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ON THE APPROXIMATE AND WEAK APPROXIMATE AMENABILITY OF THE SECOND DUAL OF BANACH ALGEBRAS

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Abstract. Let A be a Banach algebra. We investigate the relations between the bounded approximate amenability of A'' and Arens regularity of A and the role of topological centres in the approximate amenability of A''. We also find conditions under which approximate weak amenability of A'' necessitates that of A. In particular, we show that if S is an infinite weakly cancellative semigroup, then the approximate weak amenability of $l^1(S)''$ necessitates that of $l^1(S)$.

1. Introduction

In [11], Ghahramani, Loy and Willis considered the possibility of the second dual of a Banach algebra being either amenable or weakly amenable.

In particular, they showed that for a Banach algebra A, the amenability of the second dual A'' of A necessitates the amenability of A, and similarly for weak amenability provided A is a left ideal in A''.

In [10], Ghahramani and Loy introduced generalized notions of amenability, they gave examples to show that for most of these new notions, the corresponding class of Banach algebras is larger than that for the classical amenable Banach algebra introduced by Johnson in [13]. They also showed that for a Banach algebra A, the approximate amenability of the second dual A'' necessitates the approximate amenability of A. In this paper, we shall continue in the spirit as in [11] and [10] by focusing on the following questions:

(1) Does the bounded approximate amenability of the second dual A'' imply that A is Arens regular?
(2) Does the approximate weak amenability of the second dual A'' imply the approximate weak amenability of A?

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We show that under certain additional assumptions on A or A'', the answers to these questions are positive. We also explore the roles of topological centres in the approximate amenability of A''.

2. Preliminaries

First, we recall some standard notions; for further details, see [4] and [15].

Let A be an algebra. Let X be an A-bimodule. A *derivation* from A to X is a linear map $D\colon A \to X$ such that

$$D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in A).$$

For example, for $x \in X$, $\delta_x\colon a \to a \cdot x - x \cdot a$ is a derivation; derivations of this form are the *inner derivations*.

The Banach algebra A is *amenable* if, for each Banach A-bimodule X, every continuous derivation $D\colon A \to X'$ is inner and *weakly amenable* if, every continuous derivation $D\colon A \to A'$ is inner.

Let A be a Banach algebra and let X be a Banach A-bimodule. A derivation $D\colon A \to X$ is *approximately inner* if there is a net (x_α) in X such that

$$D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in A),$$

the limit being taken in $(X, \|.\|)$. That is, $D(a) = \lim_\alpha \delta_{x_\alpha}(a)$, where (δ_{x_α}) is a net of inner derivations. The Banach algebra A is *approximately amenable* if, for each Banach A-bimodule X, every continuous derivation $D\colon A \to X'$ is approximately inner and *approximately weakly amenable* if, every continuous derivation $D\colon A \to A'$ is approximately inner. A is said to be boundedly approximately amenable if the net (δ_{x_α}) can always be taken to be norm bounded in $\mathcal{B}(A, X')$. The basic properties of approximately amenable Banach algebras were established in [10], see also [3], [9] and [12].

Let A be a Banach algebra. Then the second dual A'' of A is a Banach A-bimodule for the maps $(a, \Phi) \to a \cdot \Phi$ and $(a, \Phi) \to \Phi \cdot a$ from $A \times A''$ to A'' that extend the product map $A \times A \to A$, $(a, b) \to ab$ on A. Arens in [1] defined two products, $\Box$ and $\Diamond$, on the second dual A'' of a Banach algebra A; A'' is a Banach algebra with respect to each of these products, and each algebra contains A as a closed subalgebra. The products are called the *first* and *second Arens products* on A'', respectively. For the general theory of Arens products, see [7, 8]. We recall briefly the definitions. For $\Phi \in A''$, we set

$$\langle a, \lambda \cdot \Phi\rangle = \langle \Phi, a \cdot \lambda\rangle, \quad \langle a, \Phi \cdot \lambda\rangle = \langle \Phi, \lambda \cdot a\rangle \quad (a \in A, \lambda \in A'),$$

so that $\lambda \cdot \Phi, \Phi \cdot \lambda \in A'$. Let $\Phi, \Psi \in A''$. Then

$$\langle \Phi \Box \Psi, \lambda\rangle = \langle \Phi, \Psi \cdot \lambda\rangle, \quad \langle \Phi \Diamond \Psi, \lambda\rangle = \langle \Psi, \lambda \cdot \Phi\rangle \quad (\lambda \in A').$$

Suppose that $\Phi, \Psi \in A''$ and that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$ for nets (a_α) and (b_β) in A. Then

$$\Phi \Box \Psi = \lim_\alpha \lim_\beta a_\alpha b_\beta \text{ and } \Phi \Diamond \Psi = \lim_\beta \lim_\alpha a_\alpha b_\beta,$$

where all limits are taken in the weak-$*$ topology $\sigma(A'', A')$ on A''.

3. Approximate amenability of some subalgebras of A''

We recall the definition of the topological centres of the second dual of A. For details, see [6] and [7]. Let A be a Banach algebra. The left and right topological centres, $Z_t^{(l)}(A'')$ and $Z_t^{(r)}(A'')$ of A'' are

$$Z_t^{(l)}(A'') = \{\Phi \in A'' : \Phi\Box\Psi = \Phi\Diamond\Psi \text{ for all } \Psi \in A''\},$$

$$Z_t^{(r)}(A'') = \{\Phi \in A'' : \Psi\Box\Phi = \Psi\Diamond\Phi \text{ for all } \Psi \in A''\},$$

respectively. Clearly $Z_t^{(l)}(A'')$ and $Z_t^{(r)}(A'')$ are closed subalgebras of A'' endowed with the Arens products. The Banach algebra A is Arens regular if $Z_t^{(l)}(A'') = Z_t^{(r)}(A'') = A''$.

For a Banach algebra A, we denote by A^{op} the opposite Banach algebra to A, this Banach algebra has the product $(a, b) \to ba$, $A \times A \to A$.

Proposition 3.1. *Let A be a Banach algebra. Then*

(1) *A is approximately amenable if and only if A^{op} is approximately amenable.*
(2) *Suppose that A admits a continuous anti-isomorphism. Then $(A'', \Box)$ is approximately amenable if and only if $(A'', \Diamond)$ is approximately amenable.*

Proof. (1) is trivial.

(2) Let τ be a continuous anti-isomorphism of A. Let $\Phi, \Psi \in (A'', \Box)$ and let (a_α) and (b_β) be nets in A such that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$. Let $\tau''\colon (A'', \Box) \to (A'', \Diamond)$ be the second dual of τ. Then

$$\tau''(\Phi\Box\Psi) = \lim_\alpha \lim_\beta \tau''(a_\alpha b_\beta)$$

$$= \lim_\alpha \lim_\beta \tau''(b_\beta)\tau''(a_\alpha) = \tau''(\Psi)\Diamond\tau''(\Phi).$$

Thus, τ'' is an isomorphism from $(A'', \Box)$ onto $(A'', \Diamond)^{op}$ and so, by (1), $(A'', \Box)$ is approximately amenable if and only if $(A'', \Diamond)$ is approximately amenable. $\Box$

The following is a well-known characterization of approximate amenability of A.

Theorem 3.2 (see [10])**.** *Let A be a Banach algebra.*

(1) *A is approximately amenable if and only if either of the following equivalent conditions hold:*
   (a) *there is a net $(M_v) \subset (A^\sharp \widehat{\otimes} A^\sharp)''$ such that for each $a \in A^\sharp$,*
   $$a \cdot M_v - M_v \cdot a \to 0 \text{ and } \pi''(M_v) \to e;$$
   (b) *there is a net $(M'_v) \subset (A^\sharp \widehat{\otimes} A^\sharp)''$ such that for each $a \in A^\sharp$,*
   $$a \cdot M'_v - M'_v \cdot a \to 0 \text{ and } \pi''(M'_v) = e.$$

(2) *Suppose that $(A'', \Box)$ is approximately amenable. Then A is approximately amenable.*

In the next proposition, we strengthen the result in Theorem 3.2 (b). We assume that A'' has the first Arens product and we denote by $\hat{A}$ the image of A in A'' under the canonical mapping.

Proposition 3.3. *Let B be a closed subalgebra of A'' such that $\hat{A} \subset B$. If B is approximately amenable, then A is approximately amenable.*

Proof. Let $C = A^\sharp$, where $A^\sharp$ is the unitization of A. By [11], there is a continuous linear map $\Theta : C''\widehat{\otimes}C'' \to (C\hat{\otimes}C)''$ such that for $a, b, c \in C$ and $m \in C''\widehat{\otimes}C''$,

$$\Theta(\hat{a} \otimes \hat{b}) = (a \otimes b)\hat{}; \quad \Theta(m) \cdot c = \Theta(m \cdot c);$$

$$c \cdot \Theta(m) = \Theta(c \cdot m); \quad \pi_C''(\Theta(m)) = \pi_{C''}(m),$$

where $\pi_C : C\widehat{\otimes}C \to C$, is defined by $\pi_C(c_1 \otimes c_2) = c_1c_2 \quad (c_1, c_2 \in C)$.

From the definition of projective tensor norm, we see that when both $B^\sharp\widehat{\otimes}B^\sharp$ and $C''\widehat{\otimes}C''$ are equipped with the projective tensor norm, then the map $\tau \colon B^\sharp\widehat{\otimes}B^\sharp \to C''\widehat{\otimes}C''$ defined by

$$\tau(b_1 \otimes b_2) = b_1 \otimes b_2 \quad (b_1, b_2 \in B^\sharp)$$

is norm decreasing.

Since B is approximately amenable, by using Theorem 3.2(a)(i), there is a net $(N_v) \subset (B^\sharp\widehat{\otimes}B^\sharp)''$ such that for all $b \in B^\sharp$,

$$b \cdot N_v - N_v \cdot b \to 0, \quad b \cdot \pi_{B^\sharp}''(N_v) \to b.$$

We set $\Gamma = \Theta \circ \tau \colon B^\sharp\widehat{\otimes}B^\sharp \to (C\widehat{\otimes}C)''$. Then for all $c \in C$, we have

$$\Gamma(N_v) \cdot c - c \cdot \Gamma(N_v) \to 0 \text{ and } \pi_C''(\Gamma(N_v)) \cdot c \to c.$$

Thus, A is approximately amenable using Theorem 3.2(a)(i). $\Box$

We recall that A' is said to factor on the left if $A'A = A'$, [14]. When A has a bounded approximate identity and A'' has an identity, then A' factors on the left. With this, we have the next result.

Theorem 3.4. *Let A be a commutative Banach algebra. Suppose that $(A'', \Box)$ is boundedly approximately amenable and $\hat{A}\Box A'' \subset Z_t^{(l)}(A'')$. Then A is Arens regular.*

Proof. Since A is commutative and $(A'', \Box)$ is boundedly approximately amenable, then A has a bounded approximate identity and $(A'', \Box)$ has an identity [2, Proposition 6.1], and so A' factors on the left. Let $f \in A'$, then $f = g \cdot a$, for some $g \in A'$ and $a \in A$. Let $\Phi, \Psi \in A''$, and $f \in A'$. Then, since $\hat{a}\Box\Phi \in Z_t^{(l)}(A'')$ and $\hat{a}\Box\Phi = \hat{a}\Diamond\Phi$, we have

$$\begin{aligned}\langle \Phi\Box\Psi, f\rangle &= \langle \Phi\Box\Psi, g\cdot a\rangle = \langle \hat{a}\Box(\Phi\Box\Psi), g\rangle = \langle (\hat{a}\Box\Phi)\Box\Psi, g\rangle \\ &= \langle (\hat{a}\Box\Phi)\Diamond\Psi, g\rangle = \langle (\hat{a}\Diamond\Phi)\Diamond\Psi, g\rangle \\ &= \langle \hat{a}\Diamond(\Phi\Diamond\Psi), g\rangle = \langle \Phi\Diamond\Psi, g\cdot a\rangle = \langle \Phi\Diamond\Psi, f\rangle\end{aligned}$$

and so, $\Phi\Box\Psi = \Phi\Diamond\Psi$. Thus, A is Arens regular. □

4. Approximate weak amenability of A''

Let $A^2 = \text{span}\,\{ab : a, b \in A\}$, we recall that A is essential if $\overline{A^2} = A$ (that is, A^2 is dense in A). It is known that if a Banach algebra A is weakly amenable, then A is essential. The same result is true if A is approximately weakly amenable by using the arguments of [5], Proposition 1.3(i) with proper modifications. Thus, we have the following:

Proposition 4.1. *Let A be a Banach algebra. Suppose A is approximately weakly amenable. Then A is essential.*

Proposition 4.2. *Suppose that A'' is approximately weakly amenable. Then A is essential.*

Proof. Since A'' is approximately weakly amenable, then A'' is essential by Proposition 4.1, that is, A'' is dense in $(A'')^2$. Let $a \in A$, then there exists a sequence (x_n) in $(A'')^2$ such that

$$x_n = \sum_{k=1}^{k(n)} \Phi_{n,k}\Box\Psi_{n,k} \text{ and norm-}\lim_n x_n = \hat{a} \quad (\hat{a}, \Phi_{n,k}, \Psi_{n,k} \in A'').$$

Also, for each n and k, there exist nets $\{a_{n,k,i} : i \in I\}$ and $\{b_{n,k,j} : j \in J\}$ such that

$$\lim_i \widehat{a_{n,k,i}} = \Phi_{n,k} \text{ and } \lim_j \widehat{b_{n,k,j}} = \Psi_{n,k}$$

where all limits are taken in the weak-$*$ topology $\sigma(A'', A')$ on A''. Hence

$$\Phi_{n,k}\Box\Psi_{n,k} = \lim_i \lim_j \widehat{a_{n,k,i}}\Box\widehat{b_{n,k,j}}$$

(where all limits are taking in the weak-$*$ topology $\sigma(A'', A')$ on A'',) and so

$$\hat{a} = \text{norm} - \lim_n \left(\lim_i \lim_j \widehat{a_{n,k,i}}\Box\widehat{b_{n,k,j}} \right).$$

Thus, $\hat{a}$ belongs to the closure in the weak-$*$ topology $\sigma(A'', A')$ in A'' of the set $\hat{A}\Box\hat{A}$, which means that a is in the weak closure of span(AA). Since span (AA) is convex, it follows that a belongs to the norm-closure of span (AA). Thus, A^2 is dense in A, that is, A is essential. □

We recall that a dual Banach algebra is a Banach algebra A such that $A = X'$, as a Banach space, for some Banach space X, and such that the multiplication on A is separately weak*-continuous. For a dual Banach algebra $A = X', \varphi(X)$ is a submodule of $A' = X''$, where $\varphi\colon X \to X''$ is the canonical map given by

$$\langle \varphi(x), f\rangle = \langle f, x\rangle \quad (x \in X, f \in X').$$

Thus, A is a dual Banach algebra if there is a closed submodule X of A' such that $X' = A$. We call X the predual of A.

Theorem 4.3. *Let A be a dual Banach algebra. Suppose A'' is approximately weakly amenable, then A is approximately weakly amenable.*

Proof. Let $A = X'$, for some Banach space X such that $\varphi(X) = \hat{X}$ is a submodule of the dual module $A' = X''$, where φ is the canonical map defined above. Let $i\colon X \to A'$ be the canonical map and let i' be the dual map of i. Let $a \in A$, then for $x \in X$, we have

$$\langle i'(\hat{a}), x\rangle = \langle \hat{a}, i(x)\rangle = \langle a, x\rangle.$$

Hence $i'(\hat{a}) = a$.

Let $\Phi, \Psi \in A''$, such that $\Phi = \lim_\alpha \widehat{a}_\alpha, \Psi = \lim_\beta \widehat{b}_\beta$ for nets $(a_\alpha), (b_\beta)$ in A, where the limits are taken in the weak-$*$ topology $\sigma(A'', A')$ on A''. Then we have

$$\begin{aligned} i'(\Phi\Box\Psi) &= i'(\lim_\alpha \lim_\beta \widehat{a_\alpha}\widehat{b_\beta}) = \lim_\alpha \lim_\beta i'((a_\alpha b_\beta)) = \lim_\alpha \lim_\beta (a_\alpha b_\beta) \\ &= \lim_\alpha a_\alpha \lim_\beta b_\beta = \lim_\alpha (i'(a_\alpha) \lim_\beta i'(b_\beta)) = i'(\Phi) i'(\Psi) \end{aligned}$$

where the limits are taken in the weak-$*$ topology $\sigma(A'', A')$ on A''. Thus, $i'\colon A'' \to A$ is an algebra homomorphism from A'' onto A. Let $D\colon A \to A'$ be a derivation. Set $\overline{D}\colon i'' \circ D \circ i' : A'' \to A'''$. Let $\Phi, \Phi_1, \Phi_2 \in A''$, then

$$\begin{aligned} \langle \overline{D}(\Phi_1\Box\Phi_2), \Phi\rangle &= \langle (i'' \circ D \circ i')(\Phi_1\Box\Phi_2), \Phi\rangle \\ &= \langle D(i'(\Phi_1) i'(\Phi_2), i'(\Phi)\rangle \\ &= \langle D(i'(\Phi_1) i'(\Phi_2) + i'(\Phi_1) D(i'(\Phi_2)), i'(\Phi)\rangle \\ &= \langle D(i'(\Phi_1)), i'(\Phi_2) i'(\Phi)\rangle + \langle D(i'(\Phi_2)), i'(\Phi) i'(\Phi_1)\rangle \\ &= \langle D(i'(\Phi_1)), i'(\Phi_2\Box\Phi)\rangle + \langle D(i'(\Phi_2)), i'(\Phi\Box\Phi_1)\rangle \\ &= \langle i''(D(i'(\Phi_1))), \Phi_2\Box\Phi\rangle + \langle i''(D(i'(\Phi_2))), \Phi\Box\Phi_1\rangle \\ &= \langle (i'' \circ D \circ i')(\Phi_1) \cdot \Phi_2 + \Phi_1 \cdot (i'' \circ D \circ i')(\Phi_2), \Phi\rangle \\ &= \langle \overline{D}(\Phi_1) \cdot \Phi_2 + \Phi_1 \cdot \overline{D}(\Phi_2), \Phi\rangle. \end{aligned}$$

Thus, $\overline{D}$ is a derivation. Since A'' is approximately weakly amenable, there exists a net (λ_v) in A''' such that

$$\overline{D}(\Phi) = \lim_v (\Phi \cdot \lambda_v - \lambda_v \cdot \Phi) \quad (\Phi \in A'').$$

Also, since A'' is an A-bimodule and the canonical map $j\colon A \to A''$ is an A-bimodule morphism, then $j'\colon A''' \to A'$ is also an A-bimodule morphism. Let $\gamma_v = j'(\lambda_v)$. Then for $a, b \in A$, we have

$$\begin{aligned}
\langle D(a), b\rangle &= \langle D(i'(\hat{a})), i'(\hat{b})\rangle \\
&= \langle i'' D(i'(\hat{a})), \hat{b}\rangle \\
&= \langle \overline{D}(\hat{a}), j(b)\rangle \\
&= \langle \lim_v (\hat{a}\cdot\lambda_v - \lambda_v\cdot\hat{a}), j(b)\rangle \\
&= \langle \lim_v (j'(\hat{a}\cdot\lambda_v - \lambda_v\cdot\hat{a})), b\rangle \\
&= \langle \lim_v (a\cdot j'(\lambda_v) - j'(\lambda_v)\cdot a), b\rangle \\
&= \langle \lim_v (a\cdot\gamma_v - \gamma_v\cdot a), b\rangle
\end{aligned}$$

(since we have shown earlier that $i'(\hat{a}) = a$, $(a \in A)$). Thus, $D(a) = \lim_v(a\cdot\gamma_v - \gamma_v\cdot a)$ $(a \in A)$, and so A is approximately weakly amenable. $\square$

Let S be a semigroup. For $s \in S$, we define $L_s(t) = st$, $R_s(t) = ts$ $(t \in S)$. Let F be a non-empty subset of S. Then $s^{-1}F = L_s^{-1}(F) = \{t \in S : st \in F\}$ and $Fs^{-1} = R_s^{-1}(F) = \{t \in S : ts \in F\}$. We recall that S is weakly left (respectively, right) cancellative if $s^{-1}F$ (respectively, Fs^{-1}) is finite for each $s \in S$ and each finite subset F of S, and S is weakly cancellative if it is both weakly left cancellative and weakly right cancellative. With this definition, we have the following result:

Theorem 4.4. *Let S be an infinite weakly cancellative semigroup. Then $l^1(S)$ is approximately weakly amenable if $l^1(S)''$ is approximate weakly amenable.*

Proof. Since S is weakly cancellative, then $l^1(S)$ is a dual Banach algebra [7, Theorem 4.6], and so, the result follows from Proposition 4.3. $\square$

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