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FINITE LOOP ALGEBRAS OF RA LOOPS OF ORDER 64

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ABSTRACT. Let $F[L]$ be the loop algebra of an RA loop L over a finite field F. The structure of the unit loop of the loop algebra $F[L]$ when L is an RA loop of order 64 and $\operatorname{char} F > 0$ have been obtained.

1. INTRODUCTION

An alternative ring is a ring in which $x(xy) = x^2y$ and $(yx)x = yx^2$ are identities. A loop L whose loop ring $R[L]$ over some commutative, associative ring R with unity and of characteristic different from 2 is alternative, but not associative is called an RA loop. For more definitions and terminologies, we refer the reader to [4]. GOODAIRE [2] has determined the loop of units in the integral alternative loop rings of the six smallest order loops when the loop rings have non-trivial units. The unit loop $\mathcal{U}(\mathbb{Z}[M(Q_8, 2)])$ has been studied by JESPERS and LEAL in [6], where $M(Q_8, 2) = M(Q_8, -1, 1)$ denotes the Moufang loop obtained from Q_8, the quaternion group of order 8. Recently the semi-simple loop algebras of RA loops have been studied by FERRAZ, GOODAIRE and MILIES [1].

In this paper, we first determine the Wedderburn decomposition of finite semi-simple group algebras of certain groups of order 32 and using this we describe the structure of the unit loop of finite loop algebras of all RA loops of order 64.

We begin by establishing some notations. Throughout $M(G, *, g_0)$ denotes the Moufang loop obtained from the non-abelian group G, $g_0 \in \mathcal{Z}(G)$, the center of group G and the involution $*$ on G. Also, we use the following notations:

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$K[L]$	loop algebra of a loop L over a field K
$\mathcal{U}(K[L])$	unit loop of $K[L]$
$\mathcal{Z}(L)$	center of the loop L
$\mathcal{Z}(R)$	center of the ring R
$M_2(K)$	ring of all 2×2 matrices over the field K
K_2	quadratic field extension of K
C_n	cyclic group of order n
K^*	$K\backslash\{0\}$
$\mathfrak{Z}(R)$	Zorn's vector matrix algebra over a commutative and associative ring R (with unity)
$GLL(2,R)$	General Linear Loop of degree 2 over R.

2. Some Preliminaries

The classification of indecomposable RA loops was given by Jespers, Leal and Milies [7]. There are eight indecomposable RA loops of order 64: $L_1 = M(32\Gamma_2 g, *, 1)$, $L_2 = M(32\Gamma_2 k, *, t_1)$, $L_3 = M(32\Gamma_2 f, *, 1)$, $L_4 = M(32\Gamma_2 i, *, t_1)$, $L_5 = M(32\Gamma_2 j_1, *, 1)$, $L_6 = M(32\Gamma_2 j_2, *, t_1)$, $L_7 = M(32\Gamma_2 h, *, 1)$ and $L_8 = M(32\Gamma_2 h, *, t_1)$ which are obtained from following non-abelian groups.

$$G_1 = 32\Gamma_2 g = \langle a,\ b,\ c \mid a^8,\ b^2,\ c^2,\ aba^{-1}b^{-1},\ cac^{-1}a^{-5},\ cbc^{-1}a^4b\rangle,$$
$$G_2 = 32\Gamma_2 k = \langle a,\ b \mid a^{16},\ b^2,\ baba^7\rangle,$$
$$G_3 = 32\Gamma_2 f = \langle a,\ b,\ c \mid a^4,\ b^4,\ c^2,\ aba^{-1}b^{-1},\ aca^{-1}c^{-1},\ a^2bcb^{-1}c\rangle,$$
$$G_4 = 32\Gamma_2 i = \langle a,\ b \mid a^8,\ b^4,\ bab^{-1}a^3\rangle,$$
$$G_5 = 32\Gamma_2 j_1 = \langle a,\ b \mid a^8,\ b^2,\ ba^2ba^{-2},\ (a^{-1}bab)^2\rangle,$$
$$G_6 = 32\Gamma_2 j_2 = \langle a,\ b \mid a^4,\ b^8,\ bab^{-1}a\rangle,$$
$$G_7 = 32\Gamma_2 h = \langle a,\ b,\ c \mid a^4,\ b^4,\ c^2,\ a^{-1}b^{-1}abc,\ a^{-1}c^{-1}ac,\ b^{-1}c^{-1}bc\rangle.$$

There are 8 decomposable RA loops of order 64 which will be discussed in Section 5.

We shall use the following results. If N is a sub-loop of L such that $|N|$ is invertible in K, then $\widetilde{N} = \frac{1}{|N|}\sum\limits_{n\in N} n$ is an idempotent in $K[L]$. Moreover if N is normal in L, we have

(a) $K[L] = (K[L])\widetilde{N} \oplus K[L](1-\widetilde{N})$
(b) $(K[L])\widetilde{N} \cong K[L/N]$ and $K[L](1-\widetilde{N}) = \Delta_K(L,N)$

where $\Delta_K(L,N)$ is the kernel of the algebra homomorphism $\epsilon_N\colon K[L] \to K[L/N]$.

Lemma 2.1 ([4, Ch. VI, Cor. 4.8])**.** *Let $L = M(G,*,g_0)$ be a finite RA loop with commutator-associator sub-loop $L' = \{1,s\} = G'$ and $H = L$ or G. If* char $K \nmid |H|$, *then*

$$K[H] = K[H]\left(\frac{1+s}{2}\right) \oplus K[H]\left(\frac{1-s}{2}\right),$$

where $K[H]\left(\frac{1+s}{2}\right) \cong K[H/H']$ *is a direct sum of fields and* $K[H]\left(\frac{1-s}{2}\right) = \Delta_K(H, H')$ *is a direct sum of Cayley-Dickson algebras if* $H = L$ *and a direct sum of quaternion algebras otherwise.*

Lemma 2.2 ([4, Ch. I, Cor. 4.17])**.** *Any split Cayley-Dickson algebra over a field* K *of characteristic different from* 2 *is isomorphic to Zorn's vector matrix algebra* $\mathfrak{Z}(K)$*.*

Remark 2.3. Every Cayley-Dickson algebra over a finite field is split.

Observation 2.4. The center of Zorn's vector matrix algebra $\mathfrak{Z}(K)$ is isomorphic to K.

Remark 2.5. It follows from above that every non-commutative, non-associative component in the decomposition of finite semi-simple loop algebra $F[L]$ is isomorphic to Zorn's vector matrix algebra $\mathfrak{Z}(F_i)$, where F_i denotes the extension field of F.

The following theorem gives the Wedderburn decomposition of finite semi-simple group algebras of abelian groups.

Theorem 2.6 (Perlis-Walker, [9, Th. 3.5.4])**.** *Let* G *be a finite abelian group of order* n *and* $K[G]$ *be a semi-simple group algebra. Then*

$$K[G] \cong \underset{j|n}{\oplus} a_j K(\xi_j)$$

where ξ_j *denotes a primitive root of unity of order* j *and* $a_j = \dfrac{n_j}{[K(\xi_j) : K]}$, n_j *being the number of elements of order* j *in* G*.*

Corollary 2.7. *Let* K *be a finite field having* $q = p^n$ *elements with* $p > 2$*. Then*

(1) $K[C_2] \cong K \oplus K$
(2) $K[C_4] \cong$
$4K$, *if* $q \equiv 1 \mod 4$
$2K \oplus K_2$, *if* $q \equiv 3 \mod 4$.
(3) $K[C_8] \cong$
$8K$, *if* $q \equiv 1 \mod 8$
$2K \oplus 3K_2$, *if* $q \equiv 3 \mod 8$
$4K \oplus 2K_2$, *if* $q \equiv 5 \mod 8$
$2K \oplus 3K_2$, *if* $q \equiv 7 \mod 8$.

Now we will discuss certain results for modular loop algebras $F[L]$, that is, in which char F divides $|L|$.

Theorem 2.8 ([5, Th. 3.2]). *Let F be a field of characteristic $p > 0$ and let L be an RA loop which contains an element of order p. Then $\mathcal{U}(F[L])$ is nilpotent if and only if $p = 2$.*

Theorem 2.9 ([3, Th. 4.4]). *Let $F[L]$ be the alternative loop algebra of a loop L of order 2^n over a field of characteristic 2. Then, with respect to any radical property for which nilpotent algebras are radical and algebras with 1 are not, the radical of $F[L]$ is its augmentation ideal $\Delta_F(L)$ and this is nilpotent of dimension $2^n - 1$.*

Theorem 2.10 ([10, Th. 2.10]). *Let F be a field of characteristic 2 and L be an RA loop of order 2^n. Then*

$$\mathcal{U}(F[L]) = F^* \times (1 + \Delta_F(L)).$$

3. Wedderburn decomposition of group algebras

In this section, we discuss the Wedderburn decomposition of group algebras $F[G]$ where F is a finite field of characteristic greater than 2 and G is any group of order 32 (see Table 1).

The following results will be useful.

Theorem 3.1 ([9, Prop. 3.6.11]). *Let $K[G]$ be a semi-simple group algebra. If G' denotes the commutator subgroup of G, then*

$$K[G] \cong K[G/G'] \oplus \Delta_K(G, G')$$

where $K[G/G']$ is the sum of all the commutative simple components of $K[G]$ and $\Delta_K(G, G')$ is the sum of all the others.

Theorem 3.2 (Wedderburn-Artin Theorem). *If $|G|$ is invertible in K, then*

$$K[G] \cong \overset{k_G}{\underset{i=1}{\oplus}} M(n_i, D_i)$$

where D_i is a division ring containing an isomorphic copy of K in its center and the isomorphism is an isomorphism of K-algebras.

So, we have

$$F[G] \cong F[G/G'] \oplus \left(\overset{c_G}{\underset{i=1}{\oplus}} M(n_i, K_{i,G}) \right)$$

where $K_{i,G}$ is a finite field extension of F and $n_i \geq 2$ for each $i,\ 1 \leq i \leq c_G$.

Table 2 consists of the decomposition of commutative part of $F[G]$. Since $|G/G'| = 16$, therefore $\dim_F \left(\overset{c_G}{\underset{i=1}{\oplus}} M(n_i, K_{i,G}) \right) = 16$ showing that $n_i = 2$ or 4 for $1 \leq i \leq c_G$. It remains to find k_G's, c_G's and $K_{i,G}$'s.

We divide the groups in Table 1 into three categories. The first one consists of all those groups in which the l. c. m. of order of elements is 4 and the second consists of those in which the l. c. m. of order of elements is 8 and the third one consists of those in which the l. c. m. of order of elements is 16.

Table 1: Groups of order 32

(G)	Conjugacy Classes of G
G_1	$\{1\},\ \{a,a^5\},\ \{b,a^4b\},\ \{c,a^4c\},\ \{a^2\},\ \{a^4\},\ \{ab\},\ \{ac,a^5c\},$ $\{a^3,a^7\},\ \{bc,a^4bc\},\ \{a^2b,a^6b\},\ \{a^2c,a^6c\},\ \{a^6\},\ \{abc,a^5bc\},\ \{a^3b\},$ $\{a^5b\},\ \{a^3c,a^7c\},\ \{a^2bc,a^6bc\},\ \{a^3bc,a^7bc\},\ \{a^7b\}$
G_2	$\{1\}\ ,\{a,a^9\},\ \{b,a^8b\},\ \{a^2\},\ \{a^4\},\ \{a^8\},\ \{ab,a^9b\},\ \{a^3,a^{11}\},$ $\{a^5,a^{13}\},\{a^2b,a^{10}b\},\ \{a^4b,a^{12}b\},\ \{a^6\},\ \{a^{10}\},\ \{a^{12}\},\ \{a^3b,a^{11}b\},$ $\{a^5b,a^{13}b\},\{a^7,a^{15}\},\ \{a^6b,a^{14}b\},\ \{a^{14}\},\ \{a^7b,a^{15}b\}$
G_3	$\{1\},\ \{a\},\ \{b,a^2b\},\ \{c,a^2c\},\ \{a^2\},\ \{b^2\},\ \{ab,a^3b\},\ \{ac,a^3c\},\ \{a^3\},$ $\{ab^2\},\ \{bc,a^2bc\},\ \{b^3,a^2b^3\},\ \{b^2c,a^2b^2c\},\ \{a^2b^2\},\ \{abc,a^3bc\},\ \{ab^3,a^3b^3\},$ $\{ab^2c,a^3b^2c\},\ \{a^3b^2\},\ \{b^3c,a^2b^3c\},\ \{ab^3c,a^3b^3c\}$
G_4	$\{1\},\ \{a,a^5\},\ \{b,a^4b\},\ \{a^2\},\ \{b^2\},\ \{a^4\},\ \{ab,a^5b\},\ \{a^3,a^7\},$ $\{ab^2,a^5b^2\},\ \{a^2b,a^6b\},\ \{b^3,a^4b^3\},\ \{a^2b^2\},\ \{a^6\},\ \{a^4b^2\},\ \{a^3b,a^7b\},$ $\{ab^3,a^5b^3\},\ \{a^3b^2,a^7b^2\},\ \{a^2b^3,a^6b^3\},\ \{a^6b^2\},\ \{a^3b^3,a^7b^3\}$
G_5	$\{1\},\ \{a,bab\},\ \{b,a^7ba\},\ \{a^7bab\},\ \{a^2\},\ \{a^4\},\ \{ab,ba\},\ \{a^3,a^2bab\},$ $\{a^5,a^4bab\},\ \{a^2b,aba\},\ \{a^4b,a^3ba\},\ \{abab\},\ \{a^3bab\},\ \{a^6\},$ $\{a^3b,a^2ba\},\ \{a^5b,a^4ba\},\ \{a^7,a^6bab\},\ \{a^6b,a^5ba\},\ \{a^5bab\},\ \{a^7b,a^6ba\}$
G_6	$\{1\},\ \{a,a^3\},\ \{b,a^2b\},\ \{a^2\},\ \{b^2\},\ \{b^4\},\ \{ab,a^3b\},\ \{ab^2,a^3b^2\},$ $\{ab^4,a^3b^4\},\ \{b^3,a^2b^3\},\ \{b^5,a^2b^5\},\ \{a^2b^2\},\ \{a^2b^4\},\ \{b^6\},\ \{ab^3,a^3b^3\},$ $\{ab^5,a^3b^5\},\ \{ab^6,a^3b^6\},\ \{b^7,a^2b^7\},\ \{a^2b^6\},\ \{ab^7,a^3b^7\}$
G_7	$\{1\},\ \{a,ac\},\ \{b,bc\},\ \{c\},\ \{a^2\},\ \{b^2\},\ \{ab,abc\},\ \{a^3,a^3c\},\ \{a^2c\},$ $\{ab^2,ab^2c\},\ \{a^2b,a^2bc\},\ \{b^3,b^3c\},\ \{b^2c\},\ \{a^2b^2\},\ \{a^3b,a^3bc\},$ $\{ab^3,ab^3c\},\ \{a^3b^2,a^3b^2c\},\ \{a^2b^3,a^2b^3c\},\ \{a^2b^2c\},\ \{a^3b^3,a^3b^3c\}$

To determine k_G, we shall use the Witt-Berman theorem.

An element $g \in G$ is said to be p-regular if $\gcd(p, o(g)) = 1$. Let r be the l. c. m. of the orders of the p-regular elements of G and ξ be a primitive r-th root of unity over K. Let T be the multiplicative group consisting of those integers t, taken modulo r, for which $\xi \mapsto \xi^t$ defines an automorphism of $K(\xi)$ over K. Two p-regular elements $g, h \in G$ are said to be K-conjugate if $g^t = x^{-1}hx$ for some $x \in G$ and $t \in T$. This defines an equivalence relation which partitions the p-regular elements of G into p-regular K-conjugacy classes. It is easy to see that any p-regular K conjugacy class of G is either a conjugacy class of G or a union of some conjugacy classes of G.

Theorem 3.3 (Witt-Berman [8, Ch 17, Th 5.3])**.** *The number of non-isomorphic simple $K[G]$ modules is equal to the number of K-conjugacy classes of p-regular elements of G.*

Finding D_i's in the Wedderburn decomposition of $K[G]$ is somewhat more difficult. But when $K = F$, D_i's are finite field extensions of F.

Table 2: Decomposition of $F[G/G']$

Group(G)	G'	G/G'	$F[G/G']$
G_1	$\langle a^4\rangle$	$C_4\times C_2\times C_2$	$16F$, if $q\equiv 1 \mod 4$ $8F\oplus 4F_2$, if $q\equiv 3 \mod 4$
G_2	$\langle a^8\rangle$	$C_2\times C_8$	$16F$, if $q\equiv 1 \mod 8$ $4F\oplus 6F_2$, if $q\equiv 3 \mod 8$ $8F\oplus 4F_2$, if $q\equiv 5 \mod 8$ $4F\oplus 6F_2$, if $q\equiv 7 \mod 8$
G_3	$\langle a^2\rangle$	$C_4\times C_2\times C_2$	$16F$, if $q\equiv 1 \mod 4$ $8F\oplus 4F_2$, if $q\equiv 3 \mod 4$
G_4	$\langle a^4\rangle$	$C_4\times C_4$	$16F$, if $q\equiv 1 \mod 4$ $4F\oplus 6F_2$, if $q\equiv 3 \mod 4$
G_5	$\langle a^7bab\rangle$	$C_8\times C_2$	$16F$, if $q\equiv 1 \mod 8$ $4F\oplus 6F_2$, if $q\equiv 3 \mod 8$ $8F\oplus 4F_2$, if $q\equiv 5 \mod 8$ $4F\oplus 6F_2$, if $q\equiv 7 \mod 8$
G_6	$\langle a^2\rangle$	$C_8\times C_2$	$16F$, if $q\equiv 1 \mod 8$ $4F\oplus 6F_2$, if $q\equiv 3 \mod 8$ $8F\oplus 4F_2$, if $q\equiv 5 \mod 8$ $4F\oplus 6F_2$, if $q\equiv 7 \mod 8$
G_7	$\langle c\rangle$	$C_4\times C_4$	$16F$, if $q\equiv 1 \mod 4$ $4F\oplus 6F_2$, if $q\equiv 3 \mod 4$

Table 3a: Merged p-regular F conjugacy classes of groups in which l. c. m. of order of elements is 4

G	$q\equiv 1(mod\ 4)$	$q\equiv 3(mod\ 4)$
G_3	none $k_G=20$, $c_G=4$	$\{a,a^3\},\ \{b,b^3,a^2b,a^2b^3\},\ \{ab,a^3b,ab^3,a^3b^3\}$, $\{ab^2,a^3b^2\},\ \{bc,a^2bc,b^3c,a^2b^3c\},\ \{abc,a^3bc,ab^3c,a^3b^3c\}$ $k_G=14, c_G=2$
G_7	none $k_G=20$, $c_G=4$	$\{a,ac,a^3,a^3c\},\ \{a^2b,a^2bc,a^2b^3,a^2b^3c\},\ \{ab,abc,a^3b^3,a^3b^3c\}$, $\{ab^2,ab^2c,a^3b^2,a^3b^2c\},\ \{b,bc,b^3,b^3c\},\ \{a^3b,a^3bc,ab^3,ab^3c\}$ $k_G=14, c_G=4$

As an application of Witt-Berman theorem, Table 3 give k_G, the total number of Wedderburn components in the decomposition of $F[G]$ and hence c_G for the three categories respectively. The tables also tell about the conjugacy classes of G which have merged to form a p-regular F conjugacy class of G.

Table 3b: Merged p-regular F conjugacy classes of groups in which l. c. m. of order of elements is 8

G	$q \equiv 1(mod\ 8)$	$q \equiv 3(mod\ 8)$	$q \equiv 5(mod\ 8)$	$q \equiv 7(mod\ 8)$
G_1	none $k_G = 20$, $c_G = 4$	$\{a, a^3, a^5, a^7\}$, $\{a^2, a^6\}$, $\{ac, a^3c, a^5c, a^7c\}$, $\{abc, a^3bc, a^5bc, a^7bc\}$, $\{ab, a^3b\}$, $\{a^5b, a^7b\}$ $k_G = 14, c_G = 2$	$\{ab, a^5b\}$, $\{a^3b, a^7b\}$ $k_G = 18, c_G = 2$	$\{a, a^3, a^5, a^7\}$, $\{a^2, a^6\}$, $\{ac, a^3c, a^5c, a^7c\}$, $\{abc, a^3bc, a^5bc, a^7bc\}$ $\{ab, a^7b\}$, $\{a^3b, a^5b\}$ $k_G = 14, c_G = 2$
G_4	none $k_G = 20$, $c_G = 4$	$\{a, a^3, a^5, a^7\}$, $\{a^2, a^6\}$, $\{b, b^3, a^4b, a^4b^3\}$, $\{ab^2, a^5b^2, a^3b^2, a^7b^2\}$ $\{a^2b^2, a^6b^2\}$, $\{ab, a^5b, a^3b^3, a^7b^3\}$, $\{a^2b, a^6b, a^2b^3, a^6b^3\}$, $\{a^3b, a^7b, ab^3, a^5b^3\}$ $k_G = 12, c_G = 2$	none $k_G = 20$, $c_G = 4$	$\{a, a^3, a^5, a^7\}$, $\{a^2, a^6\}$, $\{b, b^3, a^4b, a^4b^3\}$, $\{ab^2, a^5b^2, a^3b^2, a^7b^2\}$ $\{a^2b^2, a^6b^2\}$, $\{ab, a^5b, a^3b^3, a^7b^3\}$, $\{a^2b, a^6b, a^2b^3, a^6b^3\}$, $\{a^3b, a^7b, ab^3, a^5b^3\}$ $k_G = 12, c_G = 2$
G_5	none $k_G = 20$, $c_G = 4$	$\{a, a^3, bab, a^2bab\}$, $\{ab, ba, a^3b, a^2ba\}$, $\{a^5, a^4bab, a^7, a^6bab\}$, $\{a^5b, a^4ba, a^7b, a^6ba\}$ $\{a^2b, aba, a^6b, a^5ba\}$, $\{abab, a^5bab\}$, $\{a^2, a^6\}$ $k_G = 13, c_G = 3$	$\{a, a^5, bab, a^4bab\}$, $\{ab, ba, a^5b, a^4ba\}$, $\{a^3, a^2bab, a^7, a^6bab\}$ $\{a^3b, a^2ba, a^7b, a^6ba\}$ $k_G = 16, c_G = 4$	$\{a, bab, a^7, a^6bab\}$, $\{a^3, a^2bab, a^5, a^4bab\}$, $\{ab, ba, a^7b, a^6ba\}$, $\{a^3b, a^2ba, a^5b, a^4ba\}$ $\{a^2b, aba, a^6b, a^5ba\}$, $\{abab, a^5bab\}$, $\{a^2, a^6\}$ $k_G = 13, c_G = 3$
G_6	none $k_G = 20$, $c_G = 4$	$\{b, a^2b, b^3, a^2b^3\}$, $\{ab, a^3b, ab^3, a^3b^3\}$, $\{ab^2, a^3b^2, ab^6, a^3b^6\}$, $\{ab^5, a^3b^5, ab^7, a^3b^7\}$ $\{b^5, a^2b^5, b^7, a^2b^7\}$, $\{b^2, b^6\}$, $\{a^2b^2, a^2b^6\}$ $k_G = 13, c_G = 3$	$\{b, a^2b, b^5, a^2b^5\}$, $\{ab, a^3b, ab^5, a^3b^5\}$, $\{b^3, a^2b^3, b^7, a^2b^7\}$, $\{ab^3, a^3b^3, ab^7, a^3b^7\}$ $k_G = 16, c_G = 4$	$\{b, a^2b, b^7, a^2b^7\}$, $\{ab, a^3b, ab^7, a^3b^7\}$, $\{ab^2, a^3b^2, ab^6, a^3b^6\}$, $\{ab^3, a^3b^3, ab^5, a^3b^5\}$ $\{b^3, a^2b^3, b^5, a^2b^5\}$, $\{b^2, b^6\}$, $\{a^2b^2, a^2b^6\}$ $k_G = 13, c_G = 3$

Since $|G/G'| + \sum_{i=1}^{c_G} [K_{i,G} : F] = \dim_F(\mathcal{Z}(F[G]))$ = number of conjugacy classes of G and from the above discussion $K_{i,G}$'s can be easily determined. Table 4 describes the Wedderburn decomposition of these group algebras.

4. Decomposition of Loop Algebras of Indecomposable Loops

In this section, we discuss the decomposition of loop algebras of indecomposable loops of order 64 over finite fields of odd characteristic.

We will first determine $F[L/L']$ for each loop in Table 5.

It remains to find $\Delta_F(L, L')$ for each loop. For this, we find $\mathcal{Z}(\Delta_F(L, L'))$ using the center of the loop L. We shall need the following result:

Proposition 4.1. [4, Ch. XI, Prop. 1.3] *Let L be a finite RA 2-loop and K be a field such that* char $K \neq 2$. *Write $\mathcal{Z}(L) = \langle t_1 \rangle \times \cdots \times \langle t_r \rangle$, where $\langle t_i \rangle \cong C_{2^{n_i}}$,*

Table 3c: Merged p-regular F conjugacy classes of group G_2 in which l. c. m. of order of elements is 16

$q \equiv 1, 9(mod\ 16)$	none. $k_G = 20, c_G = 4$
$q \equiv 3, 11(mod\ 16)$	$\{a, a^3, a^9, a^{11}\},\ \{a^2, a^6\},\ \{a^4, a^{12}\},\ \{ab, a^9b, a^3b, a^{11}b\},\ \{a^{10}, a^{14}\},$ $\{a^5, a^{13}, a^7, a^{15}\},\ \{a^2b, a^{10}b, a^6b, a^{14}b\},\ \{a^5b, a^{13}b, a^7b, a^{15}b\}$ $k_G = 12, c_G = 2$
$q \equiv 5, 13(mod\ 16)$	$\{a, a^5, a^9, a^{13}\},\ \{a^2, a^{10}\},\ \{ab, a^9b, a^5b, a^{13}b\},$ $\{a^3, a^{11}, a^7, a^{15}\},\ \{a^6, a^{14}\},\ \{a^3b, a^{11}b, a^7b, a^{15}b\}$ $k_G = 14, c_G = 2$
$q \equiv 7, 15(mod\ 16)$	$\{a, a^7, a^9, a^{15}\},\ \{a^2, a^{14}\},\ \{a^4, a^{12}\},\ \{ab, a^9b, a^7b, a^{15}b\},\ \{a^6, a^{10}\},$ $\{a^3, a^{11}, a^5, a^{13}\},\ \{a^2b, a^{10}b, a^6b, a^{14}b\},\ \{a^3b, a^{11}b, a^5b, a^{13}b\}$ $k_G = 12, c_G = 2$

Table 4: Wedderburn decomposition of $F[G_i]$

G_i	Values of q	Wedderburn decomposition of $F[G_i]$
G_1	$q \equiv 1 \mod 8$	$16F \oplus 4M_2(F)$
	$q \equiv 3 \mod 8$	$8F \oplus 4F_2 \oplus 2M_2(F_2)$
	$q \equiv 5 \mod 8$	$16F \oplus 2M_2(F_2)$
	$q \equiv 7 \mod 8$	$8F \oplus 4F_2 \oplus 2M_2(F_2)$
G_2	$q \equiv 1 \mod 8$	$16F \oplus 4M_2(F)$
	$q \equiv 3 \mod 8$	$4F \oplus 6F_2 \oplus 2M_2(F_2)$
	$q \equiv 5 \mod 8$	$8F \oplus 4F_2 \oplus 2M_2(F_2)$
	$q \equiv 7 \mod 8$	$4F \oplus 6F_2 \oplus 2M_2(F_2)$
G_3	$q \equiv 1 \mod 4$	$16F \oplus 4M_2(F)$
	$q \equiv 3 \mod 4$	$8F \oplus 4F_2 \oplus 2M_2(F_2)$
G_4	$q \equiv 1 \mod 4$	$16F \oplus 4M_2(F)$
	$q \equiv 3 \mod 4$	$4F \oplus 6F_2 \oplus 2M_2(F_2)$
G_5	$q \equiv 1 \mod 8$	$16F \oplus 4M_2(F)$
	$q \equiv 3 \mod 8$	$4F \oplus 6F_2 \oplus 2M_2(F) \oplus M_2(F_2)$
	$q \equiv 5 \mod 8$	$8F \oplus 4F_2 \oplus 4M_2(F)$
	$q \equiv 7 \mod 8$	$4F \oplus 6F_2 \oplus 2M_2(F) \oplus M_2(F_2)$
G_6	$q \equiv 1 \mod 8$	$16F \oplus 4M_2(F)$
	$q \equiv 3 \mod 8$	$4F \oplus 6F_2 \oplus 2M_2(F) \oplus M_2(F_2)$
	$q \equiv 5 \mod 8$	$8F \oplus 4F_2 \oplus 4M_2(F)$
	$q \equiv 7 \mod 8$	$4F \oplus 6F_2 \oplus 2M_2(F) \oplus M_2(F_2)$
G_7	$q \equiv 1 \mod 4$	$16F \oplus 4M_2(F)$
	$q \equiv 3 \mod 4$	$4F \oplus 6F_2 \oplus 4M_2(F)$

for $i = 1, 2, \ldots, r$. Assume that $L' \subseteq \langle t_1 \rangle$. Then

$$\mathcal{Z}(\Delta_K(L, L')) \cong \frac{2^{n_1-1}}{[K(\xi_{2^{n_1}}) : K]} K(\xi_{2^{n_1}})[C_{2^{n_2}} \times \cdots \times C_{2^{n_r}}].$$

Table 5: Decomposition of $F[L_i/L_i']$ for $i = 1, 2, \ldots, 8$

L_i	L_i'	L_i/L_i'	$F[L_i/L_i']$
L_1	$\langle a^4 \rangle$	$C_4 \times C_2 \times C_2 \times C_2$	$32F$, if $q \equiv 1 \mod 4$ $16F \oplus 8F_2$, if $q \equiv 3 \mod 4$
L_2	$\langle a^8 \rangle$	$C_8 \times C_2 \times C_2$	$32F$, if $q \equiv 1 \mod 8$ $8F \oplus 12F_2$, if $q \equiv 3 \mod 8$ $16F \oplus 8F_2$, if $q \equiv 5 \mod 8$ $8F \oplus 12F_2$, if $q \equiv 7 \mod 8$
L_3	$\langle a^2 \rangle$	$C_4 \times C_2 \times C_2 \times C_2$	$32F$, if $q \equiv 1 \mod 4$ $16F \oplus 8F_2$, if $q \equiv 3 \mod 4$
L_4	$\langle a^4 \rangle$	$C_4 \times C_4 \times C_2$	$32F$, if $q \equiv 1 \mod 4$ $8F \oplus 12F_2$, if $q \equiv 3 \mod 4$
L_5	$\langle a^7bab \rangle$	$C_8 \times C_2 \times C_2$	$32F$, if $q \equiv 1 \mod 8$ $8F \oplus 12F_2$, if $q \equiv 3 \mod 8$ $16F \oplus 8F_2$, if $q \equiv 5 \mod 8$ $8F \oplus 12F_2$, if $q \equiv 7 \mod 8$
L_6	$\langle a^2 \rangle$	$C_8 \times C_2 \times C_2$	$32F$, if $q \equiv 1 \mod 8$ $8F \oplus 12F_2$, if $q \equiv 3 \mod 8$ $16F \oplus 8F_2$, if $q \equiv 5 \mod 8$ $8F \oplus 12F_2$, if $q \equiv 7 \mod 8$
L_7	$\langle c \rangle$	$C_4 \times C_4 \times C_2$	$32F$, if $q \equiv 1 \mod 4$ $8F \oplus 12F_2$, if $q \equiv 3 \mod 4$
L_8	$\langle c \rangle$	$C_4 \times C_4 \times C_2$	$32F$, if $q \equiv 1 \mod 4$ $8F \oplus 12F_2$, if $q \equiv 3 \mod 4$

In Table 6, we compute the center of $\Delta_F(L, L')$ for various eight loops of Table 5.

The following theorem tells us about the number of simple components in the decomposition of loop algebra of RA loop of order $2n$, once if the number of simple components in the Wedderburn decomposition of semi-simple group algebra of the corresponding group of order n is known.

Theorem 4.2 ([1, Th. 6.2]). *Let $L = M(G, *, g_0) = G\dot{\cup}Gu$ be a finite RA loop and K be a field of characteristic not dividing $|L|$. Then $K[G] = \oplus A_i$, where each A_i is either a field or a quaternion algebra. Further $K[L] = \oplus B_i$, where $B_i = A_i + A_iu$ is a field, a direct sum of two fields or a Cayley-Dickson algebra.*

Suppose that L is a loop in one of the five classes $\mathcal{L}_1, \ldots, \mathcal{L}_5$ and that there are n fields and m quaternion algebras in the decomposition of $K[G]$. Then the loop algebra $K[L]$ is the direct sum of $2n$ fields and m Cayley-Dickson algebras.

Table 6: Center of $\Delta_F(L_i, L_i')$

L_i	$\mathcal{Z}(L_i)$	$\mathcal{Z}(\Delta_F(L_i, L_i'))$
L_1	$\langle ab\rangle \cong C_8$	$4F$, if $q \equiv 1 \mod 8$ $2F_2$, if $q \equiv 3 \mod 8$ $2F_2$, if $q \equiv 5 \mod 8$ $2F_2$, if $q \equiv 7 \mod 8$
L_2	$\langle a^2\rangle \cong C_8$	$4F$, if $q \equiv 1 \mod 8$ $2F_2$, if $q \equiv 3 \mod 8$ $2F_2$, if $q \equiv 5 \mod 8$ $2F_2$, if $q \equiv 7 \mod 8$
L_3	$\langle a\rangle \times \langle b^2\rangle \cong C_4 \times C_2$	$4F$, if $q \equiv 1 \mod 4$ $2F_2$, if $q \equiv 3 \mod 4$
L_4	$\langle a^2\rangle \times \langle b^2\rangle \cong C_4 \times C_2$	$4F$, if $q \equiv 1 \mod 4$ $2F_2$, if $q \equiv 3 \mod 4$
L_5	$\langle a^7bab\rangle \times \langle a^2\rangle \cong C_2 \times C_4$	$4F$, if $q \equiv 1 \mod 4$ $2F \oplus F_2$, if $q \equiv 3 \mod 4$
L_6	$\langle a^2\rangle \times \langle b^2\rangle \cong C_2 \times C_4$	$4F$, if $q \equiv 1 \mod 4$ $2F \oplus F_2$, if $q \equiv 3 \mod 4$
L_7	$\langle a^2\rangle \times \langle b^2\rangle \times \langle c\rangle \cong C_2 \times C_2 \times C_2$	$4F$
L_8	$\langle a^2\rangle \times \langle b^2\rangle \times \langle c\rangle \cong C_2 \times C_2 \times C_2$	$4F$

The classes of loops $\mathcal{L}_1, \ldots, \mathcal{L}_5$ mentioned in the above theorem are explained in [4, Ch. V, §3].

Since loops $L_1, L_2, \ldots, L_7$ belongs to the five classes $\mathcal{L}_1, \ldots, \mathcal{L}_5$.

Using Remark 2.5, Table 3, Table 6 and Theorem 4.2, we can find decomposition of $\Delta_F(L_i, L_i')$ for $i = 1, 2, \ldots, 7$ as:

$$\Delta_F(L_1, L_1') = \Delta_F(L_2, L_2') \cong \begin{cases} 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 8 \\ 2\mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 8 \\ 2\mathfrak{Z}(F_2), & \text{if } q \equiv 5 \mod 8 \\ 2\mathfrak{Z}(F_2), & \text{if } q \equiv 7 \mod 8 \end{cases}$$

$$\Delta_F(L_3, L_3') = \Delta_F(L_4, L_4') \cong \begin{cases} 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 2\mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 4 \end{cases}$$

$$\Delta_F(L_5, L_5') = \Delta_F(L_6, L_6') \cong \begin{cases} 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 2\mathfrak{Z}(F) \oplus \mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 4 \end{cases}$$

$$\Delta_F(L_7, L_7') \cong 4\mathfrak{Z}(F).$$

Thus, we have the decomposition of $F[L_i]$ for $i = 1, 2, \ldots, 7$ as follows:

$$F[L_1] \cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 8 \\ 16F \oplus 8F_2 \oplus 2\mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 8 \\ 32F \oplus 2\mathfrak{Z}(F_2), & \text{if } q \equiv 5 \mod 8 \\ 16F \oplus 8F_2 \oplus 2\mathfrak{Z}(F_2), & \text{if } q \equiv 7 \mod 8 \end{cases}$$

$$F[L_2] \cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 8 \\ 8F \oplus 12F_2 \oplus 2\mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 8 \\ 16F \oplus 8F_2 \oplus 2\mathfrak{Z}(F_2), & \text{if } q \equiv 5 \mod 8 \\ 8F \oplus 12F_2 \oplus 2\mathfrak{Z}(F_2), & \text{if } q \equiv 7 \mod 8 \end{cases}$$

$$F[L_3] \cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 16F \oplus 8F_2 \oplus 2\mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 4 \end{cases}$$

$$F[L_4] \cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 8F \oplus 12F_2 \oplus 2\mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 4 \end{cases}$$

$$F[L_5] \cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 8 \\ 8F \oplus 12F_2 \oplus 2\mathfrak{Z}(F) \oplus \mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 8 \\ 16F \oplus 8F_2 \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 5 \mod 8 \\ 8F \oplus 12F_2 \oplus 2\mathfrak{Z}(F) \oplus \mathfrak{Z}(F_2), & \text{if } q \equiv 7 \mod 8 \end{cases}$$

$$F[L_6] \cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 8 \\ 8F \oplus 12F_2 \oplus 2\mathfrak{Z}(F) \oplus \mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 8 \\ 16F \oplus 8F_2 \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 5 \mod 8 \\ 8F \oplus 12F_2 \oplus 2\mathfrak{Z}(F) \oplus \mathfrak{Z}(F_2), & \text{if } q \equiv 7 \mod 8 \end{cases}$$

$$F[L_7] \cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 8F \oplus 12F_2 \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 3 \mod 4. \end{cases}$$

Now, it only remains to find the decomposition of $F[L_8]$. As $\mathcal{Z}(\Delta_F(L_8, L_8')) \cong 4F$, $\dim_F\ (\Delta_F(L_8, L_8')) = 32$ and $\Delta_F(L_8, L_8') \cong \overset{n}{\underset{i=1}{\oplus}} \mathfrak{Z}(K_i)$, where K_i is field extension of F. So the only possibility for $\Delta_F(L_8, L_8')$ is $4\mathfrak{Z}(F)$.

Thus

$$F[L_8] \cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 8F \oplus 12F_2 \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 3 \mod 4. \end{cases}$$

5. Decomposition of Loop Algebras of Decomposable Loops

In this section, we determine the decomposition of finite semisimple loop algebras of decomposable RA loops of order 64. The decomposable loops are: $L_9 = C_2 \times C_2 \times M(Q_8, *, t_1)$, $L_{10} = C_4 \times M(Q_8, *, t_1)$, $L_{11} = C_2 \times C_2 \times M(Q_8, 2)$, $L_{12} = C_4 \times M(Q_8, 2)$, $L_{13} = C_2 \times M(16\Gamma_2 b, *, 1)$, $L_{14} = C_2 \times M(16\Gamma_2 d, *, t_1)$, $L_{15} = C_2 \times M(16\Gamma_2 c_1, *, 1)$ and $L_{16} = C_2 \times M(16\Gamma_2 c_2, *, t_1)$.

The decomposition of finite semisimple loop algebras of RA loops of order 32 have been obtained in [10] as

$$F[M(16\Gamma_2 b, *, 1)] \cong \begin{cases} 16F \oplus 2\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 16F \oplus \mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 4, \end{cases}$$

$$F[M(16\Gamma_2 d, *, t_1)] \cong \begin{cases} 16F \oplus 2\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 8F \oplus 4F_2 \oplus \mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 4, \end{cases}$$

$$F[M(16\Gamma_2 c_1, *, 1)] \cong \begin{cases} 16F \oplus 2\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 8F \oplus 4F_2 \oplus 2\mathfrak{Z}(F), & \text{if } q \equiv 3 \mod 4, \end{cases}$$

$$F[M(16\Gamma_2 c_2, *, t_1)] \cong \begin{cases} 16F \oplus 2\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 8F \oplus 4F_2 \oplus 2\mathfrak{Z}(F), & \text{if } q \equiv 3 \mod 4. \end{cases}$$

From [4, Ch VII, Cor 2.3] and Remark 2.3, we have

$$F[M(Q_8, *, t_1)] \cong F[M(Q_8, 2)] \cong 8F \oplus \mathfrak{Z}(F).$$

It is known that if G and L are Moufang loops and F is a field, then $F[G \times L] \cong F[G] \otimes_F F[L]$. We have

(1) $F[L_9] \cong F[C_2 \times C_2 \times M(Q_8, *, t_1)]$
$$\begin{aligned} &\cong F[C_2] \otimes_F F[C_2] \otimes_F F[M(Q_8, *, t_1)] \\ &\cong 4F \otimes_F (8F \oplus \mathfrak{Z}(F)) \\ &\cong 32F \oplus 4\mathfrak{Z}(F). \end{aligned}$$

(2) $F[L_{10}] \cong F[C_4 \times M(Q_8, *, t_1)]$
$$\begin{aligned} &\cong F[C_4] \otimes_F F[M(Q_8, *, t_1)] \\ &\cong \left(\begin{cases} 4F, & \text{if } q \equiv 1 \mod 4 \\ 2F \oplus F_2, & \text{if } q \equiv 3 \mod 4 \end{cases} \right) \otimes_F (8F \oplus \mathfrak{Z}(F)) \\ &\cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 16F \oplus 8F_2 \oplus 2\mathfrak{Z}(F) \oplus \mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 4. \end{cases} \end{aligned}$$

(3) $F[L_{11}] \cong F[C_2 \times C_2 \times M(Q_8, 2)]$
$$\cong F[C_2] \otimes_F F[C_2] \otimes_F F[M(Q_8, 2)]$$
$$\cong 4F \otimes_F (8F \oplus \mathfrak{Z}(F))$$
$$\cong 32F \oplus 4\mathfrak{Z}(F).$$

(4) $F[L_{12}] \cong F[C_4 \times M(Q_8, 2)]$
$$\cong F[C_4] \otimes_F F[M(Q_8, 2)]$$
$$\cong \left(\begin{cases} 4F, & \text{if } q \equiv 1 \mod 4 \\ 2F \oplus F_2, & \text{if } q \equiv 3 \mod 4 \end{cases}\right) \otimes_F (8F \oplus \mathfrak{Z}(F))$$
$$\cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 16F \oplus 8F_2 \oplus 2\mathfrak{Z}(F) \oplus \mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 4. \end{cases}$$

(5) $F[L_{13}] \cong F[C_2 \times M(16\Gamma_2 b, *, 1)]$
$$\cong F[C_2] \otimes_F F[M(16\Gamma_2 b, *, 1)]$$
$$\cong 2F \otimes_F \begin{cases} 16F \oplus 2\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 16F \oplus \mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 4 \end{cases}$$
$$\cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 32F \oplus 2\mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 4. \end{cases}$$

(6) $F[L_{14}] \cong F[C_2 \times M(16\Gamma_2 d, *, t_1)]$
$$\cong F[C_2] \otimes_F F[M(16\Gamma_2 d, *, t_1)]$$
$$\cong 2F \otimes_F \begin{cases} 16F \oplus 2\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 8F \oplus 4F_2 \oplus \mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 4 \end{cases}$$
$$\cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 16F \oplus 8F_2 \oplus 2\mathfrak{Z}(F_2), & \text{if } q \equiv 3 \mod 4. \end{cases}$$

(7) $F[L_{15}] \cong F[C_2 \times M(16\Gamma_2 c_1, *, 1)]$
$$\cong F[C_2] \otimes_F F[M(16\Gamma_2 c_1, *, 1)]$$
$$\cong 2F \otimes_F \begin{cases} 16F \oplus 2\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 8F \oplus 4F_2 \oplus 2\mathfrak{Z}(F), & \text{if } q \equiv 3 \mod 4 \end{cases}$$
$$\cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 16F \oplus 8F_2 \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 3 \mod 4. \end{cases}$$

(8) $F[L_{16}] \cong F[C_2 \times M(16\Gamma_2 c_2, *, t_1)]$
$$\cong F[C_2] \otimes_F F[M(16\Gamma_2 c_2, *, t_1)]$$

$$\cong 2F \otimes_F \begin{cases} 16F \oplus 2\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 8F \oplus 4F_2 \oplus 2\mathfrak{Z}(F), & \text{if } q \equiv 3 \mod 4 \end{cases}$$

$$\cong \begin{cases} 32F \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 1 \mod 4 \\ 16F \oplus 8F_2 \oplus 4\mathfrak{Z}(F), & \text{if } q \equiv 3 \mod 4. \end{cases}$$

6. The Unit Loops

In this section, we describe the structure of the unit loops of loop algebras of all RA loops of order 64.

The following theorem gives the structure of the unit loops of finite loop algebras of indecomposable RA loops of order 64.

Theorem 6.1. *The following hold:*

(1) *If* $\operatorname{char} F > 2$, *then*

(a) $\mathcal{U}(F[L_1]) \cong$
$32F^* \times 4GLL(2, F)$, *if* $q \equiv 1 \mod 8$
$16F^* \times 8F_2^* \times 2GLL(2, F_2)$, *if* $q \equiv 3 \mod 8$
$32F^* \times 2GLL(2, F_2)$, *if* $q \equiv 5 \mod 8$
$16F^* \times 8F_2^* \times 2GLL(2, F_2)$, *if* $q \equiv 7 \mod 8$.

(b) $\mathcal{U}(F[L_2]) \cong$
$32F^* \times 4GLL(2, F)$, *if* $q \equiv 1 \mod 8$
$8F^* \times 12F_2^* \times 2GLL(2, F_2)$, *if* $q \equiv 3 \mod 8$
$16F^* \times 8F_2^* \times 2GLL(2, F_2)$, *if* $q \equiv 5 \mod 8$
$8F^* \times 12F_2^* \times 2GLL(2, F_2)$, *if* $q \equiv 7 \mod 8$.

(c) $\mathcal{U}(F[L_3]) \cong$
$32F^* \times 4GLL(2, F)$, *if* $q \equiv 1 \mod 4$
$16F^* \times 8F_2^* \times 2GLL(2, F_2)$, *if* $q \equiv 3 \mod 4$.

(d) $\mathcal{U}(F[L_4]) \cong$
$32F^* \times 4GLL(2, F)$, *if* $q \equiv 1 \mod 4$
$8F^* \times 12F_2^* \times 2GLL(2, F_2)$, *if* $q \equiv 3 \mod 4$.

(e) $\mathcal{U}(F[L_5) \cong$
$32F^* \times 4GLL(2, F)$, *if* $q \equiv 1 \mod 8$
$8F^* \times 12F_2^* \times 2GLL(2, F) \times GLL(2, F_2)$, *if* $q \equiv 3 \mod 8$
$16F^* \times 8F_2^* \times 4GLL(2, F)$, *if* $q \equiv 5 \mod 8$
$8F^* \times 12F_2^* \times 2GLL(2, F) \times GLL(2, F_2)$, *if* $q \equiv 7 \mod 8$.

(f) $\mathcal{U}(F[L_6]) \cong$
$32F^* \times 4GLL(2, F)$, *if* $q \equiv 1 \mod 8$
$8F^* \times 12F_2^* \times 2GLL(2, F) \times GLL(2, F_2)$, *if* $q \equiv 3 \mod 8$
$16F^* \times 8F_2^* \times 4GLL(2, F)$, *if* $q \equiv 5 \mod 8$
$8F^* \times 12F_2^* \times 2GLL(2, F) \times GLL(2, F_2)$, *if* $q \equiv 7 \mod 8$.

(g) $\mathcal{U}(F[L_7]) \cong$
$32F^* \times 4GLL(2, F)$, *if* $q \equiv 1 \mod 4$
$8F^* \times 12F_2^* \times 4GLL(2, F)$, *if* $q \equiv 3 \mod 4$.

(h) $\mathcal{U}(F[L_8]) \cong$
$32F^* \times 4GLL(2,F)$, *if* $q \equiv 1 \mod 4$
$8F^* \times 12F_2^* \times 4GLL(2,F)$, *if* $q \equiv 3 \mod 4$.

(2) *If* char $F = 2$ *and* $V_i = 1 + \Delta_F(L_i)$, *for* $i = 1, 2, \ldots, 8$, *then*
(a) $\mathcal{U}(F[L_i]) \cong F^* \times V_i$.
(b) $\mathcal{U}(F[L_i])$ *is a nilpotent loop of class* 2 *and* $dim_F(\Delta_F(L_i)) = 63$.

Proof. (1) Since $GLL(2,K)$ is the unit loop of $\mathfrak{Z}(K)$. Hence the proof follows from Section 4.
(2) The proof follows from Theorems 2.10, 2.8 and 2.9. □

The following theorem gives the structure of the unit loops of finite loop algebras of decomposable RA loops of order 64.

Theorem 6.2. *The following hold:*

(1) *If* char $F > 2$, *then*
(a) $\mathcal{U}(F[L_9]) \cong 32F^* \times 4GLL(2,F)$.
(b) $\mathcal{U}(F[L_{10}]) \cong$
$32F^* \times 4GLL(2,F)$, *if* $q \equiv 1 \mod 4$
$16F^* \times 8F_2^* \times 2GLL(2,F) \times GLL(2,F_2)$, *if* $q \equiv 3 \mod 4$.
(c) $\mathcal{U}(F[L_{11}]) \cong 32F^* \times 4GLL(2,F)$.
(d) $\mathcal{U}(F[L_{12}]) \cong$
$32F^* \times 4GLL(2,F)$, *if* $q \equiv 1 \mod 4$
$16F^* \times 8F_2^* \times 2GLL(2,F) \times GLL(2,F_2)$, *if* $q \equiv 3 \mod 4$.
(e) $\mathcal{U}(F[L_{13}]) \cong$
$32F^* \times 4GLL(2,F)$, *if* $q \equiv 1 \mod 4$
$32F^* \times 2GLL(2,F_2)$, *if* $q \equiv 3 \mod 4$.
(f) $\mathcal{U}(F[L_{14}]) \cong$
$32F^* \times 4GLL(2,F)$, *if* $q \equiv 1 \mod 4$
$16F^* \times 8F_2^* \times 2GLL(2,F_2)$, *if* $q \equiv 3 \mod 4$.
(g) $\mathcal{U}(F[L_{15}]) \cong$
$32F^* \times 4GLL(2,F)$, *if* $q \equiv 1 \mod 4$
$16F^* \times 8F_2^* \times 4GLL(2,F)$, *if* $q \equiv 3 \mod 4$.
(h) $\mathcal{U}(F[L_{16}]) \cong$
$32F^* \times 4GLL(2,F)$, *if* $q \equiv 1 \mod 4$
$16F^* \times 8F_2^* \times 4GLL(2,F)$, *if* $q \equiv 3 \mod 4$.

(2) *If* char $F = 2$ *and* $V_i = 1 + \Delta_F(L_i)$, *for* $i = 9, 10, \ldots, 16$, *then*
(a) $\mathcal{U}(F[L_i]) \cong F^* \times V_i$.
(b) $\mathcal{U}(F[L_i])$ *is a nilpotent loop of class* 2 *and* $dim_F(\Delta_F(L_i)) = 63$.

Proof. (1) Since $GLL(2,K)$ is the unit loop of $\mathfrak{Z}(K)$. Hence the proof follows from Section 5.
(2) The proof follows from Theorems 2.10, 2.8 and 2.9. □

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