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RICCI CURVATURE OF QUATERNION SLANT SUBMANIFOLDS IN QUATERNION SPACE FORMS

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Abstract. In this article, we obtain sharp estimate of the Ricci curvature of quaternion slant, bi-slant and semi-slant submanifolds in a quaternion space form, in terms of the squared mean curvature.

1. Introduction

In [15], S. Ishihara defined a quaternion manifold (or quaternion Kaehlerian manifold) as a Riemannian manifold whose holonomy group is a subgroup of $\mathbf{Sp}(1)$. It is well known that on a quaternion manifold $\tilde{M}$, there exists a 3-dimensional vector bundle E of tensors of type $(1,1)$ with local cross-section of almost Hermitian structures satisfying certain conditions [4]. A submanifold M in a quaternion manifold $\tilde{M}$ is called a quaternion submanifold if each tangent space of M is carried into itself by each section of E. In [3] authors studied quaternion CR-submanifolds of quaternion manifolds. A quaternion manifold is a quaternion space form if its quaternion sectional curvatures are constant. In [17] authors established a sharp relationship between the Ricci curvature and squared mean curvature of a quaternion CR-submanifold in a quaternion space form. Slant submanifolds of Kaehler manifolds were defined by B. Y. Chen [10] and studied by several geometers [20, 23].

On the other hand, N. Papaghiuc [18] introduced a class of submanifolds in an almost Hermitian manifold, called the semi-slant submanifolds which include proper CR-submanifolds and proper slant submanifolds as particular cases. The purpose of present paper is to study quaternion slant, bi-slant and semi-slant submanifolds in a quaternion space form.

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2. Preliminaries

Let $\tilde{M}$ be a $4m$-dimensional Riemannian manifold with metric tensor g. Then $\tilde{M}$ is said to be a quaternion Kaehlerian manifold, if there exists a 3-dimensional vector bundle E consisting of tensors of type $(1,1)$ with local basis of almost Hermitian structures J_1, J_2 and J_3 such that

(a)

$$J_1^2 = -I,\ J_2^2 = -I,\ J_3^2 = -I,$$
$$J_1J_2 = -J_2J_1 = J_3,\ J_2J_3 = -J_3J_2 = J_1,\ J_3J_1 = -J_1J_3 = J_2,$$

where I denotes the identity tensor field of type $(1,1)$ on $\tilde{M}$.

(b) for any local cross-section J of E and any vector X tangent to $\tilde{M}$, $\tilde{\nabla}_X J$ is also a local cross-section of E, where $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$.

The condition (b) is equivalent to the following condition:

(c) there exist local 1-forms p, q and r such that

$$\begin{aligned}
\tilde{\nabla}_X J_1 &= r(X)J_2 - q(X)J_3,\\
\tilde{\nabla}_X J_2 &= -r(X)J_1 + p(X)J_3,\\
\tilde{\nabla}_X J_3 &= q(X)J_1 - p(X)J_2.
\end{aligned}$$

Now, let X be an unit vector tangent to the quaternion manifold $\tilde{M}$, then X, J_1X, J_2X and J_3X form an orthonormal frame. We denote by $Q(X)$ the 4-plane spanned by them and call $Q(X)$ the quaternion section determined by X. For any orthonormal vectors X, Y tangent to $\tilde{M}$, the plane $X \wedge Y$ spanned by X, Y is said to be totally real if $Q(X)$ and $Q(Y)$ are orthogonal. Any plane in a quaternion section is called a quaternion plane. The sectional curvature of a quaternion plane is called a quaternion sectional curvature. A quaternion manifold is called a quaternion space form if its quaternion sectional curvatures are equal to a constant.

Let $\tilde{M}(c)$ be a $4m$-dimensional quaternion space form of constant quaternion sectional curvature c. The curvature tensor of $\tilde{M}(c)$ has the following expression ([15]):

$$\begin{aligned}
\tilde{R}(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y\\
&+ g(J_1Y,Z)J_1X - g(J_1X,Z)J_1Y + 2g(X,J_1Y)J_1Z\\
&+ g(J_2Y,Z)J_2X - g(J_2X,Z)J_2Y + 2g(X,J_2Y)J_2Z\\
&+ g(J_3Y,Z)J_3X - g(J_3X,Z)J_3Y + 2g(X,J_3Y)J_3Z\},
\end{aligned} \tag{2.1}$$

for any vector fields X, Y, Z tangent to $\tilde{M}$. The equation (2.1) can be written as:

$$\tilde{R}(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + \sum_{i=1}^{3}[g(J_iY,Z)J_iX - g(J_iX,Z)J_iY + 2g(X,J_iY)J_iZ]\} \tag{2.2}$$

for any vector fields X, Y, Z tangent to $\tilde{M}$.

Now, we recall

Definition 2.1 ([3])**.** Let M be a Riemannian manifold isometrically immersed in a quaternion manifold $\tilde{M}$. A distribution $D : p \to D_p \subseteq T_pM$ is called a *quaternion distribution* if we have $J_i(D) \subseteq D,\ i = 1,2,3$. In other words, D is a quaternion distribution if D is carried into itself by its quaternion structure.

Definition 2.2 ([3])**.** A submanifold M in a quaternion manifold $\tilde{M}$ is called a *quaternion CR-submanifold* if it admits a differentiable quaternion distribution D such that its orthogonal complementary distribution $D^\perp$ is totally real, i.e., $J_i(D_p^\perp) \subseteq T_p^\perp M$ and D is invariant under quaternion structure, that is, $J_i(D_p) \subseteq D_p$, $i = 1,2,3$, for any $p \in M$, where $T_p^\perp M$ denotes the normal space of M in $\tilde{M}$ at p.

A submanifold M of a quaternion manifold $\tilde{M}$ is called a *quaternion submanifold* if $\dim D_p^\perp = 0$ and a totally real submanifold if $\dim D_p = 0$. A quaternion CR-submanifold is said to be *proper* if it is neither totally real nor quaternionic.

Definition 2.3 ([10])**.** A submanifold M of a quaternion space form $\tilde{M}(c)$ is said to be *quaternion slant submanifold* if for any $p \in M$ and any $X \in T_pM$, the angle between $J_i(X),\ i = 1,2,3$ and T_pM is a constant $\theta \in [0, \frac{\pi}{2}]$, called the slant angle of quaternion submanifold M in $\tilde{M}(c)$.

In particular, quaternion submanifolds and *totally real* submanifolds of $\tilde{M}(c)$ are quaternion slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively.

Definition 2.4 ([18])**.** A submanifold M of a quaternion space form $\tilde{M}(c)$ is called a *quaternion bi-slant submanifold* if there exist two orthogonal distributions D_1 and D_2 on M such that

(i) TM admits orthogonal direct decomposition, i.e., $TM = D_1 \oplus D_2$.
(ii) For any $i = 1,2$, the distribution D_i is slant distribution with slant angle θ_i.

Let $4d_1 = \dim D_1$ and $4d_2 = \dim D_2$. If either d_1 or d_2 vanishes, the bi-slant submanifold is a slant submanifold. Thus slant submanifolds are particular cases of bi-slant submanifolds.

Definition 2.5 ([18]). Let M be a submanifold of a quaternion space form $\tilde{M}(c)$, then we say that M is a *semi-slant submanifold* if there exist two orthogonal distributions D_1 and D_2 on M such that

(i) TM admits orthogonal direct decomposition, i.e., $TM = D_1 \oplus D_2$.
(ii) The distribution D_1 is invariant by J_i, $i = 1, 2, 3$, i.e., $J_i(D_1) = D_1$.
(iii) The distribution D_2 is slant with respect to J_1, J_2, J_3 with slant angle $\theta \neq 0$, i.e. for any non-zero vector $X \in D_2(p)$, $p \in M$, the angle between J_iX, $i = 1, 2, 3$ and tangent subspace $D_2(p)$ is constant, that is, it is independent of the choice of $p \in M$ and $X \in D_2(p)$.

Now, we also recall the following Lemma of Chen [11].

Lemma 2.1 ([11]). *Let $a_1, \ldots, a_n, b$ be $(n+1), n \geq 2$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$ with equality holding if and only if

$$a_1 + a_2 = a_3 = \ldots = a_n.$$

Let M be a submanifold of a quaternion space form $\tilde{M}(c)$. We denote by g the metric tensor of $\tilde{M}(c)$ as well as that induced on M. Let ∇ be the induced connection on M. The Gauss and Weingarten formulae for M are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.3}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.4}$$

for any vector fields X, Y tangent to M and any vector field V normal to M, where h, A_V and $\nabla^{\perp}$ are the second fundamental form, the shape operator in the direction of V and the normal connection induced by ∇ on the normal bundle $T^{\perp}M$ respectively. The second fundamental form and the shape operator are related by

$$g(h(X, Y), V) = g(A_V X, Y). \tag{2.5}$$

For the second fundamental form h, we define the covariant differentiation $\tilde{\nabla}$ with respect to the connection in $TM \oplus T^{\perp}M$ by

$$(\tilde{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2.6}$$

for any vector fields X, Y, Z tangent to M.

The Gauss, Codazzi and Ricci equations for M are given by

$$\begin{aligned} R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) &+ g(h(X, W), h(Y, Z)) \\ &- g(h(X, Z), h(Y, W)), \end{aligned} \tag{2.7}$$

$$(R(X,Y),Z)^{\perp} = (\tilde{\nabla}_X h)(Y,Z) - (\tilde{\nabla}_Y h)(X,Z), \tag{2.8}$$

$$\tilde{R}(X,Y,V,\eta) = R^{\perp}(X,Y,V,\eta) - g([A_V, A_\eta]X,Y), \tag{2.9}$$

for any vector fields X, Y, Z, W tangent to M and V, η normal to M, where R and $R^{\perp}$ are the curvature tensors with respect to ∇ and $\nabla^{\perp}$ respectively.

The mean curvature vector $H(p)$ at $p \in M$ is defined by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.10}$$

where n denotes the dimension of M. If, we have

$$h(X,Y) = \lambda g(X,Y)H, \tag{2.11}$$

for any vector fields X, Y tangent to M, then M is called totally umbilical submanifold. In particular, if $h = 0$ identically, M is called a totally geodesic submanifold.

We set

$$h_{ij}^r = g(h(e_i, e_j), e_r),\ i,j \in \{1,\dots,n\},\ r \in \{n+1,\dots,4m\} \tag{2.12}$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j), h(e_i,e_j)). \tag{2.13}$$

For any $p \in M$ and X tangent to M, we put

$$J_i X = P_i X + T_i X,\ i = 1,2,3 \tag{2.14}$$

where $P_i X$ and $T_i X$ are the tangential and normal components of $J_i X$, respectively.

We recall that for a submanifold M in a Riemannian manifold, the relative null space of M at a point $p \in M$ is defined by

$$N_p = \{X \in T_pM \mid h(X,Y) = 0 \text{ for all } Y \in T_pM\}.$$

3. Quaternion slant submanifolds

In this section, we estimate the Ricci curvature of quaternion slant, bi-slant and semi-slant submanifolds of a quaternion space form.

Theorem 3.1. *Let M be an n-dimensional quaternion slant submanifold of a $4m$-dimensional quaternion space form $\tilde{M}(c)$ of constant quaternion sectional curvature c. Then*

(I) For each unit vector $X \in T_pM$, we have

$$\mathrm{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)c + 6c\cos^2\theta\}. \tag{3.1}$$

(II) If $H(p) = 0$, then an unit tangent vector X at p satisfies the equality case of (3.1) *if and only if X belongs to the relative null space N_p.*

Proof. Let $p \in M$, we choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM and $\{e_{n+1}, \ldots, e_{4m}\}$ for the normal space $T_p^{\perp}M$ at p such that $e_n = X$ and e_{n+1} is parallel to the mean curvature vector $H(p)$.

Let M be a quaternion slant submanifold of a $4m$-dimensional quaternion space form $\tilde{M}(c)$. Then using (2.2) and (2.14) in the equation of Gauss, we have

$$\begin{aligned} R(X,Y,Z,W) = \frac{c}{4}\{&g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &+ \sum_{i=1}^{3}[g(P_iY,Z)g(P_iX,W) - g(P_iX,Z)g(P_iY,W) \\ &+ 2g(X,P_iY)g(P_iZ,W)]\} \\ &+ g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \end{aligned} \tag{3.2}$$

for any vector fields X, Y, Z, W tangent to M.

Let $p \in M$ and an orthonormal basis $\{e_1, \ldots, e_n = X\}$ in T_pM. The Ricci tensor $S(X,Y)$ is given by

$$\begin{aligned} S(X,Y) &= \sum_{j=1}^{n} R(e_j, X, Y, e_j) \\ &= \frac{c}{4}\{g(X,Y)g(e_j,e_j) - g(e_j,Y)g(X,e_j) \\ &\quad + \sum_{i=1}^{3}[g(P_iX,Y)g(P_ie_j,e_j) - g(P_ie_j,Y)g(P_iX,e_j) \\ &\quad + 2g(e_j,P_iX)g(P_iY,e_j)]\} \\ &\quad + g(h(e_j,e_j),h(X,Y)) - g(h(e_j,Y),h(X,e_j)) \\ &= \frac{c}{4}\{(n-1)g(X,Y) + 3\sum_{i=1}^{3} g(P_iX,P_iY)\} \\ &\quad + \sum_{j=1}^{n}\{g(h(e_j,e_j),h(X,Y)) - g(h(e_j,Y),h(X,e_j))\}. \end{aligned} \tag{3.3}$$

The scalar curvature τ is given by

$$\tau = \sum_{l=1}^{n} S(e_l,e_l) = \frac{c}{4}\{n(n-1) + 12n\cos^2\theta\} + n^2\|H\|^2 - \|h\|^2. \tag{3.4}$$

We put

$$\epsilon = \tau - \frac{n^2}{2}\|H\|^2 - \frac{c}{4}\{n(n-1) + 12n\cos^2\theta\}. \tag{3.5}$$

Then from equations (3.4) and (3.5), we get

$$n^2\|H\|^2 = 2(\epsilon + \|h\|^2). \tag{3.6}$$

With respect to above orthonormal basis, the equation (3.6) takes the form

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left\{\epsilon + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ii}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\right\}. \tag{3.7}$$

If we set $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n-1} h_{ii}^{n+1}$ and $a_3 = h_{nn}^{n+1}$, then (3.7) becomes

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left\{\epsilon + \sum_{i=1}^{3} a_i^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 - \sum_{2\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1}\right\}. \tag{3.8}$$

Thus a_1, a_2, a_3 satisfy the Lemma 2.1 of Chen for $(n = 3)$, i.e.,

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(b + \sum_{i=1}^{3} a_i^2\right).$$

So, we have $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$.

In the case under consideration, this implies that equation (3.8) becomes

$$\sum_{1\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} \geq \epsilon + 2\sum_{i<j}(h_{ii}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2, \tag{3.9}$$

or equivalently

$$\begin{aligned}\frac{n^2}{2}\|H\|^2 &+ \frac{c}{4}[n(n-1) + 12n\cos^2\theta] \\ &\geq \tau - \sum_{1\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2.\end{aligned} \tag{3.10}$$

Using again the equation of Gauss, we have

$$\begin{aligned}\tau &- \sum_{1\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 \\ &= 2S(e_n, e_n) + \frac{c}{4}[(n-1)(n-2) + 12(n-1)\cos^2\theta] \\ &\quad + 2\sum_{i<n}(h_{in}^{n+1})^2 + \sum_{r=n+2}^{4m}\{(h_{nn}^{r})^2 + 2\sum_{i=1}^{n-1}(h_{in}^{r})^2 + (\sum_{j=1}^{n-1} h_{jj}^{r})^2\},\end{aligned} \tag{3.11}$$

where S is the Ricci tensor of M.

Combining (3.10) and (3.11), we obtain

$$\begin{aligned}(3.12)\quad & \frac{n^2}{2}\|H\|^2 + \frac{c}{4}[2(n-1) + 12\cos^2\theta] \\ & \geq 2S(e_n, e_n) + 2\sum_{i<n}(h_{in}^{n+1})^2 + \sum_{r=n+2}^{4m}\left\{\sum_{i=1}^{n}(h_{in}^r)^2 + \left(\sum_{j=1}^{n-1}h_{jj}^r\right)^2\right\}.\end{aligned}$$

Thus, we have

$$\mathrm{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)c + 6c\cos^2\theta\},$$

which proves (3.1).

(II) Assume $H(p) = 0$. Equality holds in (3.1) if and only if

$$(3.13)\quad h_{1n}^r = \ldots = h_{n-1,n}^r = 0, \quad h_{nn}^r = \sum_{i=1}^{n-1}h_{ii}^r, \; r \in \{n+1, \ldots, 4m\}.$$

Then $h_{in}^r = 0$, $\forall\, i \in \{1, \ldots, n\}$, $r \in \{n+1, \ldots, 4m\}$, i.e. X belongs to the relative null space N_p. ☐

Theorem 3.2. *Let M be an n-dimensional quaternion bi-slant submanifold of a $4m$-dimensional quaternion space form $\tilde{M}(c)$ of constant quaternion sectional curvature c. Then*

(I) For each unit vector $X \in T_pM$, if

(a) *X is tangent to D_1, we have*

$$(3.14)\quad \mathrm{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)c + 6c\cos^2\theta_1\}$$

and

(b) *X is tangent to D_2, we have*

$$(3.15)\quad \mathrm{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)c + 6c\cos^2\theta_2\}$$

(II) If $H(p) = 0$, then an unit tangent vector X at p satisfies the equality case of (3.14) *and* (3.15) *if and only if X belongs to the relative null space N_p.*

Proof. Let $p \in M$, we choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM and $\{e_{n+1}, \ldots, e_{4m}\}$ for the normal space $T_p^{\perp}M$ at p such that $e_n = X$ and e_{n+1} is parallel to the mean curvature vector $H(p)$.

From the equation of Gauss, the scalar curvature τ is given by

$$\begin{aligned}(3.16)\quad \tau &= \sum_{l=1}^{n}S(e_l, e_l) \\ &= \frac{c}{4}\{n(n-1) + 12(d_1\cos^2\theta_1 + d_2\cos^2\theta_2)\} + n^2\|H\|^2 - \|h\|^2.\end{aligned}$$

We put

$$\epsilon = \tau - \frac{n^2}{2}\|H\|^2 - \frac{c}{4}\{n(n-1) + 12(d_1 \cos^2\theta_1 + d_2 \cos^2\theta_2)\}. \tag{3.17}$$

Then from equations (3.16) and (3.17), we get

$$n^2\|H\|^2 = 2(\epsilon + \|h\|^2). \tag{3.18}$$

With respect to above orthonormal basis, the equation (3.18) takes the form

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left\{\epsilon + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ii}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\right\}. \tag{3.19}$$

If we set $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n-1} h_{ii}^{n+1}$ and $a_3 = h_{nn}^{n+1}$, then (3.19) becomes

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left\{\epsilon + \sum_{i=1}^{3} a_i^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 - \sum_{2\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1}\right\}. \tag{3.20}$$

Thus a_1, a_2, a_3 satisfy the Lemma 2.1 of Chen for ($n = 3$), i.e.,

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left(b + \sum_{i=1}^{3} a_i^2\right).$$

So, we have $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$.

In the case under consideration, this implies that equation (3.20) becomes

$$\sum_{1\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} \geq \epsilon + 2\sum_{i<j}(h_{ii}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2, \tag{3.21}$$

or equivalently

$$\begin{aligned}\frac{n^2}{2}\|H\|^2 &+ \frac{c}{4}[n(n-1) + 12(d_1\cos^2\theta_1 + d_2\cos^2\theta_2)] \\ &\geq \tau - \sum_{1\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2.\end{aligned} \tag{3.22}$$

Now, we consider two cases:

(a) If X is tangent to D_1, we have

$$(3.23)\quad \tau - \sum_{1\le\alpha\ne\beta\le n-1} h^{n+1}_{\alpha\alpha} h^{n+1}_{\beta\beta} + 2\sum_{i<j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h^r_{ij})^2$$
$$= 2S(e_n,e_n) + \frac{c}{4}[(n-1)(n-2) + 12\{(d_1-1)\cos^2\theta_1 + d_2\cos^2\theta_2\}]$$
$$+ 2\sum_{i<n}(h^{n+1}_{in})^2 + \sum_{r=n+2}^{4m}\left\{(h^r_{nn})^2 + 2\sum_{i=1}^{n-1}(h^r_{in})^2 + (\sum_{j=1}^{n-1}h^r_{jj})^2\right\},$$

where S is the Ricci tensor of M.

Combining (3.22) and (3.23), we obtain

$$(3.24)\quad \frac{n^2}{2}\|H\|^2 + \frac{c}{4}[2(n-1) + 12\cos^2\theta_1]$$
$$\ge 2S(e_n,e_n) + 2\sum_{i<n}(h^{n+1}_{in})^2 + \sum_{r=n+2}^{4m}\left\{\sum_{i=1}^{n}(h^r_{in})^2 + (\sum_{j=1}^{n-1}h^r_{jj})^2\right\}.$$

Thus, we have

$$\mathrm{Ric}(X) \le \frac{1}{4}\{n^2\|H\|^2 + (n-1)c + 6c\cos^2\theta_1\},$$

which proves (3.14).

(b) If X is tangent to D_2, we have

$$(3.25)\quad \tau - \sum_{1\le\alpha\ne\beta\le n-1} h^{n+1}_{\alpha\alpha} h^{n+1}_{\beta\beta} + 2\sum_{i<j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h^r_{ij})^2$$
$$= 2S(e_n,e_n) + \frac{c}{4}[(n-1)(n-2) + 12\{d_1\cos^2\theta_1 + (d_2-1)\cos^2\theta_2\}]$$
$$+ 2\sum_{i<n}(h^{n+1}_{in})^2 + \sum_{r=n+2}^{4m}\left\{(h^r_{nn})^2 + 2\sum_{i=1}^{n-1}(h^r_{in})^2 + (\sum_{j=1}^{n-1}h^r_{jj})^2\right\},$$

where S is the Ricci tensor of M.

Combining (3.22) and (3.25), we obtain

$$(3.26)\quad \frac{n^2}{2}\|H\|^2 + \frac{c}{4}[2(n-1) + 12\cos^2\theta_2]$$
$$\ge 2S(e_n,e_n) + 2\sum_{i<n}(h^{n+1}_{in})^2 + \sum_{r=n+2}^{4m}\left\{\sum_{i=1}^{n}(h^r_{in})^2 + (\sum_{j=1}^{n-1}h^r_{jj})^2\right\}.$$

Thus, we have

$$\mathrm{Ric}(X) \le \frac{1}{4}\{n^2\|H\|^2 + (n-1)c + 6c\cos^2\theta_2\},$$

which proves (3.15).

(II) Assume $H(p) = 0$. Equality holds in (3.14) and (3.15) if and only if

$$h^r_{1n} = \ldots = h^r_{n-1,n} = 0, \quad h^r_{nn} = \sum_{i=1}^{n-1} h^r_{ii}, \ r \in \{n+1, \ldots, 4m\}. \tag{3.27}$$

Then $h^r_{in} = 0$, $\forall i \in \{1, \ldots, n\}$, $r \in \{n+1, \ldots, 4m\}$, i.e. X belongs to the relative null space N_p. □

Now, we can state the following:

Corollary 3.3. *Let M be an n-dimensional quaternion semi-slant submanifold of a $4m$-dimensional quaternion space form $\tilde{M}(c)$ of constant quaternion sectional curvature c. Then*

(I) For each unit vector $X \in T_pM$, if

(a) *X is tangent to D_1, we have*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)c + 6c\} \tag{3.28}$$

and

(b) *X is tangent to D_2, we have*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)c + 6c\cos^2\theta\}. \tag{3.29}$$

(II) If $H(p) = 0$, then an unit tangent vector X at p satisfies the equality case of (3.28) *and* (3.29) *if and only if X belongs to the relative null space N_p.*

Corollary 3.4. *Let M be an n-dimensional quaternion submanifold of a $4m$-dimensional quaternion space form $\tilde{M}(c)$ of constant quaternion sectional curvature c. Then*

(I) For each unit vector $X \in T_pM$, we have

$$\mathrm{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)c + 6c\}. \tag{3.30}$$

(II) If $H(p) = 0$, then an unit tangent vector X at p satisfies the equality case of (3.30) *if and only if X belongs to the relative null space N_p.*

Corollary 3.5. *Let M be an n-dimensional totally real submanifold of a $4m$-dimensional quaternion space form $\tilde{M}(c)$ of constant quaternion sectional curvature c. Then*

(I) For each unit vector $X \in T_pM$, we have

$$\mathrm{Ric}(X) \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)c\}. \tag{3.31}$$

(II) If $H(p) = 0$, then an unit tangent vector X at p satisfies the equality case of (3.31) *if and only if X belongs to the relative null space N_p.*

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