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ON PRINCIPAL FIBRE BUNDLE OF THE CARTESIAN PRODUCT MANIFOLD

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Abstract. Differentiable principal fibre bundle have been defined and studied by Kobayashi and Nomizu [3] and many other geometers. In this paper, we study structures in the principal fibre bundle (P, M, G, π). Hexalinear frame bundle is also studied and it has been shown that the hexalinear frame bundle is the principal fibre bundle.

1. Preliminaries

Let M be a $(2n+r)$ dimensional differentiable manifold of class C^∞. suppose there exists on M, a tensor field ϕ of type $(1,1)$, $r(C^\infty)$ vector field $\xi_1, \xi_2, \ldots, \xi_r$ and $r(C^\infty)$ 1-forms $\eta^1, \eta^2, \ldots, \eta^r$ satisfying

$$\phi^2 = \lambda^2 I_{2n+r} + \eta^\alpha \otimes \xi_\alpha \tag{1.1}$$

where

$$\eta^\alpha \otimes \xi_\alpha = \Sigma \eta^\alpha \otimes \xi_\alpha.$$

Also

$$\begin{aligned} &\text{(i)} \quad \phi\xi_\alpha = 0 \\ &\text{(ii)} \quad \eta^\alpha \otimes \phi = 0 \\ &\text{(iii)} \quad \eta^\alpha(\xi_\beta) + a^2\delta^\alpha_\beta = 0 \end{aligned} \tag{1.2}$$

where $\alpha, \beta = 1, 2 \ldots, r$ and δ^α_β denotes the Kronecker delta.

Thus the manifold M in view of the equations (1.1) and (1.2) will be said to possess the general r-contact structure [7].

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An example of the general r-contact structure can be given as follows. Let

$$\phi = \begin{pmatrix}
0 & 0 & - & - & 0 & -\lambda & 0 & - & - & 0 & 0 & 0 & - & - & 0 \\
0 & 0 & - & - & 0 & 0 & -\lambda & - & - & 0 & 0 & 0 & - & - & 0 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
0 & 0 & - & - & 0 & 0 & 0 & - & - & -\lambda & 0 & 0 & - & - & 0 \\
+\lambda & 0 & - & - & 0 & 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 \\
0 & +\lambda & - & - & 0 & 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
0 & 0 & - & - & +\lambda & 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 \\
0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 \\
0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
0 & 0 & - & - & 0 & 0 & 0 & - & - & 0 & 0 & 0 & - & - & 0
\end{pmatrix}_{(2n+r)\times(2n+r)},$$

$$\begin{aligned}
\eta^1 &= \begin{pmatrix} 0\,0 - - 0\,0\,0 - - 0 -\lambda\, 0 - - 0 \end{pmatrix} \\
\eta^2 &= \begin{pmatrix} 0\,0 - - 0\,0\,0 - - 0\,0 -\lambda - - 0 \end{pmatrix} \\
&\vdots \\
\eta^r &= \begin{pmatrix} 0\,0 - - 0\,0\,0 - - 0\,0\,0 - - -\lambda \end{pmatrix}
\end{aligned}$$

$$\xi_1 = \begin{pmatrix} 0 \\ 0 \\ - \\ - \\ 0 \\ 0 \\ 0 \\ - \\ - \\ - \\ \lambda \\ 0 \\ - \\ - \\ 0 \end{pmatrix}, \xi_2 = \begin{pmatrix} 0 \\ 0 \\ - \\ - \\ 0 \\ 0 \\ 0 \\ - \\ - \\ - \\ 0 \\ \lambda \\ - \\ - \\ 0 \end{pmatrix}, \ldots, \xi_r = \begin{pmatrix} 0 \\ 0 \\ - \\ - \\ 0 \\ 0 \\ 0 \\ - \\ - \\ - \\ 0 \\ 0 \\ - \\ - \\ \lambda \end{pmatrix}$$

Then it can easily shown that

$$\phi^2 = \lambda^2 I_{2n+r} + \eta^\alpha \otimes \xi_a$$

let $N(X,Y)$be the Nijenhuis tensor of the structure. Then

$$N(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X,Y]$$

or

$$N(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \lambda^2[X,Y] + \eta^\alpha([X,Y])\xi_\alpha \tag{1.3}$$

The structure is called normal if

$$N(X,Y) - d\eta(X,Y)\xi = 0 \tag{1.4}$$

A differentiable principal fibre bundle is the set $\{P, M, G, \pi\}$ where P is a differentiable manifold, G is a Lie group such that

(i) G acts on P differentiable to the right is, there exists a differentiable map $P \times G \to P$ such that $(u, g) \to ug$, $u \in P$, $g \in G$ and $ug \in P$. Also $(ug)h = u(gh)$, $h \in G$

(ii) M is the quotient manifold P/G and the map $\pi \colon P \to M$ is differentiable.

(iii) For each $x \in M$ and for every coordinate neighbourhood U of x, the set $\pi^{-1}(U)$ is isomorphic to $U \times G$.

Definition. A set G is called a Lie group if G is a group as well as a differentiable manifold and two maps

(i) $G \times G \to G$ such that $(g_1, g_2) \to g_1 g_2$, $g_1, g_2 \in G$ and

(ii) $G \to G$ such that $g \to g^{-1}$ are differentiable.

Example. If $\mathrm{Gl}(n, R)$ be the set of all $n \times n$ non-singular matrices over the field of real numbers, then $\mathrm{Gl}(n, R)$ is a group under matrix multiplication. If $g \in \mathrm{Gl}(n, R)$ we can write $g = (g^a_b)$, $g^a_b \in R$, $a, b = 1, 2, 3, \ldots, n$. These n^2 real numbers g^a_b can be treated as coordinates and induce the manifold structure in $\mathrm{Gl}(n, R)$. The maps $\mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \to \mathrm{Gl}(n, R)$ and $\mathrm{Gl}(n, R) \to \mathrm{Gl}(n, R)$ are differentiable and thus $\mathrm{Gl}(n, R)$ is a Lie group. It is called the general linear group.

2. Structures in the principal fibre bundle

Let $\{P, M, G, \pi\}$ be the principal fibre bundle with the Lie group G and the projection map π. Let w be the connection 1-form in P. Let $\{\phi, \xi_p, \eta^p, \lambda\}$ be the general almost r-contact structure in M.

Suppose $\{\overline{\phi}, \overline{\xi_p}, \overline{\eta^p}, \lambda\}$ be the left invariant general almost r-contact structure on Lie group G. For tensor field J of type $(1,1)$ on P, define structure on M as follows:

$$\begin{aligned} &(i) \quad \pi(JX) = \phi\pi X + \frac{1}{r}\sum\{a\overline{\eta^p}(\omega X) + b\eta^p(\pi X)\}\xi_p \\ &(ii) \quad \omega(JX) = \overline{\phi}WX + \frac{1}{r}\sum\{a^{-1}(\frac{1}{\lambda^2} - b^2)\eta^p(\pi X) - b\overline{\eta}^p(\omega X)\}\overline{\xi}_p \end{aligned} \tag{2.1}$$

where a, b are the real numbers. Then it is easy to show

$$\begin{aligned} &(i) \quad \pi(J^2X) = \pi\lambda^2 X \\ &(ii) \quad \omega(J^2X) = \omega\lambda^2 X \end{aligned} \tag{2.2}$$

Hence J gives an almost GF-structure on P. Hence we have

Theorem 2.1. *For the principal fibre bundle* $(\{P, M, G, \pi\})$ *the* $(1,1)$ *tensor field* J *satisfying* (2.1) *defines an almost* GF*-structure on* P.

3. Hexalinear Frame Bundle

Let $M_1, M_2, \ldots, M_6$ be six C^∞ manifolds each of dimension n.If $x \in M_1, y \in M_2, z \in M_3, u \in M_4, v \in M_5, w \in M_6 \rightarrow (x, y, z, u, v, w) \in M_1 \times M_2 \times M_3 \times M_4 \times M_5 \times M_6$where $M_1 \times M_2 \times M_3 \times M_4 \times M_5 \times M_6$ is the Cartesian product manifold of $M_1, M_2, \ldots, M_6$.Let $(x^1, x^2, \ldots, x^n)$ or (x^i) be local coordinate system about x in $M_1, (y^j)$ about y in M_2,(z^k) about z in M_3,$(u^l), (v^m), (w^n)$ about u, v, w in M_4, M_5 and M_6 respectively then$\{(x^i, y^j, z^k, u^l, v^m, w^n)\}$ is the local coordinate system about (x, y, z, u, v, w) is the product manifold. Let X_a be tangent vector to $M_1$to x,Y_b to M_2 at y, $Z_c$toM_3 at z etc. Then we can write

$$X_a = X_a^i \frac{\partial}{\partial x^i}, Y_b = Y_b^j \frac{\partial}{\partial y^j}, Z_c = Z_c^k \frac{\partial}{\partial z^k}$$
$$U_d = U_d^l \frac{\partial}{\partial u^l}, V_e = V_e^m \frac{\partial}{\partial v^m}, W_f = W_f^n \frac{\partial}{\partial w^n}.$$

We call the set $(x^i, y^j, z^k \ldots w^n, X_a^i, Y_b^j, Z_c^k, U_d^l, V_e^m, W_f^n)$ the hexalinear frame at $(x^i, y^j, z^k, u^l, v^m, w^n)$ in the product manifold. Let HL be the set of all hexalinear frames at different points of the product manifold $M_1 \times M_2 \times \cdots \times M_6$. It can be shown that HL is also a differentiable manifold. Let us call the set

$$\{\mathrm{HL}, M_1 \times M_2 \times \cdots \times M_6, \pi, \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \cdots \times \mathrm{Gl}(n, R)\}$$

the hexalinear frame bundle of the product manifold $M_1 \times M_2 \times \cdots \times M_6$. Now we prove the following theorem:

Theorem 3.1. *The hexalinear frame bundle is the principal fibre bundle.*

Proof. Let

$$A = (x^i, y^j, z^k, u^l, v^m, w^n, X_a^i, Y_b^j, Z_c^k, U_d^l, V_e^m, W_f^n) \in \mathrm{HL}$$
$$B = (P_l^a, Q_m^b, R_n^c, S_o^d, T_p^e, O_q^f) \in \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \cdots \times \mathrm{Gl}(n, R)$$

Then

$$\begin{aligned}&(A, B)\\ &= ((x^i, y^j, z^k, u^l, v^m, w^n, X_a^i, Y_b^j, Z_c^k, U_d^l, V_e^m, W_f^n), (P_l^a, Q_m^b, R_n^c, S_o^d, T_p^e, O_q^f))\end{aligned}$$

is an element of $\mathrm{HL} \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \cdots \times \mathrm{Gl}(n, R)$. We can define a map

$$\mathrm{HL} \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \rightarrow \mathrm{HL}\,.$$

such that

$$(x^i, y^j, z^k, u^l, v^m, w^n, X_a^i, Y_b^j, Z_c^k, U_d^l, V_e^m, W_f^n)(P_l^a, Q_m^b, R_n^c, S_o^d, T_p^e, O_q^f)$$
$$\to (x^i, y^j, z^k, u^l, v^m, w^n, X_a^i P_l^a, Y_b^j Q_m^b, Z_c^k R_n^c, U_d^l S_o^d, V_e^m T_p^e, W_f^n O_q^f).$$

It can also be shown that if C is an element of product Lie group

$$(AB)C = A(BC)$$

Thus $\mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R)$ acts on HL differentiably to the right. The Cartesian product manifold $M_1 \times M_2 \times \cdots \times M_6$ is the quotient manifold

$$\mathrm{HL} \,/\, \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R)$$

and the map $\pi\colon \mathrm{HL} \to M_1 \times M_2 \times \cdots \times M_6$ is differentiable.

Suppose further that $(x^i, y^j, z^k, u^l, v^m, w^n)$ is a point of the Cartesian product manifold $M_1 \times M_2 \times \cdots \times M_6$ and let

$$U = \{(x^i, y^j, z^k, u^l, v^m, w^n)/1 \leq i, j, k \ldots\ldots n \leq n\}$$

be its coordinate neighbourhood. Then $\pi^{-1}(U) \subset \mathrm{HL}$ can be expressed as

$$\pi^{-1}(U) = \{(x^i, y^j, z^k, u^l, v^m, w^n, X_a^i P_l^a, Y_b^j Q_m^b, Z_c^k R_n^c, U_d^l S_o^d, V_e^m T_p^e, W_f^n O_q^f)\}$$

We can define the map

$$\pi^{-1}(U) \to U \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R)$$

such that

$$(x^i, y^j, z^k, u^l, v^m, w^n, X_a^i, Y_b^j, Z_c^k, U_d^l, V_e^m, W_f^n)$$
$$\to ((x^i, y^j, z^k, u^l, v^m, w^n), (X_a^i, Y_b^j, Z_c^k, U_d^l, V_e^m, W_f^n))$$

which is the identity map. Since identity map is always an isomorphism so $\pi^{-1}(U)$is isomorphic to

$$U \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R) \times \mathrm{Gl}(n, R)$$

Thus all the conditions for hexalinear frame bundle to be the principal fibre bundle are satisfied. Hence the theorem is proved. □

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