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SECOND ORDER PARALLEL TENSORS ON PARA r-SASAKIAN MANIFOLDS WITH A COEFFICIENT α

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ABSTRACT. Levy [11] had proved that a second order symmetric parallel non singular tensor on a space of constant curvature is a constant multiple of the metric tensor. Sharma [6] has proved that second order parallel tensor in a Kaehler Space of constant holomorphic sectional curvature is a linear combination with constant coefficients of the Kaehlerian metric and the fundamental 2-form. In this paper, we show that a second order symmetric parallel tensor on a para r-Sasakian manifold with a coefficient α is a constant multiple of the associated metric tensor and we have also proved that there is no non zero skew symmetric second order parallel tensor on a para r-Sasakian manifold.

1. Introduction

In 1923, Eisenhart [10] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of metric tensor is reducible. In 1926 Levy [11] obtained the necessary and sufficient conditions for the existence of such tensors. Sharma [13] has generalized Levy's result by showing that a second order parallel (not necessarily symmetric and non-singular) tensor on an n-dimensional $(n > 2)$ space of constant curvature is a constant multiple of the metric tensor. Sharma has also proved in [13] that on a Sasakian manifold, there is no non zero parallel 2-form. In this paper we have defined para r-Sasakian manifolds with a coefficient α (non zero scalar function) and have proved the following two theorems:

Theorem 1.1. *On a para r-Sasakian manifold with a coefficient α, a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemanian metric tensor.*

Theorem 1.2. *On a para r-Sasakisan manifold with a coefficient α, there is no non zero parallel 2-forms.*

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2. Preliminaries

Let a C^∞ differentiable manifold M be equipped with the ring of real valued differentiable functions $\mathfrak{F}(M)$ and the module of derivations $\mathfrak{F}(M)$ and a $(1,1)$ tensor field Φ as a linear map such that

$$\Phi\colon \mathfrak{X}(M) \to \mathfrak{X}(M).$$

Let there be r (C^∞) 1-forms $A_1, A_2 \dots A_r$ and r (C^∞) contravariant vector fields $T^1, T^2 \dots T^r$ satisfying the following conditions [5]

$$A_p(T^p) = \delta^p_q \text{ where } p, q = 1, 2, \dots r \tag{2.1}$$

$$\Phi(T^p) = 0 \text{ for } p = 1, 2, \dots r \tag{2.2}$$

$$A_p(\Phi X) = 0 \text{ for } p = 1, 2, \dots r \tag{2.3}$$

for any vector field $X \in \mathfrak{X}(M)$, and

$$\Phi^2 X = X - A_p(X) T^P \text{ for } p = 1, 2, \dots r. \tag{2.4}$$

Here the summation convention is employed on repeated indices where $p = 1, 2, \dots r$. If moreover M admits a positive definite Riemannian metric g such that

$$A_p(X) = g(X, T^p), \text{ for } X \in \mathfrak{X}(M) \tag{2.5}$$

$$g(\Phi X, \Phi Y) = g(X, Y) - \sum_{p=1}^{r} A_p(X) A_p(Y), \tag{2.6}$$

for any vector fields X and Y. Then a manifold satisfying conditions (2.1), (2.2), (2.3), (2.4), (2.5), and (2.6) is called an *almost r-para contact structure* (Φ, A_p, T^p, g) *on* M.

In M the following relations hold

$$\Phi(X, Y) = g(X, \Phi Y) = g(Y, \Phi X) = \Phi(Y, X) \tag{2.7a}$$

$$\Phi(X, T^p) = 0. \tag{2.7b}$$

Definition 1. If in the almost r-para contact manifold M, the following relations

$$\Phi X = \frac{1}{\alpha}\left(\nabla_X T^p\right), \ \Phi(X, Y) = \frac{1}{\alpha}\left(\nabla_X A_p(Y)\right) \tag{2.8}$$

$$\alpha(X) = \nabla_X \alpha \tag{2.9a}$$

$$g(X, \bar{\alpha}) = \alpha(X) \tag{2.9b}$$

$$\begin{aligned} \nabla_X \Phi(Y, Z) = \alpha \Bigg[\Bigg\{ -g(X, Y) + \sum_{p=1}^{r} A_p(X) A_p(Y) \Bigg\} A_p(Z) \\ + \Bigg\{ -g(X, Z) + \sum_{p=1}^{r} A_p(X) A_p(Z) \Bigg\} A_p(Y) \Bigg] \end{aligned} \tag{2.10}$$

hold where ∇ denotes the Riemannian connection of the metric tensor g, then M is called a para r-Sasakian manifold with a coefficient α.

3. Proofs of Theorem 1.1 and 1.2

In proving Theorems 1.1 and 1.2 we need the following theorems.

Theorem 3.1. *On a para r-Sasakian manifold the following holds*

$$A_p(R(X,Y)Z) = \alpha^2[g(X,Z)A_p - g(Y,Z)A_p(X)] - [\alpha(X)\Phi(Y,Z) - \alpha(Y)\Phi(X,Z)]. \tag{3.1}$$

Proof. In view of (2.8), (2.9)a and (2.10) the proof follows easily. □

Theorem 3.2. *For a para r-Sasakian manifold we have*

$$R(T^p,X)Y = \alpha^2[A_p(Y)X - g(X,Y)T^p] + \alpha(Y)\Phi X - \bar{\alpha}\Phi(X,Y), \tag{3.2}$$

where $g(X,\bar{\alpha}) = \alpha(X)$.

Proof. The proof follows immediately after making use of (3.1) and equation (2.9)b. □

Theorem 3.3. *For a para r-Sasakian manifold the following holds*

$$R(T^p,X)T^p = \beta\Phi X + \alpha^2[X - \sum_{p=1}^{r} A_p(X)T^p], \tag{3.3}$$

for $p = 1,2,\dots r$ where $\alpha(T^p) = \beta$.

Proof. In view of equation (3.2), the proof follows in an obvious manner. □

4. Proof of Theorems

Proof of Theorem (1.1). Let h denote a $(0,2)$ tensor field on a para r-Sasakian manifold M with a coefficient α such that $\nabla h = 0$, then it follows that

$$h(R(W,X)Y,Z) + h(Y,R(W,X)Z) = 0, \tag{4.1}$$

for arbitrary vector fields X,Y,Z,W on M. We can write (4.1) as

$$g(R(W,X)Y,Z) + g(Y,R(W,X)Z) = 0.$$

Substituting $W = Y = Z = T^q$ into (4.1) we get

$$g(R(T^q,X)T^q,T^q) + g(T^q,R(T^q,X)T^q) = 0. \tag{4.2}$$

In view of theorem (3.3) the above equation becomes

$$2\beta h(\Phi X,T^q) + 2\alpha^2 h(X,T^q) - 2\alpha^2 g(X,T^q)h(T^q,T^q) = 0 \tag{4.3}$$

Simplifying (4.3) we get

$$g(X,T^q)h(T^q,T^q) - h(X,T^q) - \frac{\beta}{\alpha^2}h(\Phi X,\xi) = 0. \tag{4.4}$$

Replacing X by ΦY in (4.4) we get

$$h(\Phi Y, T^q) = \frac{\beta}{\alpha^2}[h(T^q, T^q)A_p(Y) - h(Y, T^q)]. \tag{4.5}$$

Using (4.4) and (4.5) we get

$$h(T^q, T^q)A_p(Y) - h(Y, T^q) = 0 \tag{4.6}$$

if $\beta^2 \neq \alpha^4$. Differentiating (4.6) covariantly with respect to Y we get

$$h(T^q, T^q)g(X,\Phi Y) + 2g(X, T^q)h(\Phi Y, T^q) - h(X,\Phi Y) = 0. \tag{4.7}$$

From the above equation and (2.8a) we obtain

$$h(T^q, T^q)g(X,\Phi Y) = h(X,\Phi Y). \tag{4.8}$$

Replacing ΦY by Y in (4.8) we get

$$h(T^q, T^q)g(X,Y) = h(X, Y). \tag{4.9}$$

In view of the fact that $h(T^q, T^q)$ is constant along any vector on M, we have proved the theorem unless $\beta^2 \neq \alpha^4$. □

Proof of Theorem (1.2). Let us consider h to be a parallel 2-form on a para r-Sasakian manifold M with a coefficient α. Then putting $W = Y = T^q$ in (4.1) and using Theorem 3.3 and equations (2.1)–(2.6) we get

$$\begin{aligned}\beta h(Z, \Phi X) + \alpha^2[h(Z, X) - h(Z, T^q)A_p(X) + h(X, T^q)A_p(Z)]\\ = h(\bar{\alpha}, T^q)\Phi(Z, X) - h(\Phi X, T^q)\alpha(Z).\end{aligned} \tag{4.10}$$

Let us define a H to be $(2, 0)$ tensor field metrically equivalent to h then contracting (4.1) with H and using (2.3)–(2.6) we obtain

$$h(\beta, T^q) = 0. \tag{4.11}$$

Substituting (4.11) in (4.10) we get

$$\begin{aligned}\beta h(Z, \Phi, X) = \alpha^2[h(Z, X) - h(Z, T^q)A_p(X) + h(X, T^q)A_p(Z)]\\ + h(\Phi X, T^q)\alpha(Z) = 0.\end{aligned} \tag{4.12}$$

On simplifying the above equation we get

$$h(\Phi\bar{\alpha}, T^q) = 0. \tag{4.13}$$

Interchanging X and Z in (4.12) we get

$$\beta[h(Z, \Phi X) + h(X, \Phi Z)] + h(\Phi X, T^q)\alpha(Z) + h(\Phi Z, T^q)\alpha(X) = 0. \tag{4.14}$$

Replacing X by ΦY in (4.14) and making use of (2.4) and (2.6) we get

$$\begin{aligned}\beta\,[h(Z, Y) - h(Z, T^q)A_p(Y) + h(\Phi Y, \Phi Z)]\\ + h(Y, T^q)\alpha(Z) + h(\Phi Z, T^q)\alpha(\Phi Y) = 0.\end{aligned} \tag{4.15}$$

Using the fact that h is anti symmetric in (4.15) we obtain

$$h(Y,T^q)\alpha(Z)+h(Z,T^q)\alpha(Y)-\beta\left[h(Z,T^q)A_p(Y)+h(Y,T^q)A_p(Z)\right] + h(\Phi Z,T^q)\alpha(\Phi Y)+h(\Phi Y,T^q)\alpha(\Phi Z)=0. \tag{4.16}$$

Substituting $Y=\bar{\alpha}$ in (4.16) and making use of (4.13) and (4.11) we get

$$(\hat{\alpha}-\beta^2)h(Z,T^q)+\hat{\beta}h(\Phi Z,T^q)=0, \tag{4.17}$$

where $\hat{\alpha}=\alpha\bar{\alpha}$ and $\hat{\beta}=\alpha(\Phi\bar{\alpha})$. Replacing Z by ΦZ in (4.17) and in view of (1.4) and (1.6) we get

$$(\beta^2-\hat{\alpha})h(\Phi Z,T^q)=\hat{\beta}h(Z,T^q), \tag{4.18}$$

where $\beta^2\neq\hat{\alpha}$, which in view of (4.17) becomes

$$h(Z,T^q)=0 \text{ unless } (\hat{\beta})^2\neq(\hat{\alpha}-\beta^2)^2. \tag{4.19}$$

Using (4.19) in (4.12) we get

$$\beta h(Z,\Phi X)+\alpha^2 h(Z,X)=0. \tag{4.20}$$

Differentiating (4.19) covariantly along Y and using the fact that $\nabla h=0$ we get

$$h(Z,\Phi Y)=0. \tag{4.21}$$

In view of (4.21) and (4.20), we see that $h(Y,Z)=0$. □

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