*Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*
**28** (2012), 167–176
www.emis.de/journals
ISSN 1786-0091

# A NOTE ON THE FOURIER COEFFICIENTS AND PARTIAL SUMS OF VILENKIN-FOURIER SERIES

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Abstract. The aim of this paper is to investigate Paley type and Hardy-Littlewood type inequalities and strong convergence theorem of partial sums of Vilenkin-Fourier series.

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ denote a sequence of the positive numbers, not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

the additive group of integers modulo $m_k$. Define the group $G_m$ as the complete direct product of the group $Z_{m_j}$ with the product of the discrete topologies of $Z_{m_j}$'s.

The direct product $\mu$, of the measures

$$\mu_k(\{j\}) := 1/m_k, \quad (j \in Z_{m_k})$$

is the Haar measure on $G_m$, with $\mu(G_m) = 1$. If $\sup_n m_n < \infty$, then we call $G_m$ a bounded Vilenkin group. If the generating sequence $m$ is not bounded then $G_m$ is said to be an unbounded Vilenkin group. *In this paper we discuss bounded Vilenkin groups only.*

The elements of $G_m$ represented by sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots), \quad (x_k \in Z_{m_k}).$$

It is easy to give a base for the neighborhood of $G_m$ :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}, \quad (x \in G_m,\ n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, for $n \in \mathbb{N}$ and $\bar{I_n} := G_m \setminus I_n$.

2010 *Mathematics Subject Classification.* 42C10.

*Key words and phrases.* Vilenkin system, Fourier coefficients, partial sums, martingale Hardy space.

If we define the so-called generalized number system, based on $m$ in the following way :

$$M_0 := 1, \ M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum\limits_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$ and only a finite number of $n_j$'s differ from zero.

Let $|n| := \max \{j \in \mathbb{N} : n_j \neq 0\}$. Denote by $\mathbb{N}_{n_0}$ the subset of positive integers $\mathbb{N}_+$, for which $n_{|n|} = n_0 = 1$. Then every $n \in \mathbb{N}_{n_0}$, $M_k < n < M_{k+1}$ can be written as

$$n = M_0 + \sum_{j=1}^{k-1} n_j M_j + M_k = 1 + \sum_{j=1}^{k-1} n_j M_j + M_k,$$

where $n_j \in \{0, m_j - 1\}$, $(j \in \mathbb{N}_+)$.

By simple calculation we get

$$\sum_{\left\{n : M_k \leq n \leq M_{k+1}, \ n \in \mathbb{N}_{n_0}\right\}} 1 = \frac{M_{k-1}}{m_0} \geq c M_k, \tag{1}$$

where $c$ is absolute constant.

Denote by $L_1(G_m)$ the usual (one dimensional) Lebesgue space. Next, we introduce on $G_m$ an orthonormal system, which is called the Vilenkin system. At first define the complex valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi \imath x_k / m_k), \quad \left(\imath^2 = -1, \ x \in G_m, \ k \in \mathbb{N}\right).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh–Paley one if $m \equiv 2$. The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1, 14].

Now we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can establish the the Fourier coefficients, the partial sums, the Dirichlet kernels, with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu, \quad (k \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \quad (n \in \mathbb{N}_+, \ S_0 f := 0),$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+).$$

Recall that

$$D_{M_n}(x)=\begin{cases}M_n, & \text{if } x\in I_n\\ 0, & \text{if } x\notin I_n\end{cases} \tag{2}$$

and

$$D_n=\psi_n\left(\sum_{j=0}^{\infty}D_{M_j}\sum_{u=m_j-n_j}^{m_j-1}r_j^u\right). \tag{3}$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p:=\left(\int_{G_m}|f|^p\,d\mu\right)^{1/p}\qquad(0<p<\infty).$$

The space $L_{p,\infty}(G_m)$ consists of all measurable functions $f$ for which

$$\|f\|_{L_{p,\infty}}:=\sup_{\lambda>0}\lambda\mu(f>\lambda)^{1/p}<+\infty.$$

The $\sigma$-algebra, generated by the intervals $\{I_n(x):x\in G_m\}$ will be denoted by $F_n$ $(n\in\mathbb{N})$. The conditional expectation operators relative to $F_n$ $(n\in\mathbb{N})$ are denoted by $E_n$. Then

$$E_nf(x)=S_{M_n}f(x)=\sum_{k=0}^{M_n-1}\widehat{f}(k)\,w_k=\frac{1}{|I_n(x)|}\int_{I_n(x)}f(x)d\mu(x),$$

where $|I_n(x)|=M_n^{-1}$ denotes the length of $I_n(x)$.

A sequence $f=\left(f^{(n)},\ n\in\mathbb{N}\right)$ of functions $f_n\in L_1(G)$ is said to be a dyadic martingale if

(i) $f^{(n)}$ is $F_n$ measurable, for all $n\in\mathbb{N}$,
(ii) $E_nf^{(m)}=f^{(n)}$, for all $n\leq m$

(for details see e.g. [15]).

The maximal function of a martingale $f$ is denoted by

$$f^*=\sup_{n\in\mathbb{N}}\left|f^{(n)}\right|.$$

In case $f\in L_1$, the maximal functions are also be given by

$$f^*(x)=\sup_{n\in\mathbb{N}}\frac{1}{|I_n(x)|}\left|\int_{I_n(x)}f(u)\,\mu(u)\right|.$$

For $0<p<\infty$, the Hardy martingale spaces $H_p(G_m)$ consist of all martingales, for which

$$\|f\|_{H_p}:=\|f^*\|_p<\infty.$$

If $f\in L_1$, then it is easy to show that the sequence $(S_{M_n}f:n\in\mathbb{N})$ is a martingale. If $f=\left(f^{(n)},n\in\mathbb{N}\right)$ is martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty} \int_{G_m} f^{(k)}(x)\, \overline{\Psi}_i(x)\, d\mu(x)\,. \tag{4}$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n} f : n \in \mathbb{N})$ obtained from $f$.

A bounded measurable function $a$ is p-atom, if there exist a dyadic interval $I$, such that

(i) $\int_I a d\mu = 0$

(ii) $\|a\|_\infty \leq \mu(I)^{-1/p}$

(iii) $\operatorname{supp}(a) \subset I$.

The Hardy martingale spaces $H_p(G_m)$, for $0 < p \leq 1$ have an atomic characterization. Namely, the following theorem is true.

**Theorem W** (Weisz, [17])**.** *A martingale $f = \left(f^{(n)},\ n \in \mathbb{N}\right)$ is in $H_p\,(0 < p \leq 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of a real numbers, such that for every $n \in \mathbb{N}$:*

$$\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)}, \tag{5}$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

*Moreover, $\|f\|_{H_p} \backsim \inf\left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p}$, where the infimum is taken over all decomposition of $f$ of the form (5).*

When $0 < p \leq 1$, the Hardy martingale space $H_p$ is proper subspace of Lebesgue space $L_p$. It is well known that for $1 < p < \infty$ the space $H_p$ is nothing but $L_p$.

The classical inequality of Hardy type is well known in the trigonometric as well as in the Vilenkin-Fourier analysis. Namely,

$$\sum_{k=1}^{\infty} \frac{\left|\widehat{f}(k)\right|}{k} \leq c\, \|f\|_{H_1}\,,$$

where the function $f$ belongs to the Hardy space $H_1$ and $c$ is an absolute constant. This was proved in the trigonometric case by Hardy and Littlewood [6] (see also Coifman and Weiss [2]) and for Walsh system it can be found in [8].

Weisz [15, 18] generalized this result for Vilenkin system and proved:

**Theorem A** (Weisz)**.** *Let $0 < p \leq 2$. Then there is an absolute constant $c_p$, depend only $p$, such that*

$$\sum_{k=1}^{\infty} \frac{\left|\widehat{f}(k)\right|^p}{k^{2-p}} \leq c_p \|f\|_{H_p}, \tag{6}$$

*for all $f \in H_p$.*

Paley [7] proved that the Walsh–Fourier coefficients of a function $f \in L_p\,(1 < p < 2)$ satisfy the condition

$$\sum_{k=1}^{\infty} \left|\widehat{f}\left(2^k\right)\right|^2 < \infty.$$

This results fails to hold $p = 1$. However, it can be verified for functions $f \in L_1$, such that $f^*$ belongs $L_1$, i.e. $f \in H_1$ (see e.g. Coifman and Weiss [2]).

For the Vilenkin system we have the following theorem.

**Theorem B** (Weisz [11])**.** *Let $0 < p \leq 1$. Then there is an absolute constant $c_p$, depend only $p$, such that*

$$\left(\sum_{k=1}^{\infty} M_k^{2-2/p} \sum_{j=1}^{m_k-1} \left|\widehat{f}(jM_k)\right|^2\right)^{1/2} \leq c_p \|f\|_{H_p}, \tag{7}$$

*for all $f \in H_p$.*

It is well-known that Vilenkin system forms not basis in the space $L_1$. Moreover, there is a function in the dyadic Hardy space $H_1$, such that the partial sums of $f$ are not bounded in $L_1$-norm. However, in Simon [9] the following strong convergence result was obtained for all $f \in H_1$:

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0,$$

where $S_k f$ denotes the $k$-th partial sum of the Walsh–Fourier series of $f$. (For the trigonometric analogue see Smith [12], for the Vilenkin system by Gát [3]). For the Vilenkin system Simon proved:

**Theorem C** (Simon [10])**.** *Let $0 < p < 1$. Then there is an absolute constant $c_p$, depends only $p$, such that*

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p, \tag{8}$$

*for all $f \in H_p$.*

Strong convergence theorems of two-dimensional partial sums was investigate by Weisz [16], Goginava [4], Gogoladze [5], Tephnadze [13].

The main aim of this paper is to prove the following theorem:

**Theorem 1.** *Let* $\{\Phi_n\}_{n=1}^{\infty}$ *is any nondecreasing sequence, satisfying the condition* $\lim_{n\to\infty}\Phi_n = +\infty$. *Then there exists a martingale* $f \in H_p$, *such that*

$$\sum_{k=1}^{\infty}\frac{\left|\widehat{f}(k)\right|^p \Phi_k}{k^{2-p}} = \infty, \ \textit{for } 0 < p \le 2, \tag{9}$$

$$\sum_{k=1}^{\infty}\frac{\Phi_{M_k}}{M_k^{2/p-2}}\sum_{j=1}^{m_k-1}\left|\widehat{f}(jM_k)\right|^2 = \infty, \ \textit{for } 0 < p \le 1 \tag{10}$$

*and*

$$\sum_{k=1}^{\infty}\frac{\|S_k f\|_{L_{p,\infty}}^p \Phi_k}{k^{2-p}} = \infty, \ \textit{for } 0 < p < 1. \tag{11}$$

*Proof.* Let $0 < p \le 2$ and $\{\Phi_n\}_{n=1}^{\infty}$ is any nondecreasing, nonnegative sequence, satisfying condition $\lim_{n\to\infty}\Phi_n = \infty$.

For this function $\Phi(n)$, there exists an increasing sequence $\{\alpha_k \ge 2 : k \in \mathbb{N}_+\}$ of the positive integers such that:

$$\sum_{k=1}^{\infty}\frac{1}{\Phi_{M_{\alpha_k}}^{p/4}} < \infty. \tag{12}$$

Let

$$f^{(A)}(x) := \sum_{\{k;\ \alpha_k < A\}}\lambda_k a_k(x),$$

where

$$\lambda_k = \frac{1}{\Phi_{M_{\alpha_k}}^{1/4}}, \ a_k(x) = \frac{M_{\alpha_k}^{1/p-1}}{M}\left(D_{M_{\alpha_k+1}}(x) - D_{M_{\alpha_k}}(x)\right),$$

and $M = \sup_{n\in\mathbb{N}} m_n$.

It is easy to show that the martingale $f = \left(f^{(1)}, f^{(2)}, \dots, f^{(A)}, \dots\right) \in H_p$. Indeed,

$$S_{M_A}(a_k(x)) = \begin{cases} a_k(x) & \alpha_k < A \\ 0, & \alpha_k \ge A, \end{cases} \tag{13}$$

$$\text{supp}(a_k) = I_{\alpha_k}, \quad \int_{I_{\alpha_k}} a_k d\mu = 0,$$

and

$$\|a_k\|_\infty \le \frac{M_{\alpha_k}^{1/p-1}}{M}M_{\alpha_k+1} \le M_{\alpha_k}^{1/p} = \mu(\text{supp } a_k)^{-1/p}.$$

If we apply Theorem W and (12) we conclude that $f \in H_p$.

It is easy to show that

$$\widehat{f}(j)=\begin{cases}\frac{1}{M}\frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}, & \text{if } j\in\{M_{\alpha_k},\dots,\ M_{\alpha_k+1}-1\},\ k=1,2\dots,\\ 0, & \text{if } j\notin\bigcup_{k=1}^{\infty}\{M_{\alpha_k},\dots,\ M_{\alpha_k+1}-1\}.\end{cases}\tag{14}$$

First we prove equality (9). Using (14) we can

$$\begin{aligned}\sum_{l=1}^{M_{\alpha_k+1}-1}\frac{\left|\widehat{f}(l)\right|^p\Phi_l}{l^{2-p}}&=\sum_{n=1}^{k}\sum_{l=M_{\alpha_n}}^{M_{\alpha_n+1}-1}\frac{\left|\widehat{f}(l)\right|^p\Phi_l}{l^{2-p}}\\&\geq\sum_{l=M_{\alpha_k}}^{M_{\alpha_k+1}-1}\frac{\left|\widehat{f}(l)\right|^p\Phi_l}{l^{2-p}}\geq c\Phi_{M_{\alpha_k}}\sum_{l=M_{\alpha_k}}^{M_{\alpha_k+1}-1}\frac{\left|\widehat{f}(l)\right|^p}{l^{2-p}}\\&\geq c\Phi_{M_{\alpha_k}}\frac{M_{\alpha_k}^{1-p}}{\Phi_{M_{\alpha_k}}^{p/4}}\sum_{l=M_{\alpha_k}}^{M_{\alpha_k+1}-1}\frac{1}{l^{2-p}}\geq c\Phi_{M_{\alpha_k}}^{1/2}M_{\alpha_k}^{1-p}\sum_{l=M_{\alpha_k}}^{M_{\alpha_k+1}-1}\frac{1}{M_{\alpha_k+1}^{2-p}}\\&\geq c\Phi_{M_{\alpha_k}}^{1/2}M_{\alpha_k}^{1-p}\frac{1}{M_{\alpha_k+1}^{1-p}}\geq c\Phi_{M_{\alpha_k}}^{1/2}\to\infty,\ \text{when } k\to\infty.\end{aligned}$$

Next we prove equality (10). Let $0<p\leq 1$. Using (14) we get

$$\begin{aligned}\sum_{l=1}^{k}M_{\alpha_l}^{2-2/p}\Phi_{M_{\alpha_l}}\sum_{j=1}^{m_{\alpha_l}-1}\left|\widehat{f}(jM_{\alpha_l})\right|^2&\geq M_{\alpha_k}^{2-2/p}\Phi_{M_{\alpha_k}}\sum_{j=1}^{m_{\alpha_k}-1}\left|\widehat{f}(jM_{\alpha_k})\right|^2\\&\geq cM_{\alpha_k}^{2-2/p}\Phi_{M_{\alpha_k}}\sum_{j=1}^{m_{\alpha_k}-1}\frac{M_{\alpha_k}^{2/p-2}}{\Phi_{M_{\alpha_k}}^{1/2}}\\&\geq c\Phi_{M_{\alpha_k}}^{1/2}\to\infty,\ \text{when } k\to\infty.\end{aligned}$$

Finally we prove equality (11). Let $0<p<1$ and $M_{\alpha_k}\leq j<M_{\alpha_k+1}$. From (14) we have

$$\begin{aligned}S_jf(x)&=\sum_{l=0}^{M_{\alpha_{k-1}+1}-1}\widehat{f}(l)\psi_l(x)+\sum_{l=M_{\alpha_k}}^{j-1}\widehat{f}(l)\psi_l(x)\\&=\sum_{\eta=0}^{k-1}\sum_{v=M_{\alpha_\eta}}^{M_{\alpha_\eta+1}-1}\widehat{f}(v)\psi_v(x)+\sum_{v=M_{\alpha_k}}^{j-1}\widehat{f}(v)\psi_v(x)\\&=\sum_{\eta=0}^{k-1}\sum_{v=M_{\alpha_\eta}}^{M_{\alpha_\eta+1}-1}\frac{1}{M}\frac{M_{\alpha_\eta}^{1/p-1}}{\Phi_{M_{\alpha_\eta}}^{1/4}}\psi_v(x)+\sum_{v=M_{\alpha_k}}^{j-1}\frac{1}{M}\frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}\psi_v(x)\end{aligned}$$

$$= \sum_{\eta=0}^{k-1} \frac{1}{M} \frac{M_{\alpha_\eta}^{1/p-1}}{\Phi_{M_{\alpha_\eta}}^{1/4}} \left( D_{M_{\alpha_{\eta+1}}}(x) - D_{M_{\alpha_\eta}}(x) \right)$$

$$+ \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \left( D_j(x) - D_{M_{\alpha_k}}(x) \right)$$

$$= I + II.$$

Let $j \in \mathbb{N}_{n_0}$ and $x \in G_m \backslash I_1$. Since $j - M_{\alpha_k} \in \mathbb{N}_{n_0}$ and

$$D_{j+M_{\alpha_k}}(x) = D_{M_{\alpha_k}}(x) + \psi_{M_{\alpha_k}}(x) D_j(x),$$

when $j < M_{\alpha_k}$. Combining (2) and (3) we can write

$$|II| = \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \left| \psi_{M_{\alpha_k}} D_{j-M_{\alpha_k}}(x) \right| \tag{15}$$

$$= \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \left| \psi_{M_{\alpha_k}}(x) \psi_{j-M_{\alpha_k}}(x) r_0^{m_0-1}(x) D_1(x) \right|$$

$$= \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}.$$

Applying (2) and condition $\alpha_n \geq 2$ $(n \in \mathbb{N})$ for $I$ we have

$$I = 0, \text{ for } x \in G_m \backslash I_1. \tag{16}$$

It follows that

$$|S_j f(x)| = |II| = \frac{1}{M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}, \text{ for } x \in G_m \backslash I_1.$$

Hence

$$\|S_j(f(x))\|_{L_{p,\infty}} \geq \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \mu \left( x \in G_m : |S_j(f(x))| > \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \right)^{1/p}$$

$$\geq \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \mu \left( x \in G_m \backslash I_1 : |S_j(f(x))| > \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} \right)^{1/p} \tag{17}$$

$$= \frac{1}{2M} \frac{M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}} |G_m \backslash I_1|$$

$$\geq \frac{c M_{\alpha_k}^{1/p-1}}{\Phi_{M_{\alpha_k}}^{1/4}}.$$

Combining (1) and (17) we have

$$\begin{aligned}
\sum_{j=1}^{M_{\alpha_k+1}-1} \frac{\|S_j(f(x))\|_{L_{p,\infty}}^p \Phi_j}{j^{2-p}} &\geq \sum_{j=M_{\alpha_k}}^{M_{\alpha_k+1}-1} \frac{\|S_j(f(x))\|_{L_{p,\infty}}^p \Phi_j}{j^{2-p}} \\
&\geq \Phi_{M_{\alpha_k}} \sum_{\left\{j: M_k \leq j \leq M_{k+1},\ j \in \mathbb{N}_{n_0}\right\}} \frac{\|S_j(f(x))\|_{L_{p,\infty}}^p}{j^{2-p}} \\
&\geq c\Phi_{M_{\alpha_k}} \frac{M_{\alpha_k}^{1-p}}{\Phi_{M_{\alpha_k}}^{p/4}} \sum_{\left\{j: M_k \leq j \leq M_{k+1},\ j \in \mathbb{N}_{n_0}\right\}} \frac{1}{j^{2-p}} \\
&\geq c\Phi_{M_{\alpha_k}}^{3/4} M_{\alpha_k}^{1-p} \sum_{\left\{j: M_k \leq j \leq M_{k+1},\ j \in \mathbb{N}_{n_0}\right\}} \frac{1}{M_{\alpha_k+1}^{2-p}} \\
&\geq c\frac{\Phi_{M_{\alpha_k}}^{3/4}}{M_{\alpha_k+1}} \sum_{\left\{j: M_k \leq j \leq M_{k+1},\ j \in \mathbb{N}_{n_0}\right\}} 1 \\
&\geq c\Phi_{M_{\alpha_k}}^{3/4} \to \infty, \text{ when } k \to \infty. \quad \square
\end{aligned}$$

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*Received August 28, 2012.*

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