case of f-Kenmotsu manifolds satisfying a special condition called regular and show that a symmetric parallel tensor field of second order is a constant multiple of the Riemannian metric. Using this result they have obtained results on Ricci solitons concerned to f-Kenmotsu manifolds and 3-dimensional β-Kenmotsu manifolds.

2. Basic concepts of Lorentzian α-Sasakian manifolds

A differentiable manifold of dimension $(2n + 1)$ is called Lorentzian α-Sasakian manifold if it admits a $(1, 1)$ tensor field ϕ, a vector field ξ and 1-form η and Lorentzian metric g which satisfy on M respectively such that,

$$\phi^2 = I + \eta \otimes \xi,\ \eta(\xi) = -1,\ \eta \circ \phi = 0,\ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),\ g(X, \xi) = \eta(X), \tag{2.2}$$

$$\nabla_X \xi = \alpha\phi X,\ (\nabla_X \eta)Y = \alpha g(\phi X, Y), \tag{2.3}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g on M.

Further, on an Lorentzian α-Sasakian manifold M the following relations hold:

$$R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.4}$$

$$R(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X], \tag{2.5}$$

$$S(X, \xi) = 2n\alpha^2\eta(X), \tag{2.6}$$

$$Q\xi = 2n\alpha^2\xi, \tag{2.7}$$

$$S(\xi,\xi) = -2n\alpha^2, \tag{2.8}$$

where α is some constant, R is the Riemannian curvature, S is the Ricci curvature and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

2.1. **Example.** We consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3\}$, where (x,y,z) are the standard co-ordinates in R^3. Let $\{E_1,E_2,E_3\}$ be linearly independent global frame field on M given by

$$E_1 = e^z\frac{\partial}{\partial y},\ E_2 = e^z(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}),\ E_3 = k\frac{\partial}{\partial z}. \tag{2.9}$$

Let g be the Riemannian metric defined by

$$g(E_1,E_2) = g(E_2,E_3) = g(E_1,E_3) = 0,$$
$$g(E_1,E_1) = g(E_2,E_2) = 1, g(E_3,E_3) = -1,$$

where g is given by

$$g = \frac{1}{e^{2z}}[dx\otimes dx + dy\otimes dy] - \frac{1}{k^2}dz\otimes dz.$$

The (ϕ,ξ,η) is given by

$$\eta = \frac{1}{k}dz,\ \xi = E_3 = k\frac{\partial}{\partial z}, \phi E_1 = -E_1,\ \phi E_2 = -E_2,\ \phi E_3 = 0.$$

The linearity property of ϕ and g yields that

$$\eta(E_3) = -1,\ \phi^2 U = U + \eta(U)E_3,\ g(\phi U,\phi W) = g(U,W) + \eta(U)\eta(W),$$

for any vector fields U,W on M. By definition of Lie bracket, we have

$$[E_1,E_2] = 0,\quad [E_1,E_3] = -kE_1,\quad [E_2,E_3] = -kE_2.$$

Let ∇ be Levi-Civita connection with respect to the above metric g given by Koszul formula

$$\begin{aligned} 2g(\nabla_X Y,Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned} \tag{2.10}$$

Then

$$\begin{aligned} &\nabla_{E_1}E_1 = -kE_3,\ \nabla_{E_1}E_2 = 0,\ \nabla_{E_1}E_3 = -kE_1, \\ &\nabla_{E_2}E_1 = 0,\ \nabla_{E_2}E_2 = -kE_3,\ \nabla_{E_2}E_3 = -kE_2, \\ &\nabla_{E_3}E_1 = 0,\ \nabla_{E_3}E_2 = 0,\ \nabla_{E_3}E_3 = 0. \end{aligned} \tag{2.11}$$

The tangent vectors X and Y to M are expressed as linear combination of E_1,E_2,E_3, that is $X = \sum_{i=1}^3 a_iE_i$ and $Y = \sum_{i=1}^3 b_iE_i$ where $a_i,b_i (i = 1,2,3)$ are scalars. Clearly (ϕ,ξ,η,g) and X,Y satisfy equations (2.1), (2.2) and (2.3) with $\alpha = k$. Thus M is a Lorentzian α-Sasakian manifold.

Definition 1. Let M be a Riemannian manifold with metric g, ξ an unitary vector field, η the 1-form dual to ξ. Further, let h a symmetric tensor field of $(0,2)$-type on M which we suppose to be parallel with respect to ∇ that is $\nabla h = 0$. Applying the Ricci identity [16]

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0, \tag{2.12}$$

we obtain the relation [16]:

$$h(R(X,Y)Z,W) + h(Z,R(X,Y)W) = 0, \tag{2.13}$$

then by taking $Z = W = \xi$ in (2.13) it reduces to

$$A[\eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)] = 0, \tag{2.14}$$

where $A \neq 0$ is some scalar function then M is called regular (that is $M_A^{(2n+1)}(\xi)$ is called regular if $A \neq 0$).

3. Parallel symmetric second order tensors and Ricci solitons in Lorentzian α-Sasakian manifolds

Fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to ∇ that is $\nabla h = 0$. Applying the Ricci identity [16] in (2.12) we obtain (2.13). Replacing $Z = W = \xi$ in (2.13) and using (2.4) and by the symmetry of h, we have

$$2\alpha^2[\eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)] = 0. \tag{3.1}$$

Put $X = \xi$ in (3.1), we have

$$2\alpha^2[\eta(Y)h(\xi,\xi) + h(Y,\xi)] = 0. \tag{3.2}$$

Since $2\alpha^2 \neq 0$, by definition (1) Lorentzian α-Sasakian manifold is regular.

By (3.2), we have

$$h(Y,\xi) = -\eta(Y)h(\xi,\xi). \tag{3.3}$$

Differentiating (3.3) covariantly with respect to X, we have

$$\begin{aligned}(\nabla_X h)(Y,\xi) + h(\nabla_X Y,\xi) + h(Y,\nabla_X \xi) = \\ -[(\nabla_X \eta)(Y) + \eta(\nabla_X Y)]h(\xi,\xi) \\ -\eta(Y)[(\nabla_X h)(\xi,\xi) + 2h(\nabla_X \xi,\xi)].\end{aligned} \tag{3.4}$$

By using (2.2), (2.3) and (3.3), we have

$$-h(Y,\phi X) = g(Y,\phi X)h(\xi,\xi), \tag{3.5}$$

we deduce the above equation then we have

$$h(X,Y) = -g(X,Y)h(\xi,\xi), \tag{3.6}$$

which together with the standard fact that the parallelism of h implies the $h(\xi,\xi)$ is a constant and via (3.3) yields the following:

Theorem 3.1. *A symmetric parallel second order covariant tensor in a regular Lorentzian α-Sasakian manifolds is a constant multiple of the metric tensor.*

Corollary 1. *A locally Ricci symmetric* ($\nabla S = 0$) *regular Lorentzian α-Sasakian manifolds is an Einstein manifold.*

Remark: The following statements for Lorentzian α-Sasakian manifolds are equivalent. The manifold is

(i) Einstein
(ii) locally Ricci symmetric
(iii) Ricci semi-symmetric that is $R \cdot S = 0$.

The implication (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is trivial. Now we prove the implication (iii) $\Longrightarrow$ (i) and $R \cdot S = 0$ means exactly (2.13) with replaced h by S that is

$$(R(X,Y) \cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V). \tag{3.7}$$

Considering $R \cdot S = 0$ and putting $X = \xi$ in equation (3.7), we have

$$S(R(\xi,Y)U,V) + S(U,R(\xi,Y)V) = 0. \tag{3.8}$$

By using (2.5) and (2.6), we obtain

$$\begin{aligned} 2n\alpha^4 g(Y,U)\eta(V) - \alpha^2\eta(U)S(Y,V) + 2n\alpha^4 g(Y,V)\eta(U) \\ - \alpha^2\eta(V)S(U,Y) = 0. \end{aligned} \tag{3.9}$$

Again by putting $U = \xi$ in the above equation and by using (2.1), (2.2) and (2.6), we obtain

$$S(Y,V) = 2n\alpha^2 g(Y,V). \tag{3.10}$$

In conclusion:

Proposition 1. *A Ricci semi-symmetric regular Lorentzian α-Sasakian manifolds is Einstein.*

We close this section with applications of our Theorem to Ricci solitons:

Corollary 2. *Suppose that on a regular Lorentzian α-Sasakian manifolds the $(0,2)$-type field $\mathcal{L}_V g + 2S$ is parallel where V is a given vector field. Then (g,V) yield a Ricci soliton. In particular, if the given regular Lorentzian α-Sasakian manifold is Ricci-semi symmetric with $\mathcal{L}_V g$ parallel, we have the same conclusion.*

Proof. Follows from Theorem 3.1 and Corollary 1. □

Naturally, two situations appear regarding the vector field V: $V \in Span\,\xi$ and $V \perp \xi$ but the second class seems far too complex to analyse in practice. For this reason it is appropriate to investigate only the case $V = \xi$.

We are interested in expressions for $\mathcal{L}_\xi g + 2S$. A straightforward computation gives

$$(\mathcal{L}_\xi g)(X,Y) = 2\alpha g(\phi X,Y). \tag{3.11}$$

The metric g is called η-Einstein if there exists two real functions a and b such that the Ricci tensor of g is

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y). \tag{3.12}$$

Let $e_i = 1, 2, \ldots, (2n+1)$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (3.12) and taking summation over i then we get

$$r = (2n+1)a - b. \tag{3.13}$$

Again putting $X = Y = \xi$ in (3.12) then by using (2.1), (2.2) and (2.8), we have

$$-a + b = -2n\alpha^2, \tag{3.14}$$

from (3.13) and (3.14), we obtain the values of a and b

$$a = \frac{r}{2n} - \alpha^2,\ b = \frac{r}{2n} - (2n+1)\alpha^2.$$

Substituting the values of a and b in (3.12), we have

$$S(X,Y) = \left[\frac{r}{2n} - \alpha^2\right] g(X,Y) + \left[\frac{r}{2n} - (2n+1)\alpha^2\right] \eta(X)\eta(Y). \tag{3.15}$$

The above equation shows that Lorentzian α-Sasakian manifold is η-Einstein.

For $(2n+1)$-dimensional Lorentzian α-Sasakian manifolds, we have

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{3.16}$$

Then in (3.16) substituting the values of (3.11) and (3.15), we have

$$\begin{aligned} h(X,Y) = 2\alpha g(\phi X, Y) + \left[\frac{r}{n} - 2\alpha^2\right] g(X,Y) \\ + \left[\frac{r}{n} - 2(2n+1)\alpha^2\right] \eta(X)\eta(Y). \end{aligned} \tag{3.17}$$

Differentiating the above equation (3.17) with respect to Z then we have

$$\begin{aligned} (\nabla_Z h)(X,Y) = {} & \\ & 2(Z\alpha)g(\phi X,Y) + \left[\frac{\nabla_Z r}{n} - 4\alpha(Z\alpha)\right] g(X,Y) \\ & + \left[\frac{\nabla_Z r}{n} - 4(2n+1)\alpha(Z\alpha))\right] \eta(X)\eta(Y) + 2\alpha g((\nabla_Z\phi)X,Y) \\ & + \left[\frac{r}{n} - 2(2n+1)\alpha^2\right] \{\alpha g(X,\phi Z)\eta(Y) + \alpha g(Y,\phi Z)\eta(X)\}, \end{aligned} \tag{3.18}$$

by substituting $Z = \xi$ and $X = Y \in (Span\,\xi)^\perp$ in the above equation, we have

$$\nabla_\xi r = 0, \tag{3.19}$$

provided h is parallel. Thus r is constant scalar, then we state that:

Proposition 2. *An η-Einstein Lorentzian α-Sasakian Ricci soliton (g, ξ, λ) with constant scalar curvature r is shrinking.*

Proof. From equation (1.4) and (3.16), we have

$$h(X,Y) = -2\lambda g(X,Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$h(\xi,\xi) = 2\lambda. \tag{3.20}$$

Now considering (3.17), that is

$$h(X,Y) = 2\alpha g(\phi X,Y) + \left[\frac{r}{n} - 2\alpha^2\right] g(X,Y) + \left[\frac{r}{n} - 2(2n+1)\alpha^2\right] \eta(X)\eta(Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$h(\xi,\xi) = -4n\alpha^2. \tag{3.21}$$

By equating (3.20) and (3.21), we have

$$\lambda = -2n\alpha^2. \tag{3.22}$$

This shows that $\lambda < 0$ that is the Ricci soliton in $(2n+1)$-dimensional Lorentzian α-Sasakian is shrinking. □

We compute an expression for Ricci tensor for 3-dimensional Lorentzian α-Sasakian manifold as follows: The curvature tensor for 3-dimensional Riemannian manifold is given by

$$\begin{aligned} R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.23}$$

put $Z = \xi$ in the above equation that is in (3.23) and by using (2.2), (2.4) and (2.6), we obtain

$$\left[\frac{r}{2} - \alpha^2\right] [\eta(Y)X - \eta(X)Y] = \eta(Y)QX - \eta(X)QY. \tag{3.24}$$

Again put $Y = \xi$ in the equation (3.24) and by using (2.1) and (2.7), we have

$$QX = \left[\frac{r}{2} - \alpha^2\right] X + \left[\frac{r}{2} - 3\alpha^2\right] \eta(X)\xi \tag{3.25}$$

and

$$S(X,Y) = \left[\frac{r}{2} - \alpha^2\right] g(X,Y) + \left[\frac{r}{2} - 3\alpha^2\right] \eta(X)\eta(Y), \tag{3.26}$$

where r is the scalar curvature and α is a constant.

For a 3-dimensional Lorentzian α-Sasakian manifolds, we obtain

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{3.27}$$

By using (3.11) and (3.26) in (3.27), we have

$$h(X,Y) = 2\alpha g(\phi X,Y) + [r - 2\alpha^2]g(X,Y) + [r - 6\alpha^2]\eta(X)\eta(Y). \tag{3.28}$$

Differentiating the above equation with respect to Z then we have

$$\begin{aligned} (\nabla_Z h)(X,Y) = \\ 2(Z\alpha)g(\phi X,Y) + 2\alpha g((\nabla_Z \phi)X,Y) + [\nabla_Z r - 4\alpha(Z\alpha)]g(X,Y) \end{aligned} \tag{3.29}$$

$$+ [\nabla_Z r - 6(2\alpha(Z\alpha))]\eta(X)\eta(Y) + (r - 6\alpha^2)[\alpha g(X, \phi Z)\eta(Y) + \alpha g(Y, \phi Z)\eta(X)].$$

Substituting $Z = \xi$, $X = Y \in (\text{Span } \xi)^{\perp}$ in (3.29) then we have

$$\nabla_\xi r = 0, \tag{3.30}$$

provided h is parallel. Thus r is constant scalar, then we state that:

Proposition 3. *A Ricci soliton (g, ξ, λ) in 3-dimensional Lorentzian α-Sasakian manifold with constant scalar curvature r is shrinking.*

Proof. From equation (1.4) and (3.27), we have

$$h(X, Y) = -2\lambda g(X, Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$h(\xi, \xi) = 2\lambda. \tag{3.31}$$

Now considering (3.28), that is

$$h(X, Y) = 2\alpha g(\phi X, Y) + [r - 2\alpha^2]g(X, Y) + [r - 6\alpha^2]\eta(X)\eta(Y).$$

Putting $X = Y = \xi$ in the above equation, we have

$$h(\xi, \xi) = -4\alpha^2. \tag{3.32}$$

By equating (3.31) and (3.32), we have

$$\lambda = -2\alpha^2. \tag{3.33}$$

This shows that $\lambda < 0$ that is the Ricci soliton in 3-dimensional Lorentzian α-Sasakian is shrinking. □

Acknowledgement. The authors are grateful to the referee in revising the paper.

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Received March 14, 2011.

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