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ON TOTALLY UMBILICAL HYPERSURFACES OF WEAKLY PROJECTIVE SYMMETRIC SPACES

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Abstract. The object of the present paper is to study the totally umbilical hypersurfaces of weakly projective symmetric spaces and it is shown that the totally umbilical hypersurfaces of weakly projective symmetric space is also a weakly projective symmetric space.

1. Introduction

In 1989 Tamássy and Binh [12] introduced the notions of weakly symmetric and weakly projective symmetric spaces. A non-flat Riemannian space $V_n(n > 2)$ is called a weakly symmetric space if its curvature tensor R_{hijk} satisfies the condition

$$R_{hijk,l} = A_l R_{hijk} + B_h R_{lijk} + C_i R_{hljk} + D_j R_{hilk} + E_k R_{hijl}, \tag{1.1}$$

where A, B, C, D and E are 1-forms (not simultaneously zero) and the ',' denotes the covariant differentiation with respect to the metric of the space. The 1-forms are called the associated 1-forms of the space and an n-dimensional space of this kind is denoted by $(WS)_n$. The existence of a $(WS)_n$ is proved by Prvanović [8]. Then De and Bandyopadhyay [4] gave an example of $(WS)_n$ by a metric of Roter [9] and proved that in a $(WS)_n$, $B = C$ and $D = E$ [4]. Hence, the defining condition of a $(WS)_n$ reduces to the following form:

$$R_{hijk,l} = A_l R_{hijk} + B_h R_{lijk} + B_i R_{hljk} + D_j R_{hilk} + D_k R_{hijl}. \tag{1.2}$$

In this connection it may also be mentioned that although the definition of a $(WS)_n$ is similar to that of a generalized pseudosymmetric space introduced by Chaki [2], the defining condition of a $(WS)_n$ is little weaker than that of a generalized pseudosymmetric space. That is, if in (1.1) the 1-form A is replaced by $2A$ and E is replaced by A then the space will be a generalized pseudosymmetric space.

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In particular, if in (1.1) the 1-form A is replaced by $2A$ and B, C, D and E are replaced by A, then the space turns into a pseudosymmetric space of Chaki [1].

The projective curvature tensor on a space $V_n (n > 2)$ is defined by

$$P_{hijk} = R_{hijk} - \frac{1}{n-1}\Big[R_{ij}g_{hk} - R_{hj}g_{ik}\Big], \tag{1.3}$$

where R_{ij} is the Ricci tensor of the space.

A Riemannian space $V_n (n > 2)$ (the condition $n > 2$ is assumed throughout the paper) is called weakly projective symmetric space [12] if its projective curvature tensor is not identically zero and satisfies the condition

$$P_{hijk,l} = A_l P_{hijk} + B_h P_{lijk} + C_i P_{hljk} + D_j P_{hilk} + E_k P_{hijl}, \tag{1.4}$$

where A, B, C, D and E are 1-forms (not simultaneously zero). Such an n-dimensional space is denoted by $(WPS)_n$. Here W stands for the word *weakly* and P represents the *projective curvature tensor*.

Recently, Shaikh and Hui [10] studied weakly projective symmetric spaces with the existence of such notion by several proper examples. It is shown that [10] in a $(WPS)_n$ the associated 1-forms $B = C$ and $D \neq E$ and hence the defining condition (1.4) of a $(WPS)_n$ reduces to the following form:

$$P_{hijk,l} = A_l P_{hijk} + B_h P_{lijk} + B_i P_{hljk} + D_j P_{hilk} + E_k P_{hijl}, \tag{1.5}$$

where A, B, D and E are 1-forms (not simultaneously zero). If in (1.4), the 1-form A is replaced by $2A$ and B, C, D and E are replaced by A, then the space reduces to a pseudo projective-symmetric space [1].

Recently, Özen and Altay [6] studied the totally umbilical hypersurfaces of weakly symmetric and pseudosymmetric spaces. Again Özen and Altay [7] studied the totally umbilical hypersurfaces of weakly concircular and pseudo concircular symmetric spaces. In this connection it may be mentioned that Shaikh, Roy and Hui [11] studied the totally umbilical hypersurfaces of weakly conharmonically symmetric spaces.

The object of the present paper is to study totally umbilical hypersurfaces of a $(WPS)_n$, and it is shown that such a hypersurface is also a weakly projective symmetric space. It is proved that if the totally umbilical hypersurface of a weakly projective symmetric space with cyclic Ricci tensor is a weakly projective symmetric space with cyclic Ricci tensor then the mean curvature of V_n must be zero.

2. Preliminaries

Let $\{e_i : i = 1, 2, \dots, n\}$ be an orthonormal basis of the tangent space at any point of the space. Then from (1.3), we have the following [10]:

$$g^{ij} P_{hijk} = \frac{n}{n-1}\Big[R_{hk} - \frac{R}{n} g_{hk}\Big], \tag{2.1}$$

where R is the scalar curvature of the manifold and

$$g^{jk}P_{hijk} = g^{hi}P_{hijk} = g^{hk}P_{hijk} = 0. \tag{2.2}$$

Also from (1.3) it follows that [10]

$$\begin{aligned} &(i)\ \ P_{hijk} = -P_{ihjk},\\ &(ii)\ \ P_{hijk} \neq -P_{hikj},\\ &(iii)\ P_{hijk} \neq P_{jkhi},\\ &(iv)\ \ P_{hijk} + P_{ijhk} + P_{jhik} = 0. \end{aligned} \tag{2.3}$$

If we take the covariant derivative of the equation (1.3), we find

$$\begin{aligned} P_{jkhi,l} + P_{klhi,j} + P_{ljhi,k} &= R_{jkhi,l} + R_{klhi,j} + R_{ljhi,k}\\ &- \frac{1}{n-1}(R_{hk,l}g_{ij} - R_{jh,l}g_{ik} + R_{lh,j}g_{ik} - R_{kh,j}g_{il}\\ &- R_{jh,k}g_{il} - R_{lh,k}g_{ij} \end{aligned} \tag{2.4}$$

Remembering that $R_{ijkl} = R_{klij}$ and using the second Bianchi Identity, we obtain

$$\begin{aligned} P_{jkhi,l} + P_{klhi,j} + P_{ljhi,k} = \frac{1}{n-1}&((R_{lh,k} - R_{hk,l})g_{ij} + (R_{jh,l} - R_{lh,j})g_{ik}\\ &+(R_{kh,j} - R_{jh,k})g_{il}) \end{aligned} \tag{2.5}$$

If the Ricci tensor of this space is Codazzi type then we get

$$P_{jkhi,l} + P_{klhi,j} + P_{ljhi,k} = 0 \tag{2.6}$$

Thus, we have the following theorem [10]:

Theorem 2.1 (A. A. Shaikh and S. K. Hui [10])**.** *The projective curvature tensor of a Riemannian space V_n admitting Ricci tensor of Codazzi type satisfies the relation*

$$P_{jkhi,l} + P_{klhi,j} + P_{ljhi,k} = 0 \tag{2.7}$$

3. Totally umbilical hypersurfaces of weakly projective symmetric spaces

Let $(\overline{V}, \overline{g})$ be an $(n+1)$-dimensional Riemannian space covered by a system of coordinate neighbourhoods $\{U, y^\alpha\}$. Let (V, g) be a hypersurface of $(\overline{V}, \overline{g})$ defined in a locally coordinate system by means of a system of parametric equation $y^\alpha = y^\alpha(x^i)$, where Greek indices take values $1, 2, \cdots, n$ and Latin indices take values $1, 2, \cdots, (n+1)$. Let N^α be the components of a local unit normal to (V, g). Then we have

$$g_{ij} = \overline{g}_{\alpha\beta} y_i^\alpha y_j^\beta, \tag{3.1}$$

$$\overline{g}_{\alpha\beta} N^\alpha y_j^\beta = 0, \quad \overline{g}_{\alpha\beta} N^\alpha N^\beta = e = 1, \tag{3.2}$$

$$y_i^\alpha y_j^\beta g^{ij} = \overline{g}^{\alpha\beta} - N^\alpha N^\beta, \quad y_i^\alpha = \frac{\partial y^\alpha}{\partial x^i}. \tag{3.3}$$

The hypersurface (V, g) is called a totally umbilical hypersurface ([3],[5]) of $(\overline{V}, \overline{g})$ if its second fundamental form Ω_{ij} satisfies

$$\Omega_{ij} = H g_{ij}, \quad y_{i,j}^\alpha = g_{ij} H N^\alpha, \tag{3.4}$$

where the scalar function H is called the mean curvature of (V, g) given by $H = \frac{1}{n}\sum g^{ij}\Omega_{ij}$. If, in particular, $H = 0$, i.e.,

$$\Omega_{ij} = 0, \tag{3.5}$$

then the totally umbilical hypersurface is called a totally geodesic hypersurface of $(\overline{V}, \overline{g})$.

The equation of Weingarten for (V, g) can be written as $N_{,j}^\alpha = -\frac{H}{n} y_j^\alpha$. The structure equations of Gauss and Codazzi ([3],[5]) for (V, g) and $(\overline{V}, \overline{g})$ are respectively given by

$$R_{ijkl} = \overline{R}_{\alpha\beta\gamma\delta} B_{ijkl}^{\alpha\beta\gamma\delta} + H^2 G_{ijkl}, \tag{3.6}$$

$$\overline{R}_{\alpha\beta\gamma\delta} B_{ijk}^{\alpha\beta\gamma} N^\delta = (H_{,i}) g_{jk} - (H_{,j}) g_{ik}, \tag{3.7}$$

where R_{ijkl} and $\overline{R}_{\alpha\beta\gamma\delta}$ are curvature tensors of (V, g) and $(\overline{V}, \overline{g})$ respectively, and

$$B_{ijkl}^{\alpha\beta\gamma\delta} = B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta, \quad B_i^\alpha = y_i^\alpha, \quad G_{ijkl} = g_{il} g_{jk} - g_{ik} g_{jl}.$$

Also we have ([3],[5])

$$\overline{R}_{\alpha\delta} B_i^\alpha B_j^\delta = R_{ij} - (n-1) H^2 g_{ij}, \tag{3.8}$$

$$\overline{R}_{\alpha\delta} N^\alpha B_i^\delta = (n-1) H_{,i}. \tag{3.9}$$

From (1.3), (3.6) and (3.8), we obtain

$$P_{ijkl} = \overline{P}_{\alpha\beta\gamma\delta} B_{ijkl}^{\alpha\beta\gamma\delta}. \tag{3.10}$$

Also from (1.3), (3.7) and (3.9), we have

$$\overline{P}_{\alpha\beta\gamma\delta} B_{ijk}^{\alpha\beta\gamma} N^\delta = (H_{,i}) g_{jk} - (H_{,j}) g_{ik}. \tag{3.11}$$

Let $(\overline{V}, \overline{g})$ be a weakly projective symmetric space. Then we have

$$\overline{P}_{bcea,d} = A_d \overline{P}_{bcea} + B_b \overline{P}_{dcea} + B_c \overline{P}_{bdea} + D_e \overline{P}_{bcda} + E_a \overline{P}_{bced}, \tag{3.12}$$

where A, B, D and E are 1-forms (not simultaneously zero).

Multiplying both sides of (3.12) by B_{hijkr}^{abcde} and then using (3.10), we get the relation (1.5).

This leads to the following:

Theorem 3.1. *The totally umbilical hypersurface of a weakly projective symmetric space is also a weakly projective symmetric space.*

Corollary 3.1. *The totally umbilical hypersurface of pseudo projective symmetric space is also pseudo projective symmetric space.*

If we consider that the hypersurface of a weakly projective symmetric space $(\overline{V}_n, \overline{g})$ is of Codazzi type Ricci tensor, then we obtain from Theorem 2.1,

$$\overline{P}_{abcd,e} + \overline{P}_{abde,c} + \overline{P}_{abec,d} = 0 \tag{3.13}$$

and we have from (1.5)

$$\overline{P}_{abcd,e} = A_e\overline{P}_{abcd} + B_a\overline{P}_{ebcd} + B_b\overline{P}_{aecd} + D_c\overline{P}_{abed} + E_d\overline{P}_{abce} \tag{3.14}$$

where A, B, D, E are 1-forms (not simultaneously zero). Permutating the indices c, d, e in (3.14) by cyclic, adding these equations and using (3.13), we get

$$\begin{aligned} &A_e\overline{P}_{abcd} + A_c\overline{P}_{abde} + A_d\overline{P}_{abec} + B_a\overline{P}_{ebcd} + B_a\overline{P}_{cbde} + B_a\overline{P}_{dbec} + \\ &+B_b\overline{P}_{aecd} + B_b\overline{P}_{acde} + B_b\overline{P}_{adec} + D_c\overline{P}_{abed} + D_d\overline{P}_{abce} + D_e\overline{P}_{abdc} + \\ &+E_d\overline{P}_{abce} + E_e\overline{P}_{abdc} + E_c\overline{P}_{abed} = 0 \end{aligned} \tag{3.15}$$

Now, we assume that the totally umbilical hypersurface (V, g) of $(\overline{V}, \overline{g})$ is a weakly projective symmetric space whose Ricci tensor is of Codazzi type. Multiplying both sides of (3.15) by B^{abcde}_{hijkl}, from (1.5) and (3.8), we find

$$\begin{aligned} H^2((A_l - D_l - E_l)G_{hijk} &+ (A_j - D_j - E_j)G_{hikl} \\ &+ (A_k - D_k - E_k)G_{hilj}) = 0 \end{aligned} \tag{3.16}$$

Thus, it can be said that from (3.16) either $H = 0$ or

$$\begin{aligned} (A_l - D_l - E_l)G_{hijk} &+ (A_j - D_j - E_j)G_{hikl} \\ &+ (A_k - D_k - E_k)G_{hilj} = 0 \end{aligned} \tag{3.17}$$

If we suppose that $H \neq 0$ then using (3.16), we obtain

$$\begin{aligned} (A_l - D_l - E_l)G_{hijk} &+ (A_j - D_j - E_j)G_{hikl} \\ &+ (A_k - D_k - E_k)G_{hilj} = 0 \end{aligned} \tag{3.18}$$

From Walker's Lemma,[13] and (3.18), we can say that

$$A_l = D_l + E_l \quad \text{for all } l \tag{3.19}$$

In this case we have the following theorem:

Theorem 3.2. *If the totally umbilical hypersurface of a weakly projective symmetric space with Codazzi type Ricci tensor is a weakly projective symmetric space with Codazzi type Ricci tensor then the 1-forms A, D and E satisfy the condition* $A_l = D_l + E_l$

Multiplying (1.5) by $g^{il}g^{jk}$, from (2.2) and (2.3), we obtain

$$D^h P_{hl} = 0 \tag{3.20}$$

Again, multiplying (1.5) by $g^{ij}g^{hk}$, from (2.2), we also get

$$(B^h + E^h)P_{hl} = 0 \tag{3.21}$$

In addition to these equations, multiplying (1.5) by $g^{ij}g^{hl}$, from (2.2) and (2.3), we find

$$P^l_{k,l} = (A^h + B^h - D^h)P_{hk} = 0 \tag{3.22}$$

Comparing the equations (3.19), (3.20), (3.21) and (3.22), it can be easily seen that

$$P^l_{k,l} = 0 \tag{3.23}$$

This leads the following theorem:

Theorem 3.3. *If the totally umbilical hypersurface of a weakly projective symmetric space with Codazzi type Ricci tensor is a weakly projective symmetric space with Codazzi type Ricci tensor then the divergence of the tensor P_{hk} is zero, i.e. $P^l_{k,l} = 0$.*

Let us take the covariant derivative of (3.8), then we obtain

$$\overline{R}_{ab,c}B^a_i B^b_j B^c_k = R_{ij,k} - 2(n-1)HH_k g_{ij} \tag{3.24}$$

Permutating the indices a, b, c by cyclic in (3.24) and adding these equations side by side, we get

$$\begin{aligned} \overline{R}_{ab,c} + \overline{R}_{bc,a} + \overline{R}_{ca,b} = {} & R_{ij,k} + R_{jk,i} + R_{ki,j} \\ & -2(n-1)H[H_{,k}g_{ij} + H_{,i}g_{jk} + H_{,j}g_{ik}] \end{aligned} \tag{3.25}$$

Assuming that $\overline{V}_n$ is a weakly projective symmetric space with cyclic Ricci tensor and V_n, the totally umbilical hypersurface of $\overline{V}_n$, is also weakly projective symmetric space with cyclic Ricci tensor, (3.25) reduces to

$$H_{,k}g_{ij} + H_{,i}g_{jk} + H_{,j}g_{ik} = 0 \tag{3.26}$$

where $H \neq 0$. From Walker's Lemma [13], it must be from (3.26) $H_{,k} = 0$, i.e., the mean curvature is constant. Thus, we have the following theorem:

Theorem 3.4. *If the totally umbilical hypersurface of a weakly projective symmetric space with cyclic Ricci tensor is a weakly projective symmetric space with cyclic Ricci tensor then the mean curvature of V_n must be zero.*

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