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GENERALIZED SASAKIAN SPACE FORMS AND TRANS-SASAKIAN MANIFOLDS

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Abstract. In this paper we study the generalized Sasakian space forms and trans-Sasakian manifolds. We present results on generalized recurrent, generalized ϕ-recurrent, ϕ-concircular and ϕ-conharmonically recurrent trans-Sasakian manifolds and generalized Sasakian space forms.

1. Introduction

P. Alegre, D. E. Blair and A. Carriazo [1] initiated the study of generalized Sasakian space forms and presented some examples. In [2], the authors studied the structure of generalized Sasakian space forms and proved that a K-contact generalized Sasakian space form is Sasakian and if its dimension is greater than 5 then it is a Sasakian space form. Further the authors proved that any three dimensional trans-Sasakian manifold with α and β depending only on the direction of ξ is a generalized Sasakian space form. The authors U. C. De and Avijit Sarkar [7], [8] studied curvature properties of generalized Sasakian space forms and obtained important results. Motivated by the above studies, in this paper we study the trans-Sasakian manifolds and the generalized Sasakian space forms. The paper is organized as follows. After preliminaries in Section 2, we study generalized Sasakian space forms and generalized recurrent trans-Sasakian manifolds in Section 3. Generalized ϕ-recurrent trans-Sasakian manifolds are studied in Section 4. The sections 5 and 6 contain results on ϕ-concircular recurrent and ϕ-conharmonically recurrent manifolds.

2. Preliminaries

An odd dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exist on M a $(1, 1)$ tensor field ϕ, a vector

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field ξ and a 1-form η such that

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\text{(i) } \eta(\xi) = 1, \qquad \text{(ii) } g(X,\xi) = \eta(X), \tag{2.2}$$

$$\text{(i) } \eta(\phi X) = 0, \qquad \text{(ii) } \phi\xi = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.4}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.5}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y), \tag{2.6}$$

for any vector fields X, Y on M.

An almost contact metric manifold (M, ϕ, ξ, η, g) is called a trans-Sasakian manifold [11] if there exist two functions α and β on M such that

$$(\nabla_X \phi)(Y) = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.7}$$

for any vector fields X, Y on M. From (2.7), it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.8}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.9}$$

It is well known that α-Sasakian ($\beta = 0$), β-Kenmotsu ($\alpha = 0$) and cosymplectic ($\alpha = \beta = 0$) manifolds are special cases of trans-Sasakian manifolds.

A ϕ-section of an almost contact metric manifold (M, ϕ, ξ, η, g) at $p \in M$ is a section $\Pi \subseteq T_pM$ spanned by a unit vector X_p orthogonal to ξ and ϕX_p. The ϕ-sectional curvature is defined by

$$B(X, \phi X) = R(X, \phi X, \phi X, X). \tag{2.10}$$

A Sasakian manifold with constant ϕ-sectional curvature c is called a Sasakian space form and is denoted by $M(c)$. The Riemannian curvature tensor R in $M(c)$ is given by

$$R(X,Y)Z = \frac{c+3}{4}R_1(X,Y)Z + \frac{c-1}{4}R_2(X,Y)Z + \frac{c-1}{4}R_3(X,Y)Z, \tag{2.11}$$

for any vector fields X, Y and Z on M, where

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{2.12}$$

$$R_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \tag{2.13}$$

and

$$\begin{aligned} R_3(X,Y)Z \\ &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi. \end{aligned} \tag{2.14}$$

In [1], the authors introduced the notion of generalized Sasakian space form as an almost contact metric manifold (M, ϕ, ξ, η, g) whose Riemannian curvature tensor satisfies

$$R(X,Y)Z = f_1R_1(X,Y)Z + f_2R_2(X,Y)Z + f_3R_3(X,Y)Z, \tag{2.15}$$

where f_1,f_2 and f_3 are differentiable functions on M.
Throughout the paper $M(f_1, f_2, f_3)$ denote a generalized Sasakian space form, where M is a trans-Sasakian manifold.
In a generalized Sasakian space form the following hold:

$$S(Y,Z) = [(n-1)f_1 + 3f_2 - f_3]g(Y,Z) - [3f_2 + (n-2)f_3]\eta(Y)\eta(Z), \tag{2.16}$$

$$QY = [(n-1)f_1 + 3f_2 - f_3]Y - [3f_2 + (n-2)f_3]\eta(Y)\xi, \tag{2.17}$$

$$S(Y,\xi) = (n-1)(f_1 - f_3)\eta(Y), \tag{2.18}$$

$$Q\xi = (n-1)(f_1 - f_3)\xi, \tag{2.19}$$

$$r = n(n-1)f_1 + 3(n-1)f_2 - 2(n-1)f_3, \tag{2.20}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.21}$$

$$R(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{2.22}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{2.23}$$

3. Generalized recurrent manifolds

A generalized Sasakian space form $M(f_1, f_2, f_3)$ is called generalized recurrent [5], if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)(W) = A(X)R(Y,Z)W + B(X)(g(Z,W)Y - g(Y,W)Z), \tag{3.1}$$

where A and B are two 1-forms and B is non zero.
Taking $Y = W = \xi$ in (3.1), we obtain

$$(\nabla_X R)(\xi,Z)(\xi) = A(X)R(\xi,Z)\xi + B(X)(\eta(Z)\xi - Z). \tag{3.2}$$

By definition of covariant derivative, we have

$$\begin{aligned}(\nabla_X R)&(\xi,Z)\xi \\ &= \nabla_X R(\xi,Z)\xi - R(\nabla_X\xi,Z)\xi - R(\xi,\nabla_X Z)\xi - R(\xi,Z)\nabla_X\xi.\end{aligned} \tag{3.3}$$

Using (2.21), (2.22) and (2.8) in (3.3), we obtain

$$\begin{aligned}(\nabla_X R)&(\xi,Z)\xi \\ &= \nabla_X[(f_1 - f_3)(\eta(Z)\xi - Z)] - (f_1 - f_3)[\eta(\nabla_X Z)\xi - \nabla_X Z] \\ &\quad - (f_1 - f_3)[-\alpha\eta(Z)\phi X + \beta\eta(Z)X - \beta\eta(X)\eta(Z)\xi] \\ &\quad - (f_1 - f_3)[-\alpha g(Z,\phi X)\xi + \beta(g(Z,X)\xi - \eta(X)\eta(Z)\xi)].\end{aligned} \tag{3.4}$$

Using (2.8) and (2.9), equation (3.4) reduces to

$$(\nabla_X R)(\xi,Z)\xi = d((f_1 - f_3))(X)(\eta(Z)\xi - Z) + (f_1 - f_3)\nabla_X Z. \tag{3.5}$$

From (3.2) and (3.5), we have

(3.6) $$[d(f_1-f_3)(X)-A(X)(f_1-f_3)-B(X)](\eta(Z)\xi-Z)+(f_1-f_3)\nabla_X Z=0.$$

Taking Z=ξ in (3.6), we obtain

(3.7) $$(f_1-f_3)\nabla_X\xi=0.$$

From (3.7), we get that M is co-symplectic provided $f_1\neq f_3$.

Thus we have

Proposition 3.1. *A generalized recurrent $M(f_1,f_2,f_3)$ is co-symplectic provided $f_1\neq f_3$.*

As it is well known that M is generalized Ricci-recurrent [6], if its Ricci tensor S-satisfies the condition

(3.8) $$(\nabla_X S)(Y,Z)=A(X)S(Y,Z)+(n-1)B(X)g(Y,Z),$$

where A and B are two non-zero 1-forms. By definition of covariant derivative, we have

(3.9) $$(\nabla_X S)(Y,\xi)=\nabla_X S(Y,\xi)-S(\nabla_X Y,\xi)-S(Y,\nabla_X\xi).$$

Using (2.18) and (2.8) in (3.9), we have

(3.10) $$\begin{aligned}(\nabla_X S)(Y,\xi)&\\ =(n-1)[d(f_1-f_3)(X)\eta(Y)&-(f_1-f_3)(\alpha g(\phi X,Y)-\beta g(X,Y))]\\ &+\alpha S(Y,\phi X)-\beta S(Y,X).\end{aligned}$$

Taking $Z=\xi$ in (3.8) and using (2.18), we obtain

(3.11) $$(\nabla_X S)(Y,\xi)=(n-1)[(f_1-f_3)A(X)+B(X)]\eta(Y).$$

From (3.10) and (3.11), we have

(3.12) $$\begin{aligned}(n-1)(f_1-f_3)[A(X)\eta(Y)+\alpha g(\phi X,Y)-\beta g(X,Y)]+(n-1)B(X)\eta(Y)\\ -(n-1)d(f_1-f_3)(X)\eta(Y)-\alpha S(Y,\phi X)+\beta S(Y,X)=0.\end{aligned}$$

Taking $Y=\xi$ in (3.12), we obtain

(3.13) $$(f_1-f_3)A(X)+B(X)=d(f_1-f_3)(X).$$

If $f_3-f_1=c$, a constant then (3.13) reduces to

(3.14) $$B(X)=cA(X).$$

Since $B(X)$ is not zero, we have $f_1\neq f_3$. Thus we can state that

Proposition 3.2. *In a generalized recurrent $M(f_1,f_2,f_3)$, $f_1\neq f_3$ holds. Further the 1-forms A(X) and B(X) are related by* (3.14).

It is clear from (3.1) and (3.8) that, a generalized recurrent trans-Sasakian manifold is generalized Ricci- recurrent. Hence from Proposition 3.1 and Proposition 3.2 we have

Theorem 3.3. *A generalized recurrent $M(f_1, f_2, f_3)$ is always a co-symplectic manifold.*

4. Generalized ϕ-recurrent manifolds

A generalized Sasakian space form $M(f_1, f_2, f_3)$ is called generalized ϕ- Ricci recurrent [3], [6] if

$$\phi^2((\nabla_X Q)(Y)) = A(X)QY + (n-1)B(X)Y, \tag{4.1}$$

where Q is the Ricci operator, $A(X)$and $B(X)$ are non-zero 1-forms.
Using (2.1) and (4.1), we have

$$-\nabla_X QY - Q(\nabla_X Y) + \eta((\nabla_X Q)(Y))\xi = A(X)QY + (n-1)B(X)Y. \tag{4.2}$$

Taking $Y = \xi$ in (4.2) and contracting with respect to Z, we obtain

$$\begin{aligned} &- g(\nabla_X Q\xi, Z) - g(Q(\nabla_X \xi), Z) + \eta((\nabla_X Q)(\xi))\eta(Z) \\ &\qquad = A(X)g(Q\xi, Z) + (n-1)B(X)\eta(Z). \end{aligned} \tag{4.3}$$

Using (2.8) and (2.19) in (4.3), we have

$$\begin{aligned} &(n-1)[(f_1 - f_3)A(X) + B(X) + d(f_1 - f_3)(X)]\eta(Z) \\ &\quad = (n-1)(f_1 - f_3)(\alpha g(\phi X, Z) - \beta g(X, Z)) - \alpha S(\phi X, Z) + \beta S(X, Z). \end{aligned} \tag{4.4}$$

Changing Z to ϕZ and taking $\beta = 0$ in (4.4), we obtain

$$S(X, Z) = (n-1)(f_1 - f_3)g(X, Z). \tag{4.5}$$

Comparing (4.5) and (2.16), we have $(n-2)f_3 + 3f_2 = 0$. Changing Z to ϕZ and taking $\alpha = 0$ in (4.4), we obtain

$$(n-1)(f_1 - f_3)g(X, \phi Z) - S(X, \phi Z) = 0. \tag{4.6}$$

Using (2.16) in (4.6), we have $(n-2)f_3 + 3f_2 = 0$.
Thus we have

Theorem 4.1. *In an α-Sasakian (or a β-Kenmotsu) generalized Sasakian space form which is ϕ-Ricci recurrent the relation $(n-2)f_3 + 3f_2 = 0$ holds.*

If $A(X)$ and $B(X)$ are zero in (4.1), then $M(f_1, f_2, f_3)$ is called ϕ-Ricci symmetric [6].

It is easy to see that the relation (4.6) holds in ϕ-Ricci-symmetric α-Sasakian (or β-Kenmotsu) generalized Sasakian space form.

Conversely, suppose $(n-2)f_3 + 3f_2 = 0$ holds in ϕ-symmetric $M(f_1, f_2, f_3)$. Then from (2.17), we get

$$QY = (n-1)(f_1 - f_3)Y. \tag{4.7}$$

Differentiating covariantly with respect to X, we obtain

$$(\nabla_X Q)Y = (n-1)\nabla_X((f_1 - f_3)Y). \tag{4.8}$$

Applying ϕ^2 on both sides, we obtain

$$\phi^2((\nabla_X Q)Y) = (n-1)d(f_1 - f_3)(X)\phi^2 Y. \tag{4.9}$$

i.e $M(f_1, f_2, f_3)$ is ϕ-Ricci symmetric if and only if $f_1 - f_3$ is a constant.

It follows from (2.15) that in a generalized Sasakian Space form, the ξ-sectional curvature $K(X, \xi)$ is given by $K(X, \xi) = R(X, \xi, X, \xi) = f_3 - f_1$. Thus we can state that

Theorem 4.2. *An α-Sasakian (or a β-Kenmotsu) generalized Sasakian space form with constant ξ-sectional curvature is ϕ -Ricci symmetric if and only if $(n-2)f_3 + 3f_2 = 0$ holds.*

5. Concircular curvature tensor of $M(f_1, f_2, f_3)$

The concircular curvature tensor of $M(f_1, f_2, f_3)$ is given by

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{5.1}$$

Definition 5.1. $M(f_1, f_2, f_3)$ is said to be ϕ-concircular recurrent[15] if

$$\phi^2((\nabla_W \tilde{C})(X,Y)Z) = A(W)\tilde{C}(X,Y)Z, \tag{5.2}$$

where $A(W)$ is a nonzero 1-form.

Definition 5.2. $M(f_1, f_2, f_3)$ is said to be ϕ-concircular symmetric if

$$\phi^2((\nabla_W \tilde{C})(X,Y)Z) = 0. \tag{5.3}$$

Taking Covariant differentiation of (5.1), we get

$$(\nabla_W \tilde{C})(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{dr(W)}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{5.4}$$

Applying ϕ^2 on both the sides, we get

$$\begin{aligned} \phi^2((\nabla_W \tilde{C}))(X,Y)Z & \\ = \phi^2((\nabla_W R)(X,Y)Z) - \frac{dr(W)}{n(n-1)}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y]. \end{aligned} \tag{5.5}$$

Suppose $M(f_1, f_2, f_3)$ is ϕ-concircular recurrent. Then from (2.15) (2.1) and (5.2) in (5.5), we get

$$\begin{aligned}
A(W)\tilde{C}(X,Y)Z &= df_1(W)[g(X,Z)Y - g(Y,Z)X] \\
&\quad + df_2[g(Y,\phi Z)\phi X - 2g(X,\phi Y)\phi Z \\
&\quad - g(X,\phi Z)\phi Y] + f_2[g(X,\phi Z)\phi^2((\nabla_W \phi)Y) \\
&\quad - g(X,(\nabla_W \phi)\phi Z)\phi Y - g(Y,\phi Z)\phi^2((\nabla_W \phi)X) \\
&\quad + g(Y,(\nabla_W \phi)Z)\phi X + 2g(X,\phi Y)\phi^2((\nabla_W \phi)Z) \\
&\quad - 2g(X,(\nabla_W \phi)Y)\phi Z] \\
&\quad - \frac{dr(W)}{n(n-1)}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y].
\end{aligned} \tag{5.6}$$

Taking $X = \xi$ in (5.6), we get

$$\begin{aligned}
A(W)\tilde{C}(\xi,Y)Z &= -df_1(W)(g(Y,Z)\xi - \eta(Z)Y) \\
&\quad - f_2(\eta(\nabla_W \phi)\phi Z)\phi Y - f_2 g(Y,\phi Z)\phi^2((\nabla_W \phi)\xi) \\
&\quad - 2f_2\eta((\nabla_W \phi)(Y))\phi Z + \frac{dr(W)}{n(n-1)}\eta(Z)\phi^2 Y.
\end{aligned} \tag{5.7}$$

Then taking $X = \xi$ in (5.1) and using (2.15), and (2.20), we obtain

$$\tilde{C}(\xi,Y)Z = \left[\frac{(n-2)f_3 + 3f_2}{n}\right](g(Y,Z)\xi - \eta(Z)Y). \tag{5.8}$$

From (5.7) and (5.8), we obtain

$$\begin{aligned}
&- A(W)\left[\frac{(n-2)f_3 + 3f_2}{n}\right](g(Y,Z)\xi - \eta(Z)Y) \\
&= -df_1(W)(g(Y,Z)\xi - \eta(Z)Y) \\
&\quad - f_2[\alpha g(W,\phi Z) + \beta g(\phi W,\phi Z)]\phi Y \\
&\quad - f_2 g(Y,\phi Z)[-\alpha\phi^2 W - \beta\phi^3 W] + \frac{dr(W)}{n(n-1)}\eta(Z)\phi^2 Y \\
&\quad - 2f_2[\alpha(g(W,Y) - \eta(Y)\eta(W)) + \beta g(\phi W,Y)]\phi Z.
\end{aligned} \tag{5.9}$$

Taking $Z = \xi$ in (5.9), we obtain

$$\begin{aligned}
&dr(W)\phi^2 Y \\
&= -n(n-1)\left[A(W)\left[\frac{(n-2)f_3 + 3f_2}{n}\right] - df_1(W)\right](\eta(Y)\xi - Y).
\end{aligned} \tag{5.10}$$

For constant r, (5.10) yields

$$((n-2)f_3 + 3f_2)A(W) = n(df_1)(W). \tag{5.11}$$

i.e. $df_1(W) = \left[\frac{3f_2 - 2f_3}{n}\right] A(W)$ if and only if r is a constant.

Thus we have

Theorem 5.3. *A ϕ-concircular recurrent $M(f_1, f_2, f_3)$ is of constant curvature if and only if* $df_1(W) = \frac{(n-2)f_3+3f_2}{n}A(W)$.

If $M(f_1, f_2, f_3)$ is ϕ-concircular symmetric then $A(W) = 0$. From (5.10) it follows that, in a ϕ-concircular symmetric $M(f_1, f_2, f_3)$, f_1 is constant if and only if r is a constant. Thus we can state that

Theorem 5.4. *A ϕ-concircular symmetric $M(f_1, f_2, f_3)$ is of constant curvature if and only if f_1 is a constant.*

6. ϕ-CONHARMONICALLY RECURRENT $M(f_1, f_2, f_3)$

The conharmonic curvature tensor of $M(f_1, f_2, f_3)$ is given by [14]

$$N(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{6.1}$$

We know that $M(f_1, f_2, f_3)$ is said to be ϕ-conharmonically recurrent if

$$\phi^2((\nabla_W N)(X,Y)Z) = A(W)N(X,Y)Z. \tag{6.2}$$

Suppose the vector fields X, Y and Z are orthogonal to ξ. Then from (2.16) (2.17) and (2.15), we have

$$S(Y,Z) = ((n-1)f_1 + 3f_2 - f_3)g(Y,Z), \tag{6.3}$$

$$QY = ((n-1)f_1 + 3f_2 - f_3)Y, \tag{6.4}$$

$$R(\xi, Y)Z = (f_1 - f_3)g(Y,Z)\xi. \tag{6.5}$$

From the equations (6.1) (6.3) and (6.4), we have

$$N(X,Y)Z = R(X,Y)Z - \frac{2}{(n-2)}[(n-1)f_1 + 3f_2 - f_3][g(Y,Z)X - g(X,Z)Y]. \tag{6.6}$$

Taking the covariant derivative of (6.6) with respect to W, we get

$$(\nabla_W N)(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{2}{(n-2)}d[(n-1)f_1 + 3f_2 - f_3)](W)(g(Y,Z)X - g(X,Z)Y). \tag{6.7}$$

Applying ϕ^2 on both the sides of (6.7), and using (2.15) and (6.2), we obtain

$$
\begin{aligned}
(6.8)\quad & A(W)N(X,Y)Z = df_1(W)[g(X,Z)Y - g(Y,Z)X] \\
& + df_2[g(Y,\phi Z)\phi X - 2g(X,\phi Y)\phi Z \\
& - g(X,\phi Z)\phi Y] + f_2[g(X,\phi Z)\phi^2((\nabla_W\phi)Y) - g(X,(\nabla_W\phi)\phi Z)\phi Y \\
& - g(Y,\phi Z)\phi^2((\nabla_W\phi)X) + g(Y,(\nabla_W\phi)Z)\phi X - 2g(X,(\nabla_W\phi)Y)\phi Z \\
& + 2g(X,\phi Y)\phi^2((\nabla_W\phi)Z)] - \frac{2}{(n-2)}d[(n-1)f_1 + 3f_2 - f_3](W) \\
& \times [g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y].
\end{aligned}
$$

Taking $X = \xi$ in (6.8), we get

$$
\begin{aligned}
(6.9)\quad & A(W)N(\xi,Y)Z = -df_1(W)g(Y,Z)\xi \\
& - f_2\eta((\nabla_W\phi)\phi Z)\phi Y - f_2 g(Y,\phi Z)\phi^2((\nabla_W\phi)\xi) - 2\eta((\nabla_W\phi)Y)\phi Z \\
& - \frac{2}{(n-2)}d[(n-1)f_1 + 3f_2 - f_3](W)g(Y,Z)\xi.
\end{aligned}
$$

Taking $X = \xi$ in (6.1), and using (6.3), (6.4) and (6.5), we obtain

$$
N(\xi,Y)Z = \left[\frac{-nf_1 - 6f_2 + 2f_3}{n-2}\right] g(Y,Z)\xi. \tag{6.10}
$$

Using (2.7) and (6.10) in (6.9), we obtain

$$
\begin{aligned}
(6.11)\quad & A(W)\left[\frac{(-nf_1 - 6f_2 + 2f_3)}{(n-2)}\right] g(Y,Z)\xi = df_1 g(Y,Z)\xi \\
& - f_2[\alpha g(W,\phi Z) + \beta g(\phi W,\phi Z)]\phi Y \\
& - f_2 g(Y,\phi Z)[-\alpha\phi^2 W - \beta\phi^3 W] \\
& - 2[\alpha g(W,Y) - \eta(Y)\eta(W)] + \beta g(\phi W,Y)\phi Z \\
& - \frac{2}{n-2}d[(n-1)f_1 + 3f_2 - f_3](W)g(Y,Z)\xi.
\end{aligned}
$$

Contracting (6.11)with respect to ξ, we get

$$
d[(3n-4)f_1 + 6f_2 - 2f_3](W) = A(W)[nf_1 + 6f_2 - 2f_3].
$$

Thus we can state that

Theorem 6.1. *In a ϕ-conharmonically recurrent $M(f_1,f_2,f_3)$, $nf_1 + 6f_2 - 2f_3 = 0$ holds if and only if $f_1$and $3f_2 - f_3$are constants.*

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