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CHAOTIC BEHAVIOR BASED ON DISCONTINUOUS MAPS

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Abstract. In this paper a class of chaotic vector fields in R^3 is considered. We prove its chaotic behavior by using of the topological entropy of a class of interval maps with finite number of discontinuities. Semi-Lorenz maps from the viewpoint of topological entropy are studied and it is proved that they have positive topological entropies. A kind of bifurcation by presenting a class of one parameter families of interval maps is studied.

1. Introduction

One of the numerical objects which can determine the complexity of a system is "topological entropy". It is also an essential numerical invariant in application [7, 10]. The positive topological entropy of a continuous map implies to its chaotic behavior. This numerical invariant has been considered for continuous maps by Bowen and Dinaburg [1, 4, 11]. The notion of topological entropy for discontinuous maps has been studied by Ciklova [3]. One must pay attention to this point that vector fields with discontinuous components appear in nature and engineering [6]. Lorenz maps are examples of discontinuous maps which create chaotic vector fields in the nature. In fact Lorenz system [6] is one of the most important systems in R^3, and its chaotic behavior can deduce from one dimensional Lorenz maps. Chua system [2] is the other system in R^3 which is considered recently in many articles [8].

In the next section we define semi-Lorenz maps and we prove that the semi-Lorenz maps have positive topological entropy in the sense of Ciklova definition. We also present a new kind of bifurcation by using of topological entropy of a class of discontinuous maps.

By using of a class of maps with infinite topological entropies we construct a class of chaotic vector fields in R^3.

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2. Examples of discontinuous maps with positive topological entropies

In this section we assume that $T\colon [a,b] \to [a,b]$ $(a < b)$, is an interval map which may be discontinuous. As usual for a natural number n we define

$$d_n(x,y) = \max\{d(T^i(x), T^i(y)) \ : \ x, y \in [a,b] \text{ and } i \in \{0,1,2,\ldots,n\}\},$$

where T^i is the composition of T, n times, with itself. If $F \subseteq [a,b]$, and $\varepsilon > 0$, then F is called an (n,ε) spanning set for $[a,b]$ with respect to T if for given $x \in [a,b]$ there is $y \in F$ such that $d_n(x,y) \leq \varepsilon$.

A subset E of $[a,b]$ is called an (n,ε) separated if $d_n(x,y) > \varepsilon$ when x and y are different points in E. $r_n(\varepsilon,T)$ denotes the number of elements of an (n,ε) spanning set with the smallest cardinality, and $s_n(\varepsilon,T)$ denotes the number of elements of an (n,ε) separated set with the largest cardinality.

If $r(\varepsilon,T) = \limsup \frac{\log r_n(\varepsilon,T)}{n}$ then $\lim_{\varepsilon\to 0} r(\varepsilon,T)$ is called the topological entropy of T and denoted by $h(T)$. If $h(T) > 0$, then T has positive topological entropy.

If $s(\varepsilon,T) = \limsup \frac{\log s_n(\varepsilon,T)}{n}$ then $h(T) = \lim_{\varepsilon\to 0} s(\varepsilon,T)$.

Example 2.1. Let $T\colon [0,1] \to [0,1]$ be defined by

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Then for $k > 1$

$$T^k(x) = \begin{cases} 1 & \text{if } x = 1 \\ 2^k x - [2^k x] & \text{if } x \neq 1 \end{cases}$$

where

$$[k] = \max\{n \in \mathbb{Z} : n \leq k\}.$$

Let a be an irrational number in $[0,\frac{1}{2}]$, $0 < \varepsilon < \frac{1}{4}$, $n \in N$, and let $x_i = \frac{a+i-1}{2^n}$ for $1 \leq i \leq 2^n$. If $E = \{x_i \ : \ 1 \leq i \leq 2^n\}$ then E is an (n,ε) separated set. So $s_n(\varepsilon,T) \geq 2^n$. Thus $\frac{\log s_n(\varepsilon,T)}{n} \geq \log 2$. Hence $h(T) \geq \log 2$.

If

$$F = \left\{\frac{\varepsilon}{2^n}, \frac{2\varepsilon}{2^n}, \ldots, \frac{[\frac{1}{\varepsilon}]\varepsilon}{2^n}, \frac{\varepsilon+1}{2^n}, \frac{2\varepsilon+1}{2^n}, \ldots, \frac{[\frac{1}{\varepsilon}]\varepsilon+1}{2^n}, \ldots, \frac{2^n}{2^n}\right\}$$

then F is an (n,ε) spanning set. Since the cardinality of F is less than or equal to $(1+[\frac{1}{\varepsilon}])2^n$ then $h(T) \leq \log 2$. Thus $h(T) = \log 2$.

In the rest of this section we restrict ourself to a one parameter family of semi-Lorenz maps. For a parameter μ, a semi-Lorenz map is a map

$$T_\mu\colon [-1,1] \to [-1,1]$$

with the following properties (see figure 1):

(i) $T_\mu\colon [-1,0] \to [-1+\mu,1]$ is an onto, two times differentiable map with $T'_\mu > 1$ and $T''_\mu > 1$, where T'_μ is derivative of T_μ;

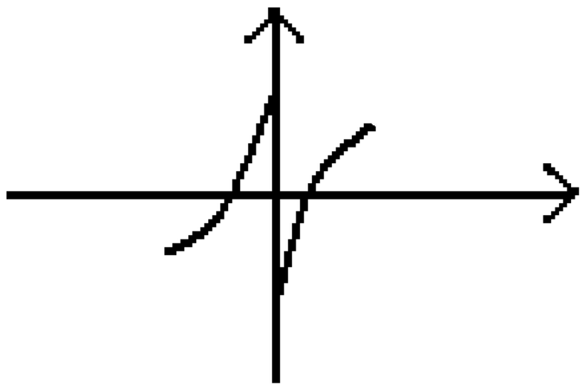

FIGURE 1. Semi-Lorenz map (the vertical axes is the dependent axes)

(ii) $T_\mu(x) = -T_\mu(-x)$ for all $x \in (0, 1]$.

In the above definition $\mu > 0$. Because, if $\mu < 0$, then $-1 + \mu < -1$. So $T_\mu : [-1, 0] \to [-1 + \mu, 1]$ is not available.

A semi-Lorenz map is called a Lorenz map [5] if: $\lim_{x\to 0^-} T'_\mu(x) = +\infty$. Lorenz map has been deduced from Lorenz system [5, 6, 12], and it determines the complexity of Lorenz system. In fact a Lorenz map has positive topological entropy, and this is the reason of the complexity of Lorenz system for special values of parameters.

Theorem 2.1 (Positive topological entropy for semi-Lorenz maps)**.** *If*

$$T_\mu : [-1, 1] \to [-1, 1]$$

is a semi-Lorenz map, then T has positive topological entropy.

Proof. Let $a \in [-1, 0]$ and $b \in [0, 1]$ be two points such that $T_\mu(a) = \frac{1}{2}$ and $T_\mu(b) = 0$. If $0 < \varepsilon < \min\{|a|, b, \frac{1}{2}\}$, $\alpha = \lim_{x\to -1^+} T'_\mu(x)$ and $x, y \in [-1, 1]$ then $d_1(x, y) > \varepsilon$ or $|T_\mu(x) - T_\mu(y)| \geq \alpha|x - y|$. For $n \in N$ and $1 \leq i \leq [\frac{2\alpha^n}{\varepsilon}]$ let $x_i = \frac{i\varepsilon}{\alpha^n} - 1$. Then $d_n(x_i, x_j) \geq \varepsilon$ when $i \neq j$, $1 \leq i \leq [\frac{2\alpha^n}{\varepsilon}]$, and $1 \leq j \leq [\frac{2\alpha^n}{\varepsilon}]$. So $s_n(\varepsilon, T_\mu) = [\frac{2\alpha^n}{\varepsilon}]$. Hence $s(\varepsilon, T_\mu) = \limsup \frac{\log s_n(\varepsilon, T_\mu}{n} \geq \log \alpha$. Thus $h(T_\mu) = \lim_{\varepsilon\to 0} s(\varepsilon, T_\mu) \geq \log \alpha > 0$. □

3. A KIND OF BIFURCATION

In this section we present an example of a one parameter family of discontinuous maps with a bifurcation point in the sense of topological entropy. For $0 < \mu < 1$ we define $T_\mu : [0, 1] \to [0, 1]$ by:

$$T_\mu(x) = \begin{cases} 2(1-\mu)x + \mu & if\ 0 \leq x \leq \frac{1}{2} \\ 2(1-\mu)x + \mu - 1 & if\ \frac{1}{2} < x \leq 1 \end{cases}.$$

We show that $\mu = \frac{1}{2}$ is a bifurcation point for the one parameter family $\{T_\mu\ :\ 0 < \mu < 1\}$. We consider the following two cases.

Case 1. $0 < \mu < \frac{1}{2}$. If $\alpha = 2(1-\mu)$, $x, y \in [0,1]$, and $x < y$, then we have the following choices:

(i) If $x, y \in [0, \frac{1}{2}]$ then $|T_\mu(x) - T_\mu(y)| = \alpha(y-x)$;

(ii)If $x \in [0, \frac{1}{2}]$, and $y \in (\frac{1}{2}, 1]$, then

$$|T_\mu(x) - T_\mu(y)| = |\alpha(x-y) + 1| \geq 1 - \alpha(y-x).$$

(iii) If $x, y \in (\frac{1}{2}, 1]$ then $|T_\mu(x) - T_\mu(y)| = \alpha(y-x)$.

So if $n \in N$, $0 < \varepsilon < \frac{1}{4}$, $1 \leq i \leq [\frac{\alpha^n}{\varepsilon}]$ and $x_i = \frac{i\varepsilon}{\alpha^n}$, then $d_n(x_i, x_j) \geq \varepsilon$, when $1 \leq j \leq [\frac{\alpha^n}{\varepsilon}]$ and $i \neq j$. Hence $s_n(\varepsilon, T_\mu) \geq [\frac{\alpha^n}{\varepsilon}]$. Thus $s(\varepsilon, T_\mu) \geq \lim_{n\to\infty} \frac{1}{n} \log[\frac{\alpha^n}{\varepsilon}] = \log\alpha$. So $h(T_\mu) \geq \log\alpha > 0$.

Case 2. $\frac{1}{2} \leq \mu < 1$. If $\alpha = 2(1-\mu)$, then

(i) $x, y \in [0, \frac{1}{2}]$ implies $|T_\mu^n(y) - T_\mu^n(x)| = \alpha^n(y-x) \leq (y-x)$;

(ii) $x, y \in (\frac{1}{2}, 1]$ implies $|T_\mu^n(y) - T_\mu^n(x)| \leq \alpha^n(y-x) \leq (y-x)$.

Let $\varepsilon > 0$ be given. Let $1 \leq i \leq [\frac{2}{\varepsilon}]$, $x_i = \frac{i\varepsilon}{2}$ and $x_{[\frac{2}{\varepsilon}]+1} = 1$. Then (i) and (ii) imply that: for all $x \in [0,1]$ there is j such that $d_n(x, x_j) < \varepsilon$. Thus $r_n(\varepsilon, T_\mu) \leq [\frac{2}{\varepsilon}] + 1$ for all $n \in N$. So $h(T_\mu) = 0$.

The above two cases imply that $\mu = \frac{1}{2}$ is a bifurcation point for the one parameter family $\{T_\mu\}$.

4. Constructing vector fields with chaotic behavior

In this section by using of topological entropy of discontinuous interval maps we present a method for constructing chaotic vector fields in R^3.
Let X be a C^2 vector field in R^3 invariant under involution

$$(x_1, x_2, x_3) \to (-x_1, -x_2, x_3)$$

with the hyperbolic fixed points $(-4,0,0)$, $(0,0,0)$, and $(4,0,0)$ with real eigenvalues $(-\lambda_1, \lambda_1, \lambda_1)$, $(\lambda_1, -\lambda_1, \lambda_3)$, and $(-\lambda_1, \lambda_1, \lambda_1)$ respectively with the following two conditions.

(i) $0 < -\lambda_3 < \lambda_1$, and

(ii) $(-1)^{\frac{-\lambda_3}{\lambda_1}}$ is a real number.

Moreover let X be linear in the cubes

$$U = \left\{(x_1, x_2, x_3) \ : \ |x_i| < \frac{3}{2} \text{ for } i = 1, 2, 3\right\}$$

and

$$V = \left\{(x_1, x_2, x_3) \ : \ |x_1 - 4| < \frac{3}{2}, \ |x_2| < \frac{3}{2}, \ |x_3| < \frac{3}{2}\right\}.$$

Theorem 4.1. *If*

$$D = \bar{U} \cap \left\{\left(x_1, x_2, \frac{3}{2}\right) \ : \ x_1, x_2 \in R\right\},$$

then the transition map

$$T_1 : \; D^+ = \left\{ \left(x_1, x_2, \frac{3}{2} \right) \in D \; : \; x_1 > 0 \right\} \to$$

$$E = \bar{U} \cap \{ \left(\frac{3}{2}, x_2, x_3 \right) \; : \; x_2, x_3 \in R\}$$

is

$$T_1 \left(x_1, x_2, \frac{3}{2} \right) = \left(\frac{3}{2}, \frac{2x_1x_2}{3}, \left(\frac{3}{2}\right)^{1-s} x_1^s \right),$$

and the transition map

$$T_3 : \; F = \bar{V} \cap \left\{ \left(x_1, x_2, \frac{3}{2} \right) \; : \; x_2, x_3 \in R \right\} \to$$

$$G = \bar{V} \cap \left\{ (x_1, x_2, x_3) \; : \; |x_2| = \frac{3}{2} \right\}$$

is

$$T_3 \left(x_1, x_2, \frac{3}{2} \right) = \begin{cases} (4 + (x_1 - 4)\frac{2x_2}{3}, \frac{3}{2}, \frac{9}{4x_2}) & \text{if } x_2 > 0 \\ (4 - (x_1 - 4)\frac{2x_2}{3}, -\frac{3}{2}, -\frac{9}{4x_2}) & \text{if } x_2 < 0 \end{cases},$$

where $s = \frac{-\lambda_3}{\lambda_1}$.

Proof. In $\bar{U}$ the system

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 \\ \dot{x}_2 = -\lambda_1 x_2 \\ \dot{x}_3 = \lambda_3 x_3 \end{cases},$$

has the integral curves

$$(x_1(0)e^{\lambda_1 t}, x_2(0)e^{-\lambda_1 t}, x_3(0)e^{\lambda_3 t}).$$

So the direct calculation implies the point (x_1, x_2, x_3) will go to the point $(\frac{3}{2}, \frac{2x_1x_2}{3}, (\frac{3}{2})^{1-s}x_1^s)$ in the set $\bar{U} \cap \{(\frac{3}{2}, x_2, x_3) \; : \; x_2, x_3 \in R\}$ by its integral curve.

In $\bar{V}$ the system

$$\begin{cases} \dot{x}_1 = -\lambda_1(x_1 - 4) \\ \dot{x}_2 = \lambda_1 x_2 \\ \dot{x}_3 = \lambda_1 x_3 \end{cases},$$

has the integral curves

$$(4 + (x_1(0) - 4)e^{-\lambda_1 t}, x_2(0)e^{\lambda_1 t}, x_3(0)e^{\lambda_1 t}).$$

If we put $x_3(0) = \frac{3}{2}$ and $x_2(0)e^{\lambda_1 t} = \frac{3}{2}$, then $t = \left(\lambda_1^{-1}\right) \log\left(\frac{3}{2x_2(0)}\right)$. So in this case $x_2(0) > 0$ and by substituting t in the integral curve we deduce

$$T_3 \left(x_1, x_2, \frac{3}{2} \right) = \left(4 + (x_1 - 4)\frac{2x_2}{3}, \frac{3}{2}, \frac{9}{4x_2} \right).$$

If we put $x_3(0) = \frac{3}{2}$ and $x_2(0)e^{\lambda_1 t} = -\frac{3}{2}$, then $t = \left(\lambda_1^{-1}\right)\log(\frac{-3}{2x_2(0)}\Big)$. So in this case $x_2(0) < 0$ and by substituting t in the integral curve we deduce

$$T_3\left(x_1, x_2, \frac{3}{2}\right) = \left(4 - (x_1 - 4)\frac{2x_2}{3}, -\frac{3}{2}, -\frac{9}{4x_2}\right). \qquad \square$$

We can take the vector field X in such manner which its transition map

$$T_2 : E^+ = \left\{\left(\frac{3}{2}, x_2, x_3\right) \in E \ : \ x_3 > 0\right\} \to F$$

be the mapping

$$T_2\left(\frac{3}{2}, x_2, x_3\right) = \left(x_3 + 4, -x_2 + 4, \frac{3}{2}\right).$$

Then the transition map $T_4 = T_3 o T_2 o T_1$ is

$$T_4\left(x_1, x_2, \frac{3}{2}\right) = \begin{cases} \left(4 + \left(\frac{3}{2}\right)^{1-s} x_1^s \left(-\frac{4}{9}x_1x_2 + \frac{8}{3}\right), \frac{3}{2}, \frac{27}{-8x_1x_2+48}\right) & \text{if } x_1x_2 < 6 \\ \left(4 + \left(\frac{3}{2}\right)^{1-s} x_1^s \left(-\frac{4}{9}x_1x_2 + \frac{8}{3}\right), \frac{-3}{2}, \frac{-27}{-8x_1x_2+48}\right) & \text{if } x_1x_2 > 6. \end{cases}$$

Now we can define the Poincare map of the vector field [9].

Now let the Poincare map P on D^+ be the combination of T_4 with a diffeomorphism from G to D^+ which carries the segment $x_1 = c_1$ in G to the segment $x_1 = c_2$ in D^+.

Since the vector field X is symmetric then we can find the Poincare map on D.

When x_2 is constant then the third component of the Poincare map as a function of x_1 is a discontinuous map with infinite entropy, because it is a kind of maps which we consider them in the next example. So the vector field is a chaotic vector field.

Example 4.1. Let $T\colon \mathbb{R} \to \mathbb{R}$ be defined by:

$$T(x) = \begin{cases} \frac{a}{bx+d} \text{ if } bx + d > 0 \\ \frac{-a}{bx+d} \text{ if } bx + d < 0 \\ c \text{ if } bx + d = 0 \end{cases}$$

$a,\ b,\ c$, and d are constants, so that a is nonzero and the case $b = d = 0$ will not happen.

If $K = [-\dfrac{d}{b}, -\dfrac{d}{b} + \dfrac{1}{b}]$ and $x_i = -\dfrac{d}{b} + \dfrac{1}{bi}$ for $i \in \mathbb{N}$, then $T^1(x_i) = ai$. So $d_1(x_i, x_j) \geq |a|$. Therefore $r_n(\epsilon, K, T)$ is infinite. Hence $h(T) = \infty$.

5. Conlusion

In this paper we introduce semi-Lorenz maps and prove that they are chaotic maps. We present examples of discontinuous maps with positive topological entropies and we find a bifurcation point in the sense of chaotic behavior. We also construct a class of geometric chaotic vector fields in R^3 based on discontinuous maps with infinite entropies.

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