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COMPLETELY PRIME IDEAL RINGS AND THEIR EXTENSIONS

V. K. BHAT

Abstract. Let R be a ring and let $I \neq R$ be an ideal of R. Then I is said to be a completely prime ideal of R if R/I is a domain and is said to be completely semiprime if R/I is a reduced ring.

In this paper, we introduce a new class of rings known as completely prime ideal rings. We say that a ring R is a completely prime ideal ring (CPI-ring) if every prime ideal of R is completely prime. We say that a ring R is a near completely prime ideal ring (NCPI-ring) if every minimal prime ideal of R is completely prime. We say that a ring R is an almost completely prime ideal ring (ACPI-ring) if every associated prime ideal of R (R viewed as a right module over itself) is completely prime.

Let now R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ is the field of rational numbers) and δ a derivation of R. Then we prove the following:

(1) R is a near completely prime ideal ring if and only if $R[x;\delta]$ is a near completely prime ideal ring.
(2) R is an almost completely prime ideal ring if and only if $R[x;\delta]$ is an almost completely prime ideal ring.

1. Introduction

We follow notation as in Bhat [3] but to make the paper self contained, we have the following:

Notation. A ring R means an associative ring with identity $1 \neq 0$, and any R-module unitary. $\mathbb{R}$ denotes the field of real numbers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}$ denotes the ring of integers and $\mathbb{N}$ denotes the set of positive integers unless other wise stated. Let R be a ring. The set of prime ideals of R is denoted by $\mathrm{Spec}(R)$, the set of associated prime ideals of R (where R is viewed as a right module over itself) is denoted by $\mathrm{Ass}(R_R)$, the

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set of minimal prime ideals of R is denoted by $\mathrm{Min.Spec}(R)$ and the set of completely prime ideals of R is denoted by $\mathrm{C.Spec}(R)$. Let K be an ideal of a ring R such that $\sigma^m(K) = K$ for some integer $m \geq 1$, we denote $\cap_{i=1}^{m}\sigma^i(K)$ by K^0.

Let R be a ring, σ an automorphisms of R and δ a σ-derivation of R; i.e. $\delta\colon R \to R$ is an additive mapping satisfying $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$.

For example for any endomorphism σ of a ring R and for any $a \in R$, $\varrho\colon R \to R$ defined as $\varrho(r) = ra - a\sigma(r)$ is a σ-derivation of R.

By a σ-derivation we mean a right σ-derivation. We note that for a left σ-derivation of δ of R, $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$.

We recall that the Ore extension

$$R[x;\sigma,\delta] = \{f = \sum x^i a_i, \quad a_i \in R, \quad 0 \leq i \leq n\}$$

with usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We would like to mention that we take coefficients of the polynomials on the right as in McConnell and Robson [12]. We denote $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable (i.e. $\sigma(I) = I$) and is also δ-invariant (i.e. $\delta(I) \subseteq I$), then clearly $I[x;\sigma,\delta]$ is an ideal of $O(R)$, and we denote it as usual by $O(I)$.

In case σ is the identity map, we denote the ring of differential operators $R[x;\delta]$ by $D(R)$. If J is an ideal of R such that J is δ-invariant (i.e. $\delta(J) \subseteq J$), then clearly $J[x;\delta]$ is an ideal of $D(R)$, and we denote it as usual by $D(J)$.

In case δ is the zero map, we denote $R[x;\sigma]$ by $S(R)$. If K is an ideal of R such that K is σ-stable (i.e. $\sigma(K) = K$), then clearly $K[x;\sigma]$ is an ideal of $S(R)$, and we denote it as usual by $S(K)$.

Completely prime ideals. Study of prime ideals in Ore extensions has been an area of active research in recent past. For more details the reader is referred to S. Annin [1], Carl Faith [6], Gabriel [8], Goodearl and Warfield [9], Leroy and Matczuk [11], H. Nordstrom [14], Bhat [3].

We shall now discuss some more types of prime ideals; i.e. completely prime ideals and minimal prime ideals.

Recall that an ideal P of a ring R is completely prime if R/P is a domain, i.e. $ab \in P$ implies $a \in P$ or $b \in P$ for a, $b \in R$ (McCoy [13]). In commutative sense completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring R is a prime ideal, but the converse need not be true.

The following example shows that a prime ideal need not be a completely prime ideal.

Example 1.1 (Bhat [3]). Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z})$. If p is a prime number, then the ideal $P = M_2(p\mathbb{Z})$ is a prime ideal of R, but is not completely prime, since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have $ab \in P$, even though $a \notin P$ and $b \notin P$.

A relation between the completely prime ideals of a ring R and those of $O(R)$ has been given in [3, Theorem 2.4.] as follows.

Theorem (Bhat [3])**.** *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then*

(1) *For any completely prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $O(P) = P[x; \sigma, \delta]$ is a completely prime ideal of $O(R)$.*
(2) *For any completely prime ideal U of $O(R)$, $U \cap R$ is a completely prime ideal of R.*

Minimal prime ideals. Towards minimal prime ideals and completely prime ideals of a ring, J. Krempa [10, Theorem 2.2.] has proved the following:

Theorem (Krempa [10])**.** *For a ring R the following conditions are equivalent:*

(1) *R is reduced.*
(2) *R is semiprime and all minimal prime ideals of R. are completely prime*
(3) *R is a subdirect product of domains.*

Towards the minimal prime ideals of $R[x; \delta]$, the following has been proved by Krempa [10, Theorem 3.1.]:

Theorem (Krempa [10])**.** *Let R be a reduced ring and let δ be a derivation of R. Then*

(1) *The differential operator ring $R[x; \delta]$ is reduced.*
(2) *Any annihilator and any minimal prime ideal of R is δ-invariant.*
(3) *Any minimal prime ideal in $R[x; \delta]$ is of the form $P[x; \delta]$ where P is a minimal prime ideal in R.*

Completely Prime Ideal Rings(CPI-rings). In this paper we introduce a new class of rings called completely prime ideal rings (CPI-rings) as follows:

Definition 1.2. Let R be a ring. We say that R is a completely prime ideal ring (CPI-ring) if every prime ideal of R is completely prime.

Definition 1.3. Let R be a ring. We say that R is a near completely prime ideal ring (NCPI-ring) if every minimal prime ideal of R is completely prime.

For example a reduced ring is a near completely primal ring.

Definition 1.4. Let R be a ring. We say that R is an almost completely prime ideal ring (ACPI-ring) if every associated prime ideal of R (R viewed as a right module over itself) is completely prime.

Our aim is to find the relation between completely prime ideal rings (CPI-rings) (near completely prime ideal rings (NCPI-rings), almost completely prime ideal rings (ACPI-rings)) and their extensions. It is known that if P is a prime ideal of a ring R, then $P[x]$ is a prime ideal of $R[x]$ (Brewer and Heinzer [5]).

It is known (Lemma 1.6 of Ferrero [7]) that for any ring R, an ideal P of $R[x]$ is prime if and only if $P \cap R$ is a prime ideal of R and

(1) either $P = (P \cap R)[x]$
(2) or P is maximal amongst ideals I of $R[x]$ such that $I \cap R = P \cap R$.

Let R be ring satisfying (1) above. Then, in Theorem (3.1), we prove the following:

R is a CPI-ring if and only if $R[x]$ is a CPI-ring.

Let R be a Noetherian $\mathbb{Q}$-algebra and δ a derivation of R. It is known that if U is a minimal prime ideal (associated prime ideal) of a ring R, then $U[x;\delta]$ is a minimal prime ideal (associated prime ideal) of $R[x;\delta]$. Conversely for any minimal prime ideal (associated prime ideal) P of $R[x;\delta]$, there exists a minimal prime ideal (associated prime ideal) V of R such that $P = V[x;\delta]$. In case of associated prime ideals a ring is viewed as a right module over itself (Bhat [4, Theorem 3.7]).

Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and δ a derivation of R. Using the above facts, in Theorem (3.3), we prove the following concerning near completely primal rings and almost completely primal rings: R is an NCPI-ring if and only if $R[x;\delta]$ is an NCPI-ring. Moreover, in Theorem (3.5), we show that R is an ACPI-ring if and only if $R[x;\delta]$ is an ACPI-ring.

2. Preliminaries

We begin with the following known results:

Lemma 2.1. *Let R be a ring and σ a an automorphism of R.*

(1) *If P is a prime ideal of $S(R)$ such that $x \notin P$, then $P \cap R$ is a prime ideal of R and $\sigma(P \cap R) = P \cap R$.*
(2) *If U is a prime ideal of R such that $\sigma(U) = U$, then $S(U)$ is a prime ideal of $S(R)$ and $S(U) \cap R = U$.*

Proof. The proof follows on the same lines as in the lemma of McConnell and Robson [10, Lemma (10.6.4)]. □

Lemma 2.2. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra. Let δ be a derivation of R. Then:*

(1) *If P is a prime ideal of $D(R)$, then $P \cap R$ is a prime ideal of R and $\delta(P \cap R) \subseteq P \cap R$.*
(2) *If U is a prime ideal of R such that $\delta(U) \subseteq U$, then $D(U)$ is a prime ideal of $D(R)$ and $D(U) \cap R = U$.*

Proof. See the theorem of Goodearl and Warfield [9, Theorem (2.22)]. □

Theorem 2.3 (Hilbert Basis Theorem)**.** *Let R be a right/left Noetherian ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then the ore extension $O(R) = R[x; \sigma, \delta]$ is right/left Noetherian. Also $R[x, x^{-1}, \sigma]$ is right/left Noetherian.*

Proof. See the theorems of Goodearl and Warfield [9, Theorem (1.12) and (1.17)]. □

Let R be a right Noetherian ring. Then we know that $\text{Min. Spec}(R)$ is finite by Theorem (2.4) of Goodearl and Warfield [9] and for any automorphism σ of R, $U \in \text{Min. Spec}(R)$ implies that $\sigma^j(U) \in \text{Min. Spec}(R)$ for all positive integers j. Therefore, there exists some $m \in \mathbb{N}$ such that $\sigma^m(U) = U$ for all $U \in \text{Min. Spec}(R)$. We denote $\cap_{i=1}^{m} \sigma^i(U)$ by U^0 as mentioned in introduction. We have a similar statement and notation for associated prime ideals of a right Noetherian ring R (where R is viewed as a right module over itself).

Theorem 2.4. *Let R be a Noetherian ring and σ an automorphism of R. Then:*

1. *$P \in \text{Ass}(S(R)_{S(R)})$ if and only if there exists $U \in \text{Ass}(R_R)$ such that $S(P \cap R) = P$ and $P \cap R = U^0$.*
2. *$P \in \text{Min. Spec}(S(R))$ if and only if there exists $U \in \text{Min. Spec}(R)$ Such that $S(P \cap R) = P$ and $P \cap R = U^0$.*

Proof. See the theorem of Bhat [2, Theorem (2.4)] □

Theorem 2.5. *Let R be a Noetherian $\mathbb{Q}$-algebra and δ a derivation of R. Then:*

1. *$P \in \text{Ass}(D(R)_{D(R)})$ if and only if $P = D(P \cap R)$ and $P \cap R \in \text{Ass}(R_R)$.*
2. *$P \in \text{Min. Spec}(D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in \text{Min. Spec}(R)$.*

Proof. See the theorem of Bhat [2, Theorem (3.7)] □

3. Completely prime ideals of Polynomial rings

Theorem 3.1. *Let R be a ring such that for any prime ideal P of $R[x]$, $P = (P \cap R)[x]$. Then R is a CPI-ring if and only if $R[x]$ is a CPI-ring.*

Proof. Let R be a CPI-ring. Let P be a prime ideal of $R[x]$. Now, Lemma (1.6) of Ferrero [7] implies that P is a prime ideal of $R[x]$ if and only if $P \cap R = V$ (say) is a prime ideal of R. Now, by hypothesis $P = V[x]$. Now, R is a CPI-ring implies that V is completely prime. Now, Theorem (2.4) of Bhat [3] implies that $V[x]$ is completely prime. Therefore $R[x]$ is a CPI-ring.

Conversely, let $R[x]$ be a CPI-ring. Let U be a prime ideal of R. Now, by hypothesis $U[x] \in \text{Spec}(R)$. Now, $R[x]$ is a CPI-ring implies that $U[x]$ is completely prime. Now, Theorem (2.4) of Bhat [3] implies that $U[x] \cap R = U$ is completely prime. Therefore, R is a CPI-ring. □

Example 3.2. Let $R = \mathbb{Z}_{(2)} = \{p/q : p, q \in \mathbb{Z}, q \text{ odd }\}$. This is a PID and the field of fractions of $\mathbb{Z}_{(2)}$ is $\mathbb{Q}$. Now it can be seen that the principal ideal generated by 2 is the unique non zero prime ideal (indeed it is unique maximal ideal) of $\mathbb{Z}_{(2)}$. Let P be any prime ideal of $R[x]$. Then $P \cap R$ is a prime ideal of R; i.e. $P \cap R = (2)$.

Theorem 3.3. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and δ a derivation of R. Then R is an NCPI-ring if and only if $R[x;\delta]$ is an NCPI-ring.*

Proof. Let R be an NCPI-ring. Let P be a minimal prime ideal of $R[x;\delta]$. Now, Theorem (3.7) of Bhat [2] implies that $P \cap R \in \mathrm{Min.Spec}(R)$ and $\delta(P \cap R) \subseteq P \cap R$ and $(P \cap R)[x;\delta] = P$. Now, R is an NCPI-ring implies that $P \cap R$ is completely prime. Now, Theorem (2.4) of Bhat [3] implies that $(P \cap R)[x;\delta] = P$ is completely prime. Therefore, $R[x;\delta]$ is an NCPI-ring.

Conversely, let $R[x;\delta]$ be an NCPI-ring. Let U be a minimal prime ideal of R. Now, Theorem (3.7) of Bhat [2] implies that $U[x;\delta] \in \mathrm{Min.Spec}(R[x;\delta])$. Now, $R[x;\delta]$ is an NCPI-ring implies that $U[x;\delta]$ is completely prime. Now, Theorem (2.4) of Bhat [3] implies that $U[x;\delta] \cap R = U$ is completely prime. Therefore R is an NCPI-ring. □

Taking $\delta = 0$ in above theorem, we get the following Corollary:

Corollary 3.4. *Let R be a Noetherian ring. Then R is an NCPI-ring if and only if $R[x]$ is NCPI-ring.*

Theorem 3.5. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and δ a derivation of R. Then R is an ACPI-ring if and only if $R[x;\delta]$ is an ACPI-ring.*

Proof. Let R be an ACPI-ring. Let $P \in \mathrm{Ass}(D(R)_{D(R)})$. Now Theorem (3.7) of Bhat [2] implies that $P \cap R \in \mathrm{Ass}(R_R)$ and $\delta(P \cap R) \subseteq P \cap R$ and $(P \cap R)[x;\delta] = P$. Now R is an ACPI-ring implies that $P \cap R$ is completely prime.

Rest is on the same lines as in Theorem (3.3) above. □

Corollary 3.6. *Let R be a Noetherian ring. Then R is an ACPI-ring if and only if $R[x]$ is an ACPI-ring.*

Remark 3.7. Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ and δ a derivation of R. Let R be a CPI-ring. Then $R[x]$ need not be a CPI-ring.

Example 3.8. Let $R = \mathbb{H}$, the ring of Quaternians. This is a CPI-ring. Now, $\mathbb{H}[x]/(x^2 + 1) \cong H \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$. Therefore, the maximal ideal $(x^2 + 1)$ is not completely prime.

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School of Mathematics,
SMVD University, P/o Kakryal,
Katra, J and K, India 182320.
E-mail address: `vijaykumarbhat2000@yahoo.com`