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ON E-CURVATURE OF R-QUADRATIC FINSLER METRICS

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Abstract. In this paper, we prove that every R-quadratic Finsler metric with constant Douglas curvature along any geodesics has vanishing $\bar{\mathbf{E}}$-curvature. It result that R-quadratic Randers metric satisfies $\mathbf{S} = 0$.

1. Introduction

Let F be a Finsler metric on a manifold M. The geodesics of F are characterized locally by the equation $\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0$, where G^i are coefficients of a spray defined on M denoted by $\mathbf{G}(x, y) = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$. A Finsler metric F is called a Berwald metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in $y \in T_xM$ for any $x \in M$. Taking a trace of Berwald curvature yields mean Berwald curvature $\mathbf{E}$. In [12], Shen find a new non-Riemannian quantity for Finsler metrics that is closely related to the mean Berwald curvature and call it $\bar{\mathbf{E}}$-curvature. Recall that $\bar{\mathbf{E}}$-curvature is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics.

The second variation of geodesics gives rise to a family of linear maps $R_y : T_xM \to T_xM$, at any point $y \in T_xM$. R_y is called the Riemann curvature in the direction y. There are many Finsler metrics whose Riemann curvature in every direction is quadratic. A Finsler metric F is said to be *R-quadratic* if R_y is quadratic in $y \in T_xM$ at each point $x \in M$. Indeed a Finsler metric is R-quadratic if and only if the h-curvature of Berwald connection depends on position only in the sense of Bácsó–Matsumoto [3]. It is remarkable that, the notion of R-quadratic Finsler metrics was introduced by Shen, which can be considered as a generalization of Berwald metrics and R-flat metrics [4, 13, 8]. In this paper, we prove the following.

Theorem 1.1. *Let F be a R-quadratic Finsler metric. Suppose that the Douglas curvature of F is constant along any Finslerian geodesics. Then $\bar{\mathbf{E}} = 0$.*

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In [1], Akbar-Zadeh considered a non-Riemannian quantity $\mathbf{H}$ which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. In the class of Weyl metrics, vanishing this quantity results that the Finsler metric is of constant flag curvature and this fact clarifies its geometric meaning [1, 10]. By the definition, if $\bar{\mathbf{E}} = 0$ then $\mathbf{H} = 0$.

In [8], it is proved that if F is a R-quadratic Finsler metric then $\mathbf{H} = 0$. Then Mo consider H-curvature of Finsler manifolds and get a new proof for this fact [7]. Recently, Li-Shen prove that every R-quadratic Randers metric has constant non-Riemannian invariant S-curvature [6]. Then Tang proved that for a Randers metric $\mathbf{H} = 0$ if and only if $\mathbf{S} = 0$ [14]. Therefore, we can conclude the following.

Corollary 1. *Let F be a R-quadratic Randers metric. Then* $\mathbf{S} = 0$.

There are many connections in Finsler geometry [15, 16]. In this paper, we use the Berwald connection and denote the h- and v- covariant derivatives of a Finsler tensor field by " $|$ " and ", " respectively.

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\left[F^2(y + su + tv)\right]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$, where $G^i = G^i(x, y)$ are local functions on TM given by

$$G^i := \frac{1}{4} g^{il}(y)\Big\{\frac{\partial^2[F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial[F^2]}{\partial x^l}\Big\}, \quad y \in T_xM.$$

$\mathbf{G}$ is called the associated spray to (M, F). The projection of an integral curve of $\mathbf{G}$ is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u,v,w) := B^i_{j\ kl}u^jv^kw^l\frac{\partial}{\partial x^i}|_x$, $\mathbf{E}_y(u,v) := E_{jk}u^jv^k$ where

$$B^i_{j\ kl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk}(y) := \frac{1}{2}B^m_{j\ km},$$

$u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. $\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature respectively. F is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = 0$ and $\mathbf{E} = 0$, respectively [12].

Let

$$D^i_{j\ kl} := B^i_{j\ kl} - \frac{1}{n+1}\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big(\frac{\partial G^m}{\partial y^m}y^i\Big).$$

It is easy to verify that $\mathcal{D} := D^i_{j\ kl}dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle TM_0. We call $\mathcal{D}$ the Douglas tensor. The Douglas tensor $\mathcal{D}$ is a non-Riemannian projective invariant, namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + Py^i$, where $P = P(x,y)$ is positively y-homogeneous of degree one, then the Douglas tensor of F is same as that of $\bar{F}$ [5, 9, 11]. Finsler metrics with vanishing Douglas tensor are called Douglas metrics. The notion of Douglas curvature was proposed by B*á*cs*ó* and Matsumoto as a generalization of Berwald curvature [2].

The quantity $\mathbf{H}_y = H_{ij}dx^i \otimes dx^j$ is defined as the covariant derivative of $\mathbf{E}$ along geodesics [10]. More precisely

$$H_{ij} := E_{ij|m}y^m$$

In local coordinates,

$$2H_{ij} = y^m\frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial x^m} - 2G^m\frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial y^m} - G^m_i B^k_{j\ km} - G^m_j B^k_{i\ km},$$

where $G^i_j := \frac{\partial G^i}{\partial y^j}$.

The Riemann curvature $\mathbf{R}_y = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$ is a family of linear maps on tangent spaces, defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

For a flag $P = \mathrm{span}\{y,u\} \subset T_xM$ with flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(P,y)$ is defined by

$$\mathbf{K}(P,y) := \frac{\mathbf{g}_y(u, \mathbf{R}_y(u))}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,u) - \mathbf{g}_y(y,u)^2},$$

where $\mathbf{g}_y = g_{ij}(x,y)dx^i \otimes dx^j$. We say that a Finsler metric F is of scalar curvature if for any $y \in T_xM$, the flag curvature $\mathbf{K} = \mathbf{K}(x,y)$ is a scalar function on the slit tangent bundle TM_0. If $\mathbf{K}$ = constant, then F is said to be of constant flag curvature.

A Finsler metric F is said to be *R-quadratic* if R_y is quadratic in $y \in T_xM$ at each point $x \in M$. Let

$$R^i_{j\ kl}(x,y) := \frac{1}{3}\frac{\partial}{\partial y^j}\{\frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k}\},$$

where $R^i{}_{jkl}$ is the Riemann curvature of Berwald connection. Then we have $R^i_k = R^i_{j\ kl}(x,y)y^jy^l$. Therefore R^i_k is quadratic in $y \in T_xM$ if and only if $R^i_{j\ kl}$ are functions of position alone. Indeed a Finsler metric is R-quadratic if and only if the h-curvature of Berwald connection depends on position only in the sense of Bácsó–Matsumoto [2].

By means of $\mathbf{E}$-curvature, we can define $\bar{\mathbf{E}}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\bar{\mathbf{E}}_y(u,v,w) := \bar{E}_{jkl}(y)u^iv^jw^k,$$

where $\bar{E}_{ijk} := E_{ij|k}$. We call it $\bar{\mathbf{E}}$-curvature. It is remarkable that, $\bar{E}_{ijk}$ is not totally symmetric in all three of its indices. By definition, if $\bar{\mathbf{E}} = 0$, then $\mathbf{E}$-curvature is covariantly constant along all horizontal directions on TM_0.

3. Proof of Theorem 1.1

To prove the Theorem 1.1, we need the following:

Lemma 1.

$$E_{jk,l|m}y^m = H_{jk,l} - \bar{E}_{jkl}. \tag{1}$$

Proof. The following Ricci identity for E_{ij} is hold:

$$E_{ij,l|k} - E_{ij|k,l} = E_{pj}B^p{}_{ikl} + E_{ip}B^p{}_{jkl}. \tag{2}$$

It follows from (2) that

$$E_{jk,l|m}y^m = E_{jk|m,l}y^m = [E_{jk|m}y^m]_{,l} - E_{jk|l}. \tag{3}$$

This yields the (1). □

Lemma 2. *Let F be a R-quadratic Finsler metric. Then the Berwald curvature of F is constant along any Finslerian geodesics.*

Proof. The curvature form of Berwald connection is

$$\Omega^i{}_j = d\omega^i{}_j - \omega^k{}_j \wedge \omega^i{}_k = \frac{1}{2}R^i{}_{jkl}\omega^k \wedge \omega^l - B^i{}_{jkl}\omega^k \wedge \omega^{n+l}. \tag{4}$$

For the Berwald connection, we have the following structure equation

$$dg_{ij} - g_{jk}\Omega^k{}_i - g_{ik}\Omega^k{}_j = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}, \tag{5}$$

where $L_{ijk} := C_{ijk|s}y^s$ is the Landsberg curvature. Differentiating (5) yields the following Ricci identity

$$\begin{aligned} g_{pj}\Omega^p{}_i - g_{pi}\Omega^p{}_j = &- 2L_{ijk|l}\omega^k \wedge \omega^l - 2L_{ijk,l}\omega^k \wedge \omega^{n+l} \\ &- 2C_{ijl|k}\omega^k \wedge \omega^{n+l} - 2C_{ijl,k}\omega^{n+k} \wedge \omega^{n+l} - 2C_{ijp}\Omega^p{}_l y^l. \end{aligned} \tag{6}$$

Differentiating of (4) yields

$$d\Omega_i^{\ j} - \omega_i^{\ k} \wedge \Omega_k^{\ j} + \omega_k^{\ j} \wedge \Omega_i^{\ k} = 0. \tag{7}$$

Define $B^i_{j\ kl|m}$ and $B^i_{j\ kl,m}$ by

$$dB^i_{jkl} - B^i_{mkl}\omega^m_i - B^i_{jml}\omega^m_k - B^i_{jkm}\omega^m_l + B^i_{jkl}\omega^i_m = B^i_{jkl|m}\omega^m + B^i_{jkl,m}\omega^{n+m}. \tag{8}$$

Similarly, we define $R^i_{\ jkl|m}$ and $R^i_{\ jkl,m}$ by

$$dR^i_{jkl} - R^i_{mkl}\omega^m_i - B^i_{jml}\omega^m_k - R^i_{jkm}\omega^m_l + R^i_{jkl}\omega^i_m = R^i_{jkl|m}\omega^m + R^i_{jkl,m}\omega^{n+m}. \tag{9}$$

From (6), (7), (8) and (9) one obtain

$$R^i_{j\ kl|m} + R^i_{j\ lm|k} + R^i_{j\ mk|l} = B^i_{j\ ku}R^u_{\ lm} + B^i_{j\ lu}R^u_{\ km} + B^i_{k\ lu}R^u_{\ jm}, \tag{10}$$

$$B^i_{j\ kl|m} - B^i_{j\ mk|l} = R^i_{j\ ml,k}, \tag{11}$$

$$B^i_{j\ kl,m} = B^i_{j\ km,l}. \tag{12}$$

By assumption and (11) we have

$$B^i_{j\ kl|m} = B^i_{j\ mk|l}, \tag{13}$$

which contacting with y^m, we conclude that

$$B^i_{j\ kl|m}y^m = 0. \tag{14}$$

By (14), we conclude that the Berwald curvature of R-quadratic Finsler metric is constant along any geodesics. □

Corollary 2. ([7, 8]) *Let F be a R-quadratic Finsler metric. Then $\mathbf{H} = 0$.*

By (11) we have

$$B^i_{j\ ml|k} - B^i_{j\ km|l} = R^i_{j\ kl,m}.$$

This implies that

$$\bar{E}_{jlk} - \bar{E}_{jkl} = 2R^m_{j\ kl,m}.$$

Thus we get the following.

Corollary 3. *Let F be a R-quadratic Finsler metric. Then $\bar{\mathbf{E}}$-curvature is totally symmetric in all three of its indices.*

Proof of Theorem 1.1:

$$D^i_{\ jkl} = B^i_{\ jkl} - \frac{2}{n+1}\{E_{jk}\delta^i_{\ l} + E_{kl}\delta^i_{\ j} + E_{lj}\delta^i_{\ k} + E_{jk,l}y^i\}. \tag{15}$$

Then

$$D^i_{\ jkl|m}y^m = B^i_{\ jkl|m}y^m - \frac{2}{n+1}\{E_{jk|m}y^m\delta^i_{\ l} + E_{kl|m}y^m\delta^i_{\ j} + E_{lj|m}y^m\delta^i_{\ k}\} - \frac{2}{n+1}E_{jk,l|m}y^m y^i. \tag{16}$$

It follows from (11) that

$$B^i_{\ jkl|m}y^m = R^i_{\ jml,k}y^m. \tag{17}$$

Then we have

$$E_{jk|m}y^m = R^p{}_{jmp,k}y^m. \tag{18}$$

We obtain

(19)

$$D^\alpha{}_{jkl|m}y^m = R^\alpha{}_{jml,k}y^m - \frac{2}{n+1}\{R^p{}_{jmp,k}y^m\delta^\alpha_l + R^p{}_{lmp,j}y^m\delta^\alpha{}_k + R^p{}_{kmp,l}y^m\delta^\alpha{}_j\} - \frac{2}{n+1}E_{jk,l|m}y^m y^i.$$

By assumptions we have

$$E_{jk,l|m}y^m y^i = 0. \tag{20}$$

Contracting (20) with y_i yields

$$E_{jk,l|m}y^m = 0. \tag{21}$$

Considering (1), we conclude that $\bar{E}_{ijk} = 0$. □

Corollary 4. *Let F be a R-quadratic Douglas metric. Then* $\bar{\mathbf{E}} = 0$.

It is remarkable that, the assumption of R-quadraticness of a Finsler metric is necessary in Theorem 1.1 and can not be dropped. For example, see the following.

Example 1. Let

$$F := |y| + \frac{< x, y >}{\sqrt{1+|x|^2}}, \quad y \in T_x\mathbb{R}^n \simeq \mathbb{R}^n$$

where $|.|$ and $<, >$ denote the Euclidean norm and inner product on $\mathbb{R}^n$ respectively. F is indeed a Randers metric on the whole of $\mathbb{R}^n$ and it is a projectively flat Randers metric on $\mathbb{R}^n$ i.e., the spray coefficients are in the form $G^i = Py^i$, for a scalar function on TM_0 given by

$$P = c\big(|y| - \frac{< x, y >}{\sqrt{1+|x|^2}}\big),$$

where $c = 1/2(\sqrt{1+|x|^2})$. Then F is a Douglas metric. The flag curvature of F given by

$$K = \frac{3}{4(1+|x|^2)}.\frac{|y|\sqrt{1+|x|^2} - < x, y >}{|y|\sqrt{1+|x|^2} + < x, y >}.$$

Therefore, this Randers metric is not R-quadratic. By a simple calculation, we get $\bar{E}_{ijk} = (n+1)P_{ij|k} \neq 0$.

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