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SOME RESULTS FOR CYCLIC NONLINEAR CONTRACTIVE MAPPINGS IN METRIC SPACES

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ABSTRACT. In this paper, the concept of cyclic $(\phi - \psi)$-Kannan and cyclic $(\phi - \psi)$-Chatterjea contractions, and fixed point theorems for these types of mappings in the context of complete metric spaces have been introduced. The results proved here extend some fixed point theorems in the literature.

1. Introduction and Preliminaries

The Banach contraction mapping principle [2] is a very popular tool for solving the existence problem in many branches of mathematical analysis. Generalizations of this principle have been established in various settings, see for example [6]-[10] and references therein. In 1968, Kannan [7] proved a fixed point theorem for contractions extending the well-known Banach's contraction principle that need not to be continuous (but are continuous at their fixed point), by considering the next definition.

Definition 1.1 ([7])**.** A mapping $T\colon X \to X$, where (X, d) is a metric space, is said to be a Kannan contraction if there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that for all $x, y \in X$, the inequality

$$d\left(Tx, Ty\right) \leq \alpha \left[d\left(x, Tx\right) + d\left(y, Ty\right)\right],$$

holds.

Kannan proved that if X is complete, then every Kannan contraction has a unique fixed point. The cyclical extension for the Kannan's theorem was obtained by Rus in [12] using fixed point structure arguments.

Theorem 1.1 (See [12])**.** *Let $\{A_i\}_{i=1}^{p}$ be non-empty closed subsets of a complete metric space X and suppose $T\colon \bigcup\limits_{i=1}^{p} A_i \to \bigcup\limits_{i=1}^{p} A_i$ is a cyclical operator, i.e., satisfies the condition*

$$T\left(A_i\right) \subseteq A_{i+1} \text{ for all } i \in \{1, 2, \ldots, p\},$$

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such that

$$d\left(Tx,Ty\right)\leq\alpha\left[d\left(x,Tx\right)+d\left(y,Ty\right)\right],\ \textit{for all } x\in A_i, y\in A_{i+1},$$

where $1\leq i\leq p$ *and* $\alpha\in\left[0,\frac{1}{2}\right)$ *is a constant. Then*

(i) T *has a unique fixed point* $z\in\bigcap\limits_{i=1}^{p}A_i$.

(ii) *The Picard iteration* $\{x_n\}$ *given by* $x_{n+1}=Tx_n$, $n\geq 0$, *converges to* z *for any starting point* $x_0\in\bigcup\limits_{i=1}^{p}A_i$.

Following the Kannan's contraction, a lot of papers were devoted to obtain fixed point theorems for various classes of contractive type conditions that do not require the continuity of T. One of them, actually a sort of dual of Kannan contraction, due to Chatterjea [3] as follows.

Definition 1.2 ([3])**.** A mapping $T\colon X\to X$, where (X,d) is a metric space, is said to be a Chatterjea contraction if there exists $\alpha\in\left[0,\frac{1}{2}\right)$ such that for all $x,y\in X$, the inequality

$$d\left(Tx,Ty\right)\leq\alpha\left[d\left(x,Ty\right)+d\left(y,Tx\right)\right],$$

holds.

Chatterjea [3] proved that if X is complete, then every Chatterjea contraction has a unique fixed point. The cyclical extension for the Chatterjea theorem was obtained by Petric [11] using fixed point structure arguments.

Theorem 1.2 (See [11])**.** *Let* $\{A_i\}_{i=1}^{p}$ *be non-empty closed subsets of a complete metric space* X, *and suppose* $T\colon\bigcup\limits_{i=1}^{p}A_i\to\bigcup\limits_{i=1}^{p}A_i$ *is a cyclical operator, i.e., satisfies the condition*

$$T\left(A_i\right)\subseteq A_{i+1}\ \textit{for all } i\in\{1,2,\ldots,p\},$$

such that

$$d\left(Tx,Ty\right)\leq\alpha\left[d\left(x,Ty\right)+d\left(y,Tx\right)\right],\ \textit{for all } x\in A_i, y\in A_{i+1},$$

where $1\leq i\leq p$ *and* $\alpha\in\left[0,\frac{1}{2}\right)$ *is a constant. Then*

(i) T *has a unique fixed point* $z\in\bigcap\limits_{i=1}^{p}A_i$.

(ii) *The Picard iteration* $\{x_n\}$ *given by* $x_{n+1}=Tx_n$, $n\geq 0$, *converges to* z *for any starting point* $x_0\in\bigcup\limits_{i=1}^{p}A_i$.

In [15] Zamfirescu obtained a very interesting fixed point theorem which gathers the three contractive conditions i.e., conditions of Banach, of Kannan, and of Chatterjea, in a rather unexpected way.

Definition 1.3 ([15])**.** A self mapping $T\colon X \to X$ is said to be Zamfirescu contraction if there exist real numbers α, β, and γ satisfying $0 \leq \alpha < 1$, $0 \leq \beta, \gamma < \frac{1}{2}$, such that for $x, y \in X$ at least one of the following is true.

(i) $d(Tx, Ty) \leq \alpha d(x, y)$,
(ii) $d(Tx, Ty) \leq \beta [d(x, Tx) + d(y, Ty)]$,
(iii) $d(Tx, Ty) \leq \gamma [d(x, Ty) + d(y, Tx)]$.

Zamfirescu [15] proved that if X is complete, then every Zamfirescu contraction has a unique fixed point, and the cyclical extension for this result was obtained by Petric [11] as well, using fixed point structure arguments.

In [4] Choudhury redefined the concept of Chatterjea contraction as follows.

Definition 1.4 ([4])**.** A mapping $T\colon X \to X$, where (X, d) is a metric space, is said to be a weak Chatterjea contraction if for all $x, y \in X$, the inequality

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx)),$$

holds, where $\psi\colon [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

In [4], Choudhury proved the following theorem.

Theorem 1.3 (See [4])**.** *If X is a complete metric space, then every weak Chatterjea contraction T has a unique fixed point.*

A new category of fixed point problems with the help of a control function in terms of altering distances was addressed by Khan *et. al.* [8]. Altering distances have been used in metric fixed point theory in many papers, see for example [9]–[13] and references therein.

We define in what follows, an altering distance function which will be used throughout the paper to get new fixed point theorems.

Definition 1.5. The function $\phi\colon [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied.

(i) ϕ is continuous and nondecreasing,
(ii) $\phi(t) = 0$ if and only if $t = 0$.

By the use of the continuous function ψ given in Definition 1.4 and the altering distance function ϕ given in Definition 1.5, we present in the next section new fixed point theorems for cyclic nonlinear contractive mappings.

2. Main results

Definition 2.1. Let $\{A_i\}_{i=1}^{p}$ be non-empty closed subsets of a metric space X, and suppose $T\colon \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is a cyclical operator. Then T is said to be a cyclic $(\phi - \psi)$-Kannan contraction if for any $x \in A_i, y \in A_{i+1}, i = 1, 2, \ldots, p$,

$$\phi(d(Tx, Ty)) \leq \phi\left(\frac{1}{2}[d(x, Tx) + d(y, Ty)]\right) - \psi(d(x, Tx), d(y, Ty)),$$

where $\phi\colon [0,\infty) \to [0,\infty)$ is an altering distance function, and $\psi\colon [0,\infty)^2 \to [0,\infty)$ is a continuous function with $\psi(t,s) = 0$ if and only if $t = s = 0$.

Theorem 2.1. *Let $\{A_i\}_{i=1}^p$ be non-empty closed subsets of a complete metric space (X,d). If $T\colon \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is a cyclic $(\phi-\psi)$-Kannan contraction, then T has a unique fixed point $z \in \bigcap_{i=1}^p A_i$.*

Proof. Take $x_0 \in X$ and consider the sequence given by $x_{n+1} = Tx_n, n \geq 0$. If there exists $n_0 \in N$ such that $x_{n_0+1} = x_{n_0}$, then the point of existence of the fixed point is proved. So, suppose that $x_{n+1} \neq x_n$ for any $n = 0, 1, \ldots$. Then, there exists $i_n \in \{1, \ldots, p\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Since T is a cyclic $(\phi-\psi)$-Kannan contraction, we have

$$\begin{aligned}
\phi\left(d\left(x_n, x_{n+1}\right)\right) &= \phi\left(d\left(Tx_{n-1}, Tx_n\right)\right) \\
&\leq \phi\left(\frac{1}{2}\left[d\left(x_{n-1}, Tx_{n-1}\right) + d\left(x_n, Tx_n\right)\right]\right) \\
&\qquad -\psi\left(d\left(x_{n-1}, Tx_{n-1}\right), d\left(x_n, Tx_n\right)\right) \\
&\leq \phi\left(\frac{1}{2}\left[d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right)\right]\right) \\
&\qquad -\psi\left(d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right)\right) \\
&\leq \phi\left(\frac{1}{2}\left[d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right)\right]\right).
\end{aligned}$$

Since ϕ is a nondecreasing function, we get

$$d\left(x_n, x_{n+1}\right) \leq \frac{1}{2}\left[d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right)\right],$$

which implies

$$d\left(x_n, x_{n+1}\right) \leq d\left(x_{n-1}, x_n\right), \forall n. \tag{1}$$

So, we get that $d\left(x_n, x_{n+1}\right)$ is a nonincreasing sequence of nonnegative real numbers. Hence, there is $r \geq 0$ such that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = r.$$

Using the continuity of ϕ and ψ, we get

$$\begin{aligned}
\phi(r) &\leq \phi\left(\frac{1}{2}2r\right) - \psi(r,r) \\
&= \phi(r) - \psi(r,r),
\end{aligned}$$

which implies that $\psi(r,r) = 0$, and hence, $r = 0$.

In the sequel, we show that $\{x_n\}$ is a Cauchy sequence in X. To do so, we need to prove first, the claim that for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \geq n$ with $p - q \equiv 1\,(m)$, then $d\left(x_p, x_q\right) < \epsilon$. Suppose the contrary case, i.e., there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, we can find $p_n > q_n \geq n$ with $p_n - q_n \equiv 1\,(m)$ satisfying $d\left(x_{p_n}, x_{q_n}\right) \geq \epsilon$. Now, we take $n > 2m$. Then, corresponding to $q_n \geq n$, we can choose p_n in such a way that it is the smallest integer with $p_n > q_n$ satisfying $p_n - q_n \equiv 1\,(m)$ and $d\left(x_{p_n}, x_{q_n}\right) \geq \epsilon$. Therefore, $d\left(x_{q_n}, x_{p_{n-m}}\right) < \epsilon$. Using the triangular inequality,

$$\begin{aligned}\epsilon \leq d\left(x_{p_n}, x_{q_n}\right) \leq d\left(x_{q_n}, x_{p_{n-m}}\right) + \sum_{i=1}^{m} d\left(x_{p_{n-i}}, x_{p_{n-i+1}}\right)\\ < \epsilon + \sum_{i=1}^{m} d\left(x_{p_{n-i}}, x_{p_{n-i+1}}\right).\end{aligned}$$

Letting $n \to \infty$ in the last inequality, and taking into account that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0,$$

we obtain $\lim\limits_{n\to\infty} d\left(x_{p_n}, x_{q_n}\right) = \epsilon$. Again, by triangle inequality, we have

$$\begin{aligned}\epsilon &\leq d(x_{q_n}, x_{p_n})\\ &\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{p_{n+1}}) + d(x_{p_{n+1}}, x_{p_n})\\ &\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{q_n}) + d(x_{q_n}, x_{p_n}) + d(x_{p_n}, x_{p_{n+1}}) + d(x_{p_{n+1}}, x_{p_n})\\ &\leq 2d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_n}, x_{p_n}) + 2d(x_{p_n}, x_{p_{n+1}}).\end{aligned}$$

Taking the limit as $n \to \infty$, and taking into account that $\lim\limits_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0$, we get $\lim\limits_{n\to\infty} d\left(x_{q_{n+1}}, x_{p_{n+1}}\right) = \epsilon$. Since x_{p_n} and x_{q_n} lie in different adjacently labelled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, using the fact that T is a cyclic $(\phi - \psi)$-Kannan contraction, we have

$$\begin{aligned}\phi\left(d\left(x_{q_{n+1}}, x_{p_{n+1}}\right)\right) &= \phi\left(d\left(Tx_{q_n}, Tx_{p_n}\right)\right)\\ &\leq \phi\left(\frac{1}{2}\left[d\left(x_{q_n}, Tx_{q_n}\right) + d\left(x_{p_n}, Tx_{p_n}\right)\right]\right)\\ &\quad -\psi\left(d\left(x_{q_n}, Tx_{q_n}\right), d\left(x_{p_n}, Tx_{p_n}\right)\right).\end{aligned}$$

Letting $n \to \infty$ in the last inequality, we obtain

$$\phi\left(\epsilon\right) \leq \phi\left(0\right) - \psi\left(0, 0\right) = 0.$$

Therefore, we get $\epsilon = 0$ which is a contradiction. From the above proved claim, and for arbitrary $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that if $p, q > n_0$ with $p - q = 1(m)$, then $d\left(x_p, x_q\right) < \epsilon$. Since $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$, we can find $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{\epsilon}{m}, \text{ for } n > n_1.$$

Now, for $r, s > \max\{n_0, n_1\}$ and $s > r$, there exists $k \in \{1, 2, \ldots, m\}$ such that $s - r = k(m)$. Therefore, $s - r + j = 1(m)$ for $j = m - k + 1$. So, we have

$$d(x_r, x_s) \leq d(x_r, x_{s+j}) + d(x_{s+j}, x_{s+j-1}) + \cdots + d(x_{s+1}, x_s).$$

This implies

$$d(x_r, x_s) \leq \epsilon + \frac{\epsilon}{m} \sum_{j=1}^{m} 1 = 2\epsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in $\bigcup_{i=1}^{p} A_i$. Consequently, $\{x_n\}$ converges to some $z \in \bigcup_{i=1}^{p} A_i$. However, in view of cyclical condition, the sequence $\{x_n\}$ has an infinite number of terms in each A_i, for $i = 1, 2, \ldots, p$. Therefore, $z \in \bigcap_{i=1}^{p} A_i$. Now, we will prove that z is a fixed point of T. Suppose $z \in A_i$, $Tz \in A_{i+1}$, and we take a subsequence x_{n_k} of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$. Then,

$$\begin{aligned}
\phi\left(d\left(x_{n_{k+1}}, Tz\right)\right) &= \phi\left(d\left(Tx_{n_k}, Tz\right)\right) \\
&\leq \phi\left(\frac{1}{2}\left[d\left(x_{n_k}, Tx_{n_k}\right) + d\left(z, Tz\right)\right]\right) \\
&\qquad -\psi\left(d\left(x_{n_k}, Tx_{n_k}\right), d\left(z, Tz\right)\right) \\
&\leq \phi\left(\frac{1}{2}\left[d\left(x_{n_k}, Tx_{n_k}\right) + d\left(z, Tz\right)\right]\right).
\end{aligned}$$

Letting $k \to \infty$, we have

$$\phi\left(d\left(z, Tz\right)\right) \leq \phi\left(\frac{1}{2}\left[d\left(z, z\right) + d\left(z, Tz\right)\right]\right),$$

and since ϕ is a nondecreasing function, we get

$$d\left(z, Tz\right) \leq \frac{1}{2} d\left(z, Tz\right).$$

Thus, $d\left(z, Tz\right) = 0$, and hence, $z = Tz$. □

Definition 2.2. Let $\{A_i\}_{i=1}^{p}$ be non-empty closed subsets of a metric space X, and suppose $T\colon \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is a cyclical operator. Then T is said to be a cyclic $(\phi - \psi)$-Chatterjea contraction if for any $x \in A_i, y \in A_{i+1}, i = 1, 2, \ldots, p$,

$$\phi\left(d\left(Tx, Ty\right)\right) \leq \phi\left(\frac{1}{2}\left[d\left(x, Ty\right) + d\left(y, Tx\right)\right]\right) - \psi\left(d\left(x, Ty\right), d\left(y, Tx\right)\right),$$

where $\phi\colon [0, \infty) \to [0, \infty)$ is an altering distance function, and $\psi\colon [0, \infty)^2 \to [0, \infty)$ is a continuous function with $\psi\left(t, s\right) = 0$ if and only if $t = s = 0$.

Theorem 2.2. *Let $\{A_i\}_{i=1}^{p}$ be non-empty closed subsets of a complete metric space (X, d). If $T\colon \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is a cyclic $(\phi - \psi)$-Chatterjea contraction, then T has a unique fixed point $z \in \bigcap_{i=1}^{p} A_i$.*

Proof. The proof follows straightforwardly as the proof of Theorem 2.1. Take $x_0 \in X$, and consider the sequence given by $x_{n+1} = Tx_n, n \geq 0$. If there exists $n_0 \in N$ such that $x_{n_0+1} = x_{n_0}$, then the point of existence of the fixed point is proved. So, suppose that $x_{n+1} \neq x_n$ for any $n = 0, 1, \ldots$. Then, there exists $i_n \in \{1, \ldots, p\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Since, T is a cyclic $(\phi - \psi)$-Chatterjea contraction, we have

$$\begin{aligned}
\phi\left(d\left(x_n, x_{n+1}\right)\right) &= \phi\left(d\left(Tx_{n-1}, Tx_n\right)\right) \\
&\leq \phi\left(\frac{1}{2}\left[d\left(x_{n-1}, Tx_n\right) + d\left(x_n, Tx_{n-1}\right)\right]\right) \\
&\qquad -\psi\left(d\left(x_{n-1}, Tx_n\right), d\left(x_n, Tx_{n-1}\right)\right) \\
&\leq \phi\left(\frac{1}{2}\left[d\left(x_{n-1}, x_{n+1}\right) + d\left(x_n, x_n\right)\right]\right) \\
&\qquad -\psi\left(d\left(x_{n-1}, x_{n+1}\right), d\left(x_n, x_n\right)\right) \\
&\leq \phi\left(\frac{1}{2} d\left(x_{n-1}, x_{n+1}\right)\right).
\end{aligned}$$

Since, ϕ is a nondecreasing function, we get

$$d\left(x_n, x_{n+1}\right) \leq \frac{1}{2} d\left(x_{n-1}, x_{n+1}\right), \tag{2}$$

and by triangular inequality, we have

$$\begin{aligned}
d\left(x_n, x_{n+1}\right) &\leq \frac{1}{2} d\left(x_{n-1}, x_{n+1}\right) \\
&\leq \frac{1}{2}\left[d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right)\right],
\end{aligned}$$

which implies

$$d\left(x_n, x_{n+1}\right) \leq d\left(x_{n-1}, x_n\right). \tag{3}$$

So, we get that $\{d\left(x_n, x_{n+1}\right)\}$ is a nonincreasing sequence of nonnegative real numbers. Hence, there is $r \geq 0$ such that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = r.$$

From (2), we have

$$d\left(x_{n-1}, x_{n+1}\right) \geq 2d\left(x_n, x_{n+1}\right),$$

and hence,

$$\lim_{n\to\infty} d\left(x_{n-1}, x_{n+1}\right) \geq 2r,$$

but,

$$d\left(x_{n-1}, x_{n+1}\right) \leq d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right),$$

and as $n \to \infty$, we have

$$\lim_{n\to\infty} d\left(x_{n-1}, x_{n+1}\right) \leq 2r.$$

Therefore, $\lim_{n\to\infty} d\left(x_{n-1}, x_{n+1}\right) = 2r$. Using the continuity of ϕ and ψ, we get

$$\begin{aligned}\phi\left(r\right) &\leq \phi\left(\frac{1}{2}2r\right) - \psi\left(2r, 0\right)\\ &= \phi\left(r\right) - \psi\left(2r, 0\right),\end{aligned}$$

which implies that $\psi\left(2r, 0\right) = 0$, and hence, $r = 0$.

Now, using similar arguments as those used in the proof of Theorem 2.1, one can show that $\{x_n\}$ is a Cauchy sequence in X and that $\{x_n\}$ converges to some $z \in \bigcup_{i=1}^{p} A_i$, but now with the use of the fact that T is a cyclic $(\phi - \psi)$-Chatterjea contraction, where we have

$$\begin{aligned}\phi\left(d\left(x_{q_{n+1}}, x_{p_{n+1}}\right)\right) &= \phi\left(d\left(Tx_{q_n}, Tx_{p_n}\right)\right)\\ &\leq \phi\left(\frac{1}{2}\left[d\left(x_{q_n}, Tx_{p_n}\right) + d\left(x_{p_n}, Tx_{q_n}\right)\right]\right)\\ &\qquad -\psi\left(d\left(x_{q_n}, Tx_{p_n}\right), d\left(x_{p_n}, Tx_{q_n}\right)\right),\end{aligned}$$

and, as $n \to \infty$,

$$\phi\left(\epsilon\right) \leq \phi\left(\epsilon\right) - \psi\left(\epsilon, \epsilon\right),$$

which implies $\psi\left(\epsilon, \epsilon\right) = 0$, and hence, $\epsilon = 0$.

Similar to Theorem 2.1, in view of cyclical condition the sequence $\{x_n\}$ has an infinite number of terms in each A_i, for $i = 1, 2, \ldots, p$. Therefore, $z \in \bigcap_{i=1}^{p} A_i$.

To prove that z is a fixed point of T, we suppose $z \in A_i$, $Tz \in A_{i+1}$, and we take a subsequence x_{n_k} of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$. Then,

$$\begin{aligned}\phi\left(d\left(x_{n_{k+1}}, Tz\right)\right) &= \phi\left(d\left(Tx_{n_k}, Tz\right)\right)\\ &\leq \phi\left(\frac{1}{2}\left[d\left(x_{n_k}, Tz\right) + d\left(z, Tx_{n_k}\right)\right]\right) - \psi\left(d\left(x_{n_k}, Tz\right), d\left(z, Tx_{n_k}\right)\right)\\ &\leq \phi\left(\frac{1}{2}\left[d\left(x_{n_k}, Tz\right) + d\left(z, Tx_{n_k}\right)\right]\right).\end{aligned}$$

Letting $k \to \infty$, we have

$$\phi\left(d\left(z, Tz\right)\right) \leq \phi\left(\frac{1}{2}\left[d\left(z, Tz\right) + d\left(z, z\right)\right]\right),$$

since ϕ is a nondecreasing function, we get

$$d\left(z, Tz\right) \leq \frac{1}{2} d\left(z, Tz\right).$$

Thus, $d\left(z, Tz\right) = 0$, and hence, $z = Tz$. $\square$

Corollary 1. *Let X be a complete metric space, m positive integer, $A_1, \ldots, A_m$ non-empty closed subsets of X, and $X = \bigcup_{i=1}^{m} A_i$. Let $T\colon X \to X$ be an operator such that*

(i) *$X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T.*

(ii) *for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$ and $\rho\colon [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping satisfies*

$$\int_0^t \rho(s)\, ds > 0$$

for $t > 0$, we have one of the following:

$$\int_0^{d(Tx,Ty)} \rho(t)\, dt \le \int_0^{\frac{1}{2}[d(x,Tx)+d(y,Ty)]} \rho(t)\, dt,$$

or

$$\int_0^{d(Tx,Ty)} \rho(t)\, dt \le \int_0^{\frac{1}{2}[d(Tx,y)+d(Ty,x)]} \rho(t)\, dt.$$

Then T has a unique fixed point $z \in \bigcap_{i=1}^{m} A_i$.

Proof. Let $\phi\colon [0,\infty) \to [0,\infty)$ be defined as $\phi(t) = \int_0^t \rho(s)\, ds > 0$. Then ϕ is alternating distance function, and by taking $\psi(t) = 0$, we get the result. $\square$

Example 2.1. Let $X = [-1,1] \subseteq \mathbb{R}$ with $d(x,y) = |x-y|$. Let $T\colon [-1,1] \to [-1,1]$ be given by

$$T(x) = \begin{cases} -\frac{1}{2}xe^{-\frac{1}{|x|}}, & x \in [-1,0) \cup (0,1], \\ 0, & x = 0. \end{cases}$$

By taking $\psi(t) = 0$, $\phi(t) = t$, and $x \in [0,1]$, $y \in [-1,0]$, we have

$$\begin{aligned} |Tx - Ty| &= \left| -\tfrac{1}{2}xe^{-\frac{1}{|x|}} + \tfrac{1}{2}ye^{-\frac{1}{|y|}} \right| \\ &\le \tfrac{1}{2}|x| + \tfrac{1}{2}|y| \\ &\le \tfrac{1}{2}\left|x + \tfrac{1}{2}xe^{-\frac{1}{|x|}}\right| + \tfrac{1}{2}\left|y + \tfrac{1}{2}ye^{-\frac{1}{|y|}}\right| \\ &= \tfrac{1}{2}|Tx - x| + \tfrac{1}{2}|Ty - y| \\ &= \tfrac{1}{2}(|Tx - x| + |Ty - y|), \end{aligned}$$

which implies that T has a unique fixed point in $[-1,0] \cap [0,1]$ which is $z = 0$.

3. Conclusions

The concept of cyclic $(\phi - \psi)$-Kannan contraction and cyclic $(\phi - \psi)$-Chatterjea contraction has been presented, and some fixed point theorems for these types of mappings in the context of complete metric spaces have been proved to extend other results in the literature.

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