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GENERALIZED WARPED PRODUCT MANIFOLDS AND CRITICAL RIEMANNIAN METRIC

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Abstract. In this paper, we present some properties of generalized warped product manifold. We establish the relationship between the scalar curvature of a generalized warped product $M \times_f N$ of Riemannian manifolds and those ones of M an N, and we obtain a necessary condition for a critical Riemannian metrics on $M \times_f N$.

1. Introduction

Let (M^m, g) be a compact oriented Riemannian manifold. A critical Riemannian metric is a critical point of the functional:

$$H(g) = \int_M |S_g|^2 v_g$$

where S_g is the scalar curvature of (M, g) and v_g is the volume element measured by g.

Locally, if $(x_i)_{i=1}^m$ denote a local coordinates on M, then g is a critical Riemannian metric, if and only if we have:

$$m\nabla_i\nabla_j S_g - mS_g \operatorname{Ric}_{ij}^g - (\Delta S_g)g_{ij} + S_g^2 g_{ij} = 0 \tag{1}$$

where $\operatorname{Ric}_{ij}^g$ denote the components of Ricci tensors with respect to g. For more details, we can refer to [1] and [7].

2. Results on generalized warped product

In this section, we give the definition and some geometric properties of generalized warped product manifolds.

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Definition 1. Let (M^m, g) and (N^n, h) be two Riemannian manifolds, and $f\colon M \times N \to \mathbb{R}$ be a smooth positive function. The generalized warped metric on $M \times_f N$ is defined by

$$G_f = \pi^* g + (f)^2 \eta^* h \tag{2}$$

where $\pi\colon (x, y) \in M \times N \to x \in M$ and $\eta\colon (x, y) \in M \times N \to y \in N$ are the canonical projections.

For all $X, Y \in T(M \times N)$, we have

$$G_f(X, Y) = g(d\pi(X), d\pi(Y) + (f)^2 h(d\eta(X), d\eta(Y))$$

and we denote by $X \wedge_{G_{f^2}} Y$, the linear map:

(3) $Z \in \mathcal{H}(M) \times \mathcal{H}(N) \to \Big(X \wedge_{G_{f^2}} Y\Big)Z = G_{f^2}\Big(Z, Y\Big)X - G_{f^2}\Big(Z, X\Big)Y.$

Theorem 1. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds. If $\overline{\nabla}$ denote the Levi-Civita connection on $(M \times_f N, G_f)$, then for all $X_1, Y_1 \in \mathcal{H}(M)$ and $X_2, Y_2 \in \mathcal{H}(N)$ we have:*

$$\begin{aligned} \overline{\nabla}_X Y = \nabla_X Y &+ X(\ln f)(0, Y_2) + Y(\ln f)(0, X_2) \\ &- \frac{1}{2} h(X_2, Y_2)(\operatorname{grad}_M f^2, \frac{1}{f^2}\operatorname{grad}_N f^2) \end{aligned} \tag{4}$$

where $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ and $\nabla_X Y = (\nabla^M_{X_1} Y^1, \nabla^N_{X_2} Y^2)$

The geometry of product manifolds is considered in [2], [5], [8].

Proof of Theorem 1 follows from the Kosul formula and the following formulas:

$$\begin{aligned} X(f^2).h(Y_2, Z_2) &= 2X(\ln f) G_f((0, Y_2), Z) \\ Y(f^2).h(X_2, Z_2) &= 2Y(\ln f) G_f((0, X_2), Z) \\ Z(f^2).h(X_2, Y_2) &= h(X_2, Y_2) G_f((\operatorname{grad}_M f^2, \frac{1}{f^2}\operatorname{grad}_N f^2), Z) \\ G_f(\nabla_X Y, Z) &= g(\nabla^M_{X_1} Y_1, Z_1) \circ \pi + f^2.h(\nabla^N_{X_2} Y_2, Z_2) \circ \eta \end{aligned}$$

where $Z = (Z_1, Z_2) \in \mathcal{H}(M) \times \mathcal{H}(N)$.

Remark 1. (1) If $f\colon (x, y) \in M \times N \mapsto f(x, y) = f(x)$, then

$$\begin{aligned} \overline{\nabla}_X Y = \nabla_X Y + X_1(\ln f)\Big(0, Y_2\Big) + Y_1(\ln f)\Big(0, X_2\Big) \\ - \frac{1}{2} h(X_2, Y_2)\Big(\operatorname{grad}_M(f^2), 0\Big) \end{aligned}$$

is the Levi-Civita connection of warped product manifolds.

(2) If $f\colon (x,y) \in M \times N \mapsto f(x,y) = f(y)$, then

$$\overline{\nabla}_X Y = \nabla_X Y + X_2(\ln f)\Big(0, Y_2\Big) + Y_2(\ln f)\Big(0, X_2\Big) - \frac{1}{2}h(X_2,Y_2)\Big(0, \frac{1}{f^2}\operatorname{grad}_N(f^2)\Big) = \Big(\nabla^M_{X_1}Y_1, \widehat{\nabla}_{X_2}Y_2\Big)$$

is the Levi-Civita connection of product Riemannian manifolds (M,g) and (N, f^2h), where

$$\widehat{\nabla}_{X_2}Y_2 = \nabla^N_{X_2}Y_2 + X_2(\ln f)Y_2 + Y_2(\ln f)X_2 - h(X_2,Y_2)\operatorname{grad}_N \ln f.$$

From Theorem 1, we obtain

Corollary 1. *For all $X_1, Y_1 \in \mathcal{H}(M)$ and $X_2, Y_2 \in \mathcal{H}(N)$, we have:*

$$\begin{aligned}
\overline{\nabla}_{(X_1,0)}(Y_1,0) &= (\nabla^M_{X_1}Y_1, 0)\\
\overline{\nabla}_{(X_1,0)}(0,Y_2) &= X_1(\ln f)(0,Y_2)\\
\overline{\nabla}_{(0,X_2)}(Y_1,0) &= Y_1(\ln f)(0,X_2)
\end{aligned}$$

and

$$\overline{\nabla}_{(0,X_2)}(0,Y_2) = (0, \nabla^N_{X_2}Y_2) + X_2(\ln f)(0,Y_2) + Y_2(\ln f)(0,X_2) - \frac{1}{2}h(X_2,Y_2)(\operatorname{grad}_M f^2, \frac{1}{f^2}\operatorname{grad}_N f^2).$$

By Theorem 1, Corollary 1 and formula of curvature tensor, we have

Theorem 2. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds and $f\colon M \times N \to \mathbb{R}$ be smooth positive function. If R and $\overline{R}$ denote the curvatures tensors of product manifold $(M \times N, G)$ and generalized warped product manifold $(M \times_f N, G_f)$ respectively, then*

$$\begin{aligned}
(5)\ \overline{R}(X,Y)Z = R(X,Y)Z &+ \frac{1}{f}\Big((\nabla^M_{Y_1}\operatorname{grad}_M f, 0) \wedge_{G_f} (0, X_2)\Big)Z\\
&- \frac{1}{f}\Big((\nabla^M_{X_1}\operatorname{grad}_M f, 0) \wedge_{G_f} (0, Y_2)\Big)Z\\
&+ \frac{1}{f^2}\Bigg[(0, \nabla^N_{Y_2}\operatorname{grad}_N \ln f - Y_2(\ln f)\operatorname{grad}_N \ln f) \wedge_{G_f} (0, X_2)\\
&- (0, \nabla^N_{X_2}\operatorname{grad}_N \ln f - X_2(\ln f)\operatorname{grad}_N \ln f) \wedge_{G_f} (0, Y_2)\\
&- (f^2 \mid \operatorname{grad}_M \ln f\mid^2 + \mid \operatorname{grad}_N \ln f\mid^2)(0,X_2) \wedge_{G_f} (0,Y_2)\Bigg]Z\\
&+ \Big[X_1(Z_2(\ln f)) + X_2(Z_1(\ln f))\Big](0, Y_2)\\
&- \Big[Y_1(Z_2(\ln f)) + Y_2(Z_1(\ln f))\Big](0, X_2)
\end{aligned}$$

for all $X, Y, Z \in \mathcal{H}(M) \times \mathcal{H}(N)$, *where* $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, $Z = (Z_1, Z_2)$ *and* $R(X,Y)Z = (R^M(X_1,Y_1)Z_1, R^N(X_2,Y_2)Z_2)$.

Theorem 3. *Let* (M^m, g) *and* (N^n, h) *be two Riemannian manifolds and* $f\colon M \times N \to \mathbb{R}$ *be smooth positive function. The Ricci curvature from generalized warped product manifolds* $(M \times_f N, G_f)$ *is given by the following formulas:*

$$\begin{aligned}
\mathrm{Ric}((X_1,0),(Y_1,0)) &= \mathrm{Ric}^M(X_1,Y_1) \\
&\quad -ng(\nabla^M_{X_1}\operatorname{grad}_M \ln f + X_1(\ln f)\operatorname{grad}_M \ln f, Y_1) \\
&= \mathrm{Ric}^M(X_1,Y_1) - \frac{n}{f} g(\nabla^M_{X_1}\operatorname{grad}_M f, Y_1) \\
\mathrm{Ric}((X_1,0),(0,Y_2)) &= -nX_1(Y_2(\ln f)) \\
\mathrm{Ric}((0,X_2),(Y_1,0)) &= h(X_2, \operatorname{grad}_N(Y_1(\ln f))) - nX_2(Y_1(\ln f)) \\
&= (1-n)X_2(Y_1(\ln f)) \\
\mathrm{Ric}((0,X_2),(0,Y_2)) &= \mathrm{Ric}^N(X_2,Y_2) + (2-n)h(\nabla^N_{X_2}\operatorname{grad}_N \ln f, Y_2) \\
&\quad +(2-n)h(X_2,Y_2) \mid \operatorname{grad}_N \ln f \mid^2 \\
&\quad -(2-n)X_2(\ln f)h(\operatorname{grad}_N \ln f, Y_2) \\
&\quad -nf^2 h(X_2,Y_2) \mid \operatorname{grad}_M \ln f \mid^2 \\
&\quad -h(X_2,Y_2)\big[\Delta_N(\ln f) + f^2\Delta_M(\ln f)\big]
\end{aligned}$$

for all $X_1, Y_1 \in \mathcal{H}(M)$ *and* $X_2, Y_2 \in \mathcal{H}(N)$.

Lemma 1. *Any local orthonormal frame* $\{e_i, i = 1,\ldots,m\}$ *and* $\{\overline{e}_j, j = 1,\ldots,n\}$ *with respect to* (M^m,g) *and* (N^n,h) *, such* $\nabla^M_{e_s} e_i = 0$ *and* $\nabla^N_{\overline{e}_k}\overline{e}_j = 0$, *induces a local orthonormal frame on* $(M \times_f N, G_f)$ *by:*

$$\{(e_i,0), \quad (0,\frac{1}{f}\overline{e}_j): \quad i = \overline{1,\ldots,m}, \quad j = \overline{1,\ldots,n}\}$$

and followings formulae:

$$\sum_j h(\nabla^N_{\overline{e}_j}\operatorname{grad}_N \ln f), \overline{e}_j) = \Delta_N(\ln f)$$

$$\sum_j h(\overline{e}_j, Y_2)h((\nabla^N_{X_2}\operatorname{grad}_N \ln f), \overline{e}_j) = h(\nabla^N_{X_2}\operatorname{grad}_N \ln f, Y_2)$$

$$\sum_j h(\overline{e}_j, Y_2)h(X_2, \overline{e}_j) = h(X_2, Y_2)$$

Proof of Theorem 3. Using Theorem 2 and the formula of Ricci curvature, by summing over the indexes i and j, we obtain:

$$\begin{aligned}\operatorname{Ric}((X_1,0),(Y_1,0)) = {} & G_f(\overline{R}((e_i,0),(X_1,0))((Y_1,0),(e_i,0)) \\ & + \frac{1}{f^2}G_f(\overline{R}((0,\overline{e}_i),(X_1,0))((Y_1,0),(0,\overline{e}_i)) \\ = {} & g(R_M(e_i,X_1)Y_1,e_i) \\ & - g(\nabla^M_{X_1}\operatorname{grad}_M \ln f + X_1(\ln f)\operatorname{grad}_M \ln f, Y_1)h(\overline{e}_j,\overline{e}_j) \\ = {} & \operatorname{Ric}^M(X_1,Y_1) \\ & - ng(\nabla^M_{X_1}\operatorname{grad}_M \ln f + X_1(\ln f)\operatorname{grad}_M \ln f, Y_1),\end{aligned}$$

$$\begin{aligned}\operatorname{Ric}((X_1,0),(0,Y_2)) = {} & G_f(\overline{R}((e_i,0),(X_1,0))(0,Y_2),(e_i,0)) \\ & + \frac{1}{f^2}G_f(\overline{R}((0,\overline{e}_j),(X_1,0))(0,Y_2),(0,\overline{e}_j)) \\ = {} & \frac{1}{f^2}h(Y_2,\overline{e}_j)G_f((\nabla^M_{X_1}\operatorname{grad}_M \ln f,(0,\overline{e}_j)) \\ & + \frac{1}{f^2}h(Y_2,\overline{e}_j)G_f((X_1(\ln f)\operatorname{grad}_M \ln f,0),(0,\overline{e}_j)) \\ & - \frac{1}{f^2}X_1(Y_2(\ln f))G_f((0,\overline{e}_j),(0,\overline{e}_j)) \\ = {} & -nX_1(Y_2(\ln f)),\end{aligned}$$

$$\begin{aligned}\operatorname{Ric}((0,X_2),(Y_1,0)) = {} & G_f(\overline{R}((e_i,0),(0,X_2))(Y_1,0),(e_i,0)) \\ & + \frac{1}{f^2}G_f(\overline{R}((0,\overline{e}_j),(0,X_2))(Y_1,0),(0,\overline{e}_j)) \\ = {} & g(\nabla^M_{e_i}\operatorname{grad}_M \ln f, Y_1)G_f((0,X_2),(e_i,0)) \\ & + g(e_i(\ln f)\operatorname{grad}_M \ln f, Y_1)G_f((0,X_2),(e_i,0)) \\ & + \frac{1}{f^2}\overline{e}_j(Y_1(\ln f))G_f((0,X_2),(0,\overline{e}_j)) \\ & - \frac{1}{f^2}X_2(Y_1(\ln f))G_f((0,\overline{e}_j),(0,\overline{e}_j)) \\ = {} & h(X_2,\operatorname{grad}_N(Y_1(\ln f))) - nX_2(Y_1(\ln f)),\end{aligned}$$

and

$$\begin{aligned}\operatorname{Ric}((0,X_2),(0,Y_2)) = {} & G_f(\overline{R}((e_i,0),(0,X_2))(0,Y_2),(e_i,0)) \\ & + \frac{1}{f^2}G_f(\overline{R}((0,\overline{e}_j),(0,X_2))(0,Y_2),(0,\overline{e}_j)).\end{aligned} \tag{6}$$

By Theorem 2, formula (3) and Lemma 1, we obtain

$$
\begin{aligned}
(7)\qquad & G_f(\overline{R}((e_i,0),(0,X_2))(0,Y_2),(e_i,0)) \\
&= -f^2h(X_2,Y_2)G_f((\nabla^M_{e_i}\operatorname{grad}_M \ln f,0),(e_i,0)) \\
&\quad -f^2h(X_2,Y_2)G_f((e_i(\ln f)\operatorname{grad}_M \ln f,0),(e_i,0)) \\
&\quad +e_i(Y_2(\ln f))G_f((0,X_2),(e_i,0)) \\
&= -f^2h(X_2,Y_2)\Big[\Delta_M(\ln f)+\mid \operatorname{grad}_M \ln f\mid^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
& G_f(\overline{R}((0,\overline{e}_j),(0,X_2))(0,Y_2),(0,\overline{e}_j)) = f^2\operatorname{Ric}^N(X_2,Y_2) \\
&\quad + \frac{1}{f^2}G_f((0,\nabla^N_{X_2}\operatorname{grad}_N \ln f))\wedge_{G_f}(0,\overline{e}_j))(0,Y_2),(0,\overline{e}_j)) \\
&\quad - \frac{1}{f^2}G_f((0,X_2(\ln f)\operatorname{grad}_N \ln f))\wedge_{G_f}(0,\overline{e}_j))(0,Y_2),(0,\overline{e}_j)) \\
&\quad - \frac{1}{f^2}G_f((0,\nabla^N_{\overline{e_j}}\operatorname{grad}_N \ln f))\wedge_{G_f}(0,X_2))(0,Y_2),(0,\overline{e}_j)) \\
&\quad + \frac{1}{f^2}G_f((0,\overline{e_j}(\ln f)\operatorname{grad}_N \ln f))\wedge_{G_f}(0,X_2))(0,Y_2),(0,\overline{e}_j)) \\
&\quad - \frac{1}{f^2}(f^2\mid \operatorname{grad}_M \ln f\mid^2)G_f(((0,\overline{e}_j)\wedge_{G_f}(0,X_2))(0,Y_2),(0,\overline{e}_j)) \\
&\quad - \frac{1}{f^2}(\mid \operatorname{grad}_N \ln f\mid^2)G_f(((0,\overline{e}_j)\wedge_{G_f}(0,X_2))(0,Y_2),(0,\overline{e}_j)) \\
&= f^2\operatorname{Ric}^N(X_2,Y_2) \\
&\quad + f^2h(Y_2,\overline{e_j})h(\nabla^N_{X_2}\operatorname{grad}_N \ln f - X_2(\ln f)\operatorname{grad}_N \ln f),\overline{e}_j) \\
&\quad - f^2h(Y_2,\nabla^N_{X_2}\operatorname{grad}_N \ln f - X_2(\ln f)\operatorname{grad}_N \ln f)h(\overline{e_j},\overline{e}_j) \\
&\quad - f^2h(Y_2,X_2)h(\nabla^N_{\overline{e_j}}\operatorname{grad}_N \ln f - \overline{e_j}(\ln f)\operatorname{grad}_N \ln f,\overline{e_j}) \\
&\quad + f^2h(Y_2,\nabla^N_{\overline{e_j}}\operatorname{grad}_N \ln f - \overline{e_j}(\ln f)\operatorname{grad}_N \ln f)h(X_2,\overline{e}_j) \\
&\quad - f^2(f^2\mid \operatorname{grad}_M \ln f\mid^2)(h(X_2,Y_2)h(\overline{e}_j,\overline{e}_j) - h(\overline{e}_j,Y_2)h(X_2,\overline{e}_j)) \\
&\quad - f^2(\mid \operatorname{grad}_N \ln f\mid^2)(h(X_2,Y_2)h(\overline{e}_j,\overline{e}_j) - h(\overline{e}_j,Y_2)h(X_2,\overline{e}_j))
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{f^2}G_f(\overline{R}((0,\overline{e}_j),(0,X_2))(0,Y_2),(0,\overline{e}_j)) = \\
&= \operatorname{Ric}^N(X_2,Y_2) + h(Y_2,(\nabla^N_{X_2}\operatorname{grad}_N \ln f - X_2(\ln f)\operatorname{grad}_N \ln f)) \\
&\quad - nh(Y_2,\nabla^N_{X_2}\operatorname{grad}_N \ln f - X_2(\ln f)\operatorname{grad}_N \ln f) \\
&\quad - h(Y_2,X_2)(\Delta_N(\ln f) - \mid \operatorname{grad}_N \ln f\mid^2) \\
&\quad + h(Y_2,\nabla^N_{X_2}\operatorname{grad}_N \ln f) - h(Y_2,\operatorname{grad}_N \ln f)h(X_2,\operatorname{grad}_N \ln f) \\
&\quad - (f^2\mid \operatorname{grad}_M \ln f\mid^2 + \mid \operatorname{grad}_N \ln f\mid^2)(nh(X_2,Y_2) - h(Y_2,X_2))
\end{aligned}
$$

then

$$\begin{aligned}(8)\qquad \frac{1}{f^2}G_f(\overline{R}((0,\overline{e}_j),(0,X_2))(0,Y_2),(0,\overline{e}_j)) &= \mathrm{Ric}^N(X_2,Y_2)\\ &+(2-n)h(Y_2,(\nabla^N_{X_2}\operatorname{grad}_N\ln f - X_2(\ln f)\operatorname{grad}_N\ln f))\\ &-h(Y_2,X_2)(\Delta_N(\ln f) - |\operatorname{grad}_N\ln f|^2)\\ &-(n-1)(f^2\mid\operatorname{grad}_M\ln f\mid^2 + \mid\operatorname{grad}_N\ln f\mid^2)h(X_2,Y_2).\end{aligned}$$

Substituting (7) and (8) in (6), we deduce:

$$\begin{aligned}\mathrm{Ric}((0,X_2),(0,Y_2)) &= \mathrm{Ric}^N(X_2,Y_2) + (2-n)h(X_2,Y_2)|\operatorname{grad}_N\ln f\mid^2\\ &+(2-n)h(\nabla^N_{X_2}\operatorname{grad}_N\ln f - X_2(\ln f)\operatorname{grad}_N\ln f, Y_2)\\ &-h(X_2,Y_2)\Big[nf^2\mid\operatorname{grad}_M\ln f\mid^2 + \Delta_N(\ln f) + f^2\Delta_M(\ln f)\Big]. \quad\square\end{aligned}$$

Corollary 2. *Locally, the component of Ricci tensors from generalized warped product manifolds $(M\times_f N, G_f)$ are given by the following formulas:*

$$(9)\qquad \mathrm{Ric}_{ij} = \mathrm{Ric}^M_{ij} - \frac{n}{f}\nabla^M_i\nabla^M_j f$$

$$(10)\qquad \mathrm{Ric}_{ib} = -n\nabla^M_i\nabla^N_b(\ln f))$$

$$(11)\qquad \mathrm{Ric}_{aj} = (1-n)\nabla^N_a\nabla^M_j(\ln f))$$

$$\begin{aligned}(12)\qquad \mathrm{Ric}_{ab} &= \mathrm{Ric}^N_{ab} - h_{ab}\big[\Delta_N(\ln f) + f^2\Delta_M(\ln f)\big]\\ &-h_{ab}\big[nf^2\mid\operatorname{grad}_M\ln f\mid^2 + (n-2)\mid\operatorname{grad}_N\ln f\mid^2\big]\\ &+(2-n)\big[\nabla^N_a\nabla^N_b\ln f - \nabla^N_a(\ln f)\nabla^N_b(\ln f))\big]\end{aligned}$$

$i,j=1,\dots,m$ and $a,b=1,\dots,n$.

2.1. Scalar curvature.

Theorem 4. *Let S^M, S^N and $\overline{S}$ denote the scalar curvature on (M^m,g), (N^n,h) and $(M\times_{G_f}N,G_f)$ respectively. Then the following equality holds:*

$$\begin{aligned}(13)\quad \overline{S} &= S^M + \frac{1}{f^2}S^N - 2n\Delta_M(\ln f) + \frac{2(1-n)}{f^2}\Delta_N(\ln f)\\ &- n(n-1)|\operatorname{grad}_M\ln f|^2 - \frac{(n-1)(n-2)}{f^2}|\operatorname{grad}_N\ln f|^2.\end{aligned}$$

Proof. Let $\{e_i, i=1,\dots,m\}$ and $\{\overline{e}_a, a=1,\dots,n\}$ with respect to (M^m,g) and (N^n,h) , such $\nabla^M_{e_i}e_j=0$ and $\nabla^N_{\overline{e}_a}\overline{e}_b=0$, we have:

$$(14)\qquad \overline{S} = \sum_{i=1}^m \mathrm{Ric}((e_i,0),(e_i,0)) + \frac{1}{f^2}\sum_{a=1}^n \mathrm{Ric}((0,\overline{e}_a),(0,\overline{e}_a)).$$

From Theorem 3, we obtain

$$(15)\quad \sum_{i=1}^{m}\mathrm{Ric}((e_i,0),(e_i,0)) = S^M - n\sum_{i=1}^{m} g(\nabla^M_{e_i}\operatorname{grad}_M \ln f, e_i) - n\sum_{i=1}^{m} g(e_i(\ln f)\operatorname{grad}_M \ln f, e_i) = S^M - n|\operatorname{grad}_M \ln f|^2 - n\Delta_M(\ln f)$$

$$\sum_{a=1}^{n}\mathrm{Ric}((0,\overline{e}_a),(0,\overline{e}_a)) = S^N + (2-n)\sum_{a=1}^{n} h(\nabla^N_{\overline{e}_a}\operatorname{grad}_N \ln f, \overline{e}_a) + (2-n)\sum_{a=1}^{n}\big[h(\overline{e}_a,\overline{e}_a) \mid \operatorname{grad}_N \ln f\mid^2 - \overline{e}_a(\ln f)h(\operatorname{grad}_N \ln f, \overline{e}_a)\big] - \sum_{a=1}^{n} h(\overline{e}_a,\overline{e}_a)\big[nf^2 \mid \operatorname{grad}_M \ln f\mid^2 + \Delta_N(\ln f) + f^2\Delta_M(\ln f)\big]$$

$$(16)\quad \sum_{a=1}^{n}\mathrm{Ric}((0,\overline{e}_a),(0,\overline{e}_a)) = S^N + 2(1-n)\Delta_N(\ln f) + (2-n)(n-1)\mid \operatorname{grad}_N \ln f\mid^2 - n^2 f^2 \mid \operatorname{grad}_M \ln f\mid^2 - nf^2\Delta_M(\ln f)$$

Substituting (22) and (16) in (14), we deduce the equality (13). □

Corollary 3. *Let $U = f^{\frac{n+1}{2}}$ and $V = f^{\frac{n-2}{2}}$, $(n \geq 3)$, then*

$$(17)\quad \overline{S} = S^M + \frac{1}{f^2}S^N - \frac{4n}{n+1}U^{-1}\Delta_M U - \frac{4(n-1)}{n-2}V^{-\frac{n+2}{n-2}}\Delta_N V.$$

Particular cases:

(1) If $f(x,y) = f(y)$, $n \geq 3$ and $g = 0$. From the formula (17), we obtain the Yamabe equation associated to conformal metrics

$$\overline{S}.V^{\frac{n+2}{n-2}} = S^N.V - \frac{4(n-1)}{n-2}\Delta_N V.$$

(2) If $f(x,y) = f(x)$. From the formula (17), we obtain the scalar curvature equation of warped product manifolds:

$$\overline{S}.U = S^M.U + S^N.U^{\frac{n-3}{n+1}} - \frac{4n}{n+1}\Delta_M U.$$

the result is obtained in [3]

(3) If $n = 1$ then $f = U$ and

$$\overline{S}.U = S^M.U + S^N.U^{-1} - 2\Delta_M U.$$

(4) If $n = 2$ and $\gamma = \ln f$ then

$$\overline{S} = S^M + e^{-2\gamma}S^N - 4\Delta_M(\gamma) - 2e^{-2\gamma}\Delta_N(\gamma) - 2|\operatorname{grad}_M \gamma|^2.$$

From Theorem 4 and Corollary 2 we deduce:

Corollary 4. *If* $\ln f(x,y) = f_1(x) + f_2(y)$, *then the following formulas holds*

$$\text{(18)} \quad \mathrm{Ric}_{ij} = \mathrm{Ric}^M_{ij} - \frac{n}{f}\nabla^M_i \nabla^M_j f$$

$$\text{(19)} \quad \mathrm{Ric}_{ib} = 0$$

$$\text{(20)} \quad \mathrm{Ric}_{aj} = 0$$

$$\text{(21)} \quad \begin{aligned} \mathrm{Ric}_{ab} = \mathrm{Ric}^N_{ab} &- h_{ab}\big[nf^2 \mid \mathrm{grad}_M f_1 \mid^2 + \Delta_N(f_2) + f^2\Delta_M(f_1)\big] \\ &+ (2-n)\big[\nabla^N_a \nabla^N_b f_2 + h_{ab} \mid \mathrm{grad}_N f_2 \mid^2 - \nabla^N_a(f_2)\nabla^N_b(f_2))\big] \end{aligned}$$

and

$$\text{(22)} \quad \begin{aligned} \overline{S} = S^M &+ e^{-2(f_1+f_2)}S^N - 2n\Delta_M(f_1) + 2(1-n)e^{-2(f_1+f_2)}\Delta_N(f_2) \\ &- n(n-1)|\,\mathrm{grad}_M f_1|^2 - (n-1)(n-2)e^{-2(f_1+f_2)}|\,\mathrm{grad}_N f_2|^2. \end{aligned}$$

Theorem 5. *Let* (M^m, g) *and* (N^n, h) *be compact manifolds with scalar curvatures* S^M *and* S^N *respectively and* $\ln f(x,y) = f_1(x) + f_2(y)$. *If* G_f *is critical Riemannian metric on* $M \times N$, *then the warped product space* $(M \times_f N, G_f)$ *is Riemannian product space or*

$$\text{(23)} \qquad S^N = e^{2.f_2} + 2(n-1)\Delta_N(f_2) + (n-1)(n-2)|\,\mathrm{grad}_n f_2|^2.$$

Proof. Let

$$H = S^N + 2(1-n)\Delta_N(f_2) - (n-1)(n-2)|\,\mathrm{grad}_N f_2|^2$$

For $i = 1, \ldots, n$ and $a = 1, \ldots, n$ and using formula (22), we obtain

$$\begin{aligned} \nabla_i \nabla_a \overline{S} &= \partial_i(\partial_a(e^{-2(f_1+f_2)}H)) \\ &= \partial_i(e^{-2(f_1+f_2)}(\partial_a(H) - 2H\partial_a(f_2))) \end{aligned}$$

$$\text{(24)} \qquad \nabla_i \nabla_a \overline{S} = -2e^{-2(f_1+f_2)}\partial_i(f_1)[\partial_a(H) - 2H\partial_a(f_2)]$$

Other hand, from formula (1) and Corollary 4, we have

$$\text{(25)} \qquad \nabla_i \nabla_a \overline{S} = 0.$$

then

$$\text{(26)} \qquad \partial_i(f_1)[\partial_a(H) - 2H\partial_a(f_2)] = 0.$$

Hence, the solution of equation 2 is given by

$$f_1 = \text{constant or } H = e^{2.f_2}.$$

If $f_1 = \textit{constant}$, then the warped product space $(M \times_f N, G_f)$ is the product space $(M \times N, g \oplus f^2 h)$ of Riemannian manifolds (M^m, g) and $(N^n, f^2.h)$. □

Remark 2. In the case where $f_2 = \textit{constant}$, we recover the result obtained in [6].

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