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# METRIC PROPERTIES OF CONVERGENCE IN MEASURE WITH RESPECT TO A MATRIX-VALUED MEASURE

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Abstract. A notion of convergence in measure with respect to a matrix-valued measure $M$ is discussed and a corresponding metric space denoted by $L_0(M)$ is introduced. There are given some conditions on $M$ under which $L_0(M)$ is locally convex or normable. Some density results are obtained and applied to the description of shift invariant sub-modules of $L_0(M)$ if $M$ is defined on the $\sigma$-algebra of Borel sets of $(-\pi, \pi]$.

## 1. Introduction

For $r, s, t \in \mathbb{N}$, let $\mathbf{M}_{s,t}$ be the linear space of $s \times t$ matrices with complex entries, which is equipped with an arbitrary norm, $\mathbf{M}_{t,t} =: \mathbf{M}_t$, and $M$, $M := (m_{jk})_{j=1,\dots,t}^{k=1,\dots,r}$, be an $\mathbf{M}_{t,r}$-valued measure. In [12] it was introduced a notion of convergence in measure $M$ of a sequence of $\mathbf{M}_{s,t}$-valued functions. However, since the main goal of [12] was a discussion of a problem of linear algebra, which arose in connection with this notion, measure-theoretic or functional-analytic aspects of convergence in measure $M$ were not studied thoroughly there. The present paper is devoted to such questions.

We mention that notions of convergence in measure with respect to rather general vector measures were defined in several papers, see e.g. [4]. Applying these definitions to our situation, we obtain that a sequence of $\mathbf{M}_{s,t}$-valued functions converges in measure $M$ if and only if it converges in measure $m_{j,k}$ for $j = 1, \dots, t, k = 1, \dots, r$. Thus, the fact that the measures $m_{j,k}$ form a matrix is ignored by these definitions. In [12] we proposed a different way of introducing convergence in measure $M$, which, to some extent, takes into account the matrix structure of $M$. Its main idea, which goes back to I. S. Kac [8] and was applied by M. Rosenberg [16] independently and in a slightly more

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general way, is to deal with $M$ in the form $dM = \frac{dM}{d\mu} d\mu$, where $\mu$ is a finite non-negative (scalar) measure, with respect to which $M$ is absolutely continuous, and $\frac{dM}{d\mu}$ denotes the corresponding Radon-Nikodym derivative.

In Section 2 of the present paper we define convergence in measure $M$ (see Definition 2.8) and give an equivalent formulation (see Proposition 2.14), which is sometimes more convenient. To do this we have to describe the set of those $\mathbf{M}_{s,t}$-valued functions, for which convergence in measure $M$ can be defined and to introduce a certain equivalence relation on this set. We supplement results of [12] discussing some questions arising if $M$ is defined on a non-complete $\sigma$-algebra.

Analogously to convergence in measure with respect to a non-negative measure, convergence in measure $M$ can be defined by a metric. The corresponding metric space is denoted by $L_0(M)$ and is studied in Sections 3 and 4. We give necessary and sufficient conditions on $M$ such that $L_0(M)$ is locally convex or can be normed. Similar results for non-negative measures were obtained in [17].

In Section 4 we derive some density results and apply them to the description of shift invariant sub-modules of $L_0(M)$ if $M$ is defined on the Borel $\sigma$-algebra of $(-\pi, \pi]$.

As usual, by $\mathbb{N}$ and $\mathbb{C}$ we denote the set of positive integers and complex numbers, resp. For $X \in \mathbf{M}_{t,r}$, denote by $\mathcal{R}(X)$ and $X^*$ its range and adjoint matrix, resp. The unit matrix of $\mathbf{M}_t$ is denoted by $I_t$ and any zero matrix by 0.

## 2. Definition and basic properties

Let $(\Omega, \mathfrak{A})$ be a measurable space. A function $F\colon \Omega \to \mathbf{M}_{s,t}$ is called *measurable* if it is $(\mathfrak{A}, \mathfrak{B}_{s,t})$-measurable, where $\mathfrak{B}_{s,t}$ denotes the $\sigma$-algebra of Borel subsets of the Banach space $\mathbf{M}_{s,t}$.

For an $\mathbf{M}_{t,r}$-valued measure $M$, $M := (m_{jk})_{j=1,\dots,t}^{k=1,\dots,r}$, let $\Delta_M$ be the set of all non-negative finite measures on $\mathfrak{A}$, with respect to which $M$ is absolutely continuous. Note that $\Delta_M$ is not empty since the measure

$$\mu_M := \sum_{j=1}^{t} \sum_{k=1}^{r} |m_{jk}| \tag{2.1}$$

where $|m_{jk}|$ denotes the variation of the $\mathbb{C}$-valued measure $m_{jk}$, is an element of $\Delta_M$. Note further that $\mu_M$ is absolutely continuous with respect to $\mu$ if $\mu \in \Delta_M$.

For a certain set of $\mathbf{M}_{s,t}$-valued functions on $\Omega$ we shall define a notion of $M$-equivalence and then for these $M$-equivalence classes a notion of convergence in measure $M$. As Remark 2.7 below shows it would be enough to deal with measurable $\mathbf{M}_{s,t}$-valued functions. However, since sometimes it is convenient to enlarge the $M$-equivalence classes, cf. [12], our first task will be to describe the set of functions we shall study.

If $\mu \in \Delta_M$ and $\frac{dM}{d\mu}$ is a corresponding Radon-Nikodym derivative, denote by $P_\mu(\omega)$ the orthogonal projection in $\mathbb{C}^t$ onto $\mathcal{R}(\frac{dM}{d\mu}(\omega))$, $\omega \in \Omega$. Recall that from the measurability of $\frac{dM}{d\mu}$ it follows the measurability of the function $P_\mu$, cf. [1]. Let $\mathcal{P}_\mu$ be the set of all orthoprojection-valued functions differing from $P_\mu$ on some set of $\mu$-measure 0 and $\Phi_s(M,\mu)$ be the set of all functions $F\colon \Omega \to \mathbf{M}_{s,t}$ such that $FP_\mu$ is measurable for some $P_\mu \in \mathcal{P}_\mu$.

**Lemma 2.1.** *If $\mu,\nu \in \Delta_M$, then $\Phi_s(M,\mu) = \Phi_s(M,\nu)$.*

*Proof.* For $\mu,\nu \in \Delta_M$, choose a Radon-Nikodym derivative $\frac{d\mu}{d(\mu+\nu)}$ and set $A := \{\omega \in \Omega\colon \frac{d\mu}{d(\mu+\nu)}(\omega) \neq 0\}$. Let $F \in \Phi_s(M,\mu)$ and $P_\mu \in \mathcal{P}_\mu$ be such that $FP_\mu$ is measurable. The chain rule leads to $\frac{dM}{d(\mu+\nu)} = \frac{dM}{d\mu}\frac{d\mu}{d(\mu+\nu)}$ $(\mu+\nu)$-a.e., cf. [6, §32, Theorem A]. It follows that there exists $P_{\mu+\nu} \in \mathcal{P}_{\mu+\nu}$ satisfying $P_{\mu+\nu} = P_\mu$ on $A$ and $P_{\mu+\nu} = 0$ on $\Omega \setminus A$. Denoting by $\mathbf{1}_A$ the indicator function of $A$, we get $FP_{\mu+\nu} = \mathbf{1}_A FP_\mu$, which yields the measurability of $FP_{\mu+\nu}$. Conversely, if $F \in \Phi_s(M,\mu+\nu)$ and $P_{\mu+\nu} \in \mathcal{P}_{\mu+\nu}$ are such that $FP_{\mu+\nu}$ is measurable, we set $P_\mu := P_{\mu+\nu}$ on $A$ and $P_\mu = 0$ on $\Omega \setminus A$. Since $\mu(\Omega \setminus A) = 0$, we obtain that $P_\mu \in \mathcal{P}_\mu$ and the function $FP_\mu = \mathbf{1}_A FP_{\mu+\nu}$ is measurable. Thus, the equality $\Phi_s(M,\mu) = \Phi_s(M,\mu+\nu)$ is proved and the result follows by symmetry. □

According to the preceding lemma it is correct to set $\Phi_s(M) := \Phi_s(M,\mu)$, $\mu \in \Delta_M$, and to call the elements of $\Phi_s(M)$ $M$-measurable functions. The set $\Phi_s(M)$ can be described with the aid of the completion of $\mathfrak{A}$ under $M$, which is denoted by $\mathfrak{A}_M$ and is, by definition, the completion of $\mathfrak{A}$ under $\mu_M$. Recall that $\mathfrak{A}_M := \{A \cup A_0\colon A \in \mathfrak{A}, A_0 \in \mathfrak{A}_0\}$, where $\mathfrak{A}_0$ is the $\sigma$-algebra of $\mu_M$-negligible sets, i. e., $\mathfrak{A}_0 := \{A_0\colon \text{There exists } A \in \mathfrak{A} \text{ satisfying } \mu_M(A) = 0 \text{ and } A_0 \subseteq A\}$, cf. [2, Section 1.5]. A measure $M$ is called *complete* if $\mathfrak{A}_M = \mathfrak{A}$.

**Proposition 2.2.** *Let $\mu \in \Delta_M$ and $F$ be an $\mathbf{M}_{s,t}$-valued function on $\Omega$. If $F \in \Phi_s(M)$, then $FQ_\mu$ is $(\mathfrak{A}_M, \mathfrak{B}_{s,t})$-measurable for every $Q_\mu \in \mathcal{P}_\mu$. If $FQ_\mu$ is $(\mathfrak{A}_M, \mathfrak{B}_{s,t})$-measurable for some $Q_\mu \in \mathcal{P}_\mu$, then $F \in \Phi_s(M)$.*

*Proof.* Let $F \in \Phi_s(M)$ and $P_\mu \in \mathcal{P}_\mu$ be such that $FP_\mu$ is measurable. For $Q_\mu \in \mathcal{P}_\mu$, set $A := \{\omega \in \Omega\colon P_\mu(\omega) \neq Q_\mu(\omega)\}$ and write $FQ_\mu = \mathbf{1}_{\Omega\setminus A}FQ_\mu + \mathbf{1}_A FQ_\mu = \mathbf{1}_{\Omega\setminus A}FP_\mu + \mathbf{1}_A FQ_\mu$. Since $\Omega\setminus A \in \mathfrak{A}$ and $\mu(A) = 0$, the first assertion is proved. Now assume that $FQ_\mu$ is $(\mathfrak{A}_M, \mathfrak{B}_{s,t})$-measurable for some $Q_\mu \in \mathcal{P}_\mu$. There exists a set $B \in \mathfrak{A}$ such that $\mu(B) = 0$ and $\mathbf{1}_{\Omega\setminus B}FQ_\mu$ is measurable, cf. [2, Proposition 2.2.3]. Let $P_\mu = Q_\mu$ on $\Omega \setminus B$ and $P_\mu = 0$ on $B$. Then $P_\mu \in \mathcal{P}_\mu$ and $FP_\mu$ is measurable, hence, $F \in \Phi_s(M)$. □

**Proposition 2.3.** *A measure $M$ is complete if and only if $FP_{\mu_M}$ is measurable for all $F \in \Phi_s(M)$ and $P_{\mu_M} \in \mathcal{P}_{\mu_M}$.*

*Proof.* If $\mathfrak{A}_M = \mathfrak{A}$ and $\mu \in \Delta_M$, then by the first assertion of Proposition 2.2, $FP_\mu$ is measurable for $F \in \Phi_s(M)$ and $P_\mu \in \mathcal{P}_\mu$. If $\mathfrak{A}_M \neq \mathfrak{A}$, there exist

$A_0 \in (\mathfrak{A}_0 \setminus \mathfrak{A})$ and $A \in (\mathfrak{A}_0 \cap \mathfrak{A})$ satisfying $A_0 \subseteq A$. Let $X \in \mathbf{M}_{s,t}$, $X \neq 0$, and $F := \mathbf{1}_{A_0} X$. Then $F$ belongs to $\Phi_s(M)$, however, $FQ_{\mu_M}$ is not measurable if $Q_{\mu_M} = I_t$ on $A$, $Q_{\mu_M} \in \mathcal{P}_{\mu_M}$. □

In what follows for $\mu \in \Delta_M$ and $F \in \Phi_s(M)$, we denote by $P_\mu$ such an element of $\mathcal{P}_\mu$ that $FP_\mu$ is measurable. A simple result will be useful.

**Lemma 2.4.** *Let $\mu \in \Delta_M$ and $\{F_n\}_{n \in \mathbb{N}}$ be a sequence of functions of $\Phi_s(M)$. There exists $P_\mu \in \mathcal{P}_\mu$ such that $F_n P_\mu$ is measurable for all $n \in \mathbb{N}$.*

*Proof.* If $P_{\mu,j} \in \mathcal{P}_\mu$ and $F_j P_{\mu,j}$ is measurable, set $A_{j,k} := \{\omega \in \Omega\colon P_{\mu,j}(\omega) \neq P_{\mu,k}(\omega)\}$, $j,k \in \mathbb{N}$, $A := \bigcup_{j,k \in \mathbb{N}} A_{j,k}$, $P_\mu := P_{\mu,1}$ on $\Omega \setminus A$ and $P_\mu = 0$ on $A$. Then $P_\mu \in \mathcal{P}_\mu$ and $F_n P_\mu$ is measurable for $n \in \mathbb{N}$. □

**Lemma 2.5.** *Let $\mu, \nu \in \Delta_M$ and $F, G \in \Phi_s(M)$. Then $FP_\mu = GP_\mu$ $\mu$-a.e. if and only if $FP_\nu = GP_\nu$ $\nu$-a.e.*

*Proof.* Applying the chain rule similarly to the proof of Lemma 2.11, one can show that $FP_\mu = GP_\mu$ $\mu$-a.e. if and only if $FP_{\mu+\nu} = GP_{\mu+\nu}$ $(\mu+\nu)$-a.e., which yields the result. □

The preceding lemma justifies the following definition.

**Definition 2.6.** Two functions $F, G \in \Phi_s(M)$ are called *$M$-equivalent* if for some and, hence, for all $\mu \in \Delta_M$, $FP_\mu = GP_\mu$ $\mu$-a.e.

The set of $M$-equivalence classes of functions of $\Phi_s(M)$ is denoted by $\widetilde{\Phi}_s(M)$. It is obvious that if $F_1, F_2, G_1, G_2 \in \Phi_s(M)$, $X \in \mathbf{M}_s$, and $F_1$ and $F_2$ as well as $G_1$ and $G_2$ are $M$-equivalent, then $F_1 + G_1$ and $F_2 + G_2$ as well as $XF_1$ and $XF_2$ are $M$-equivalent. Therefore, $\widetilde{\Phi}_s(M)$ forms a left $\mathbf{M}_s$-module. As is common practice, studying $M$-equivalence classes we shall work with their representatives, i.e. with functions from $\Phi_s(M)$.

*Remark* 2.7. If $F \in \Phi_s(M)$, then $F$ and $FP_\mu$, $\mu \in \Delta_M$, belong to the same $M$-equivalence class. Therefore, any $M$-equivalence class contains a measurable function and we could confine ourselves to measurable functions $F$ from the very beginning.

**Definition 2.8.** Let $M$ be an $\mathbf{M}_{t,r}$-valued measure on $\mathfrak{A}$, $\mu \in \Delta_M$, and let $\|\cdot\|$ be an arbitrary norm on $\mathbf{M}_{s,t}$. A sequence $\{F_n\}_{n \in \mathbb{N}}$ of elements of $\widetilde{\Phi}_s(M)$ is called *fundamental in measure $M$* if the sequence $\{F_n P_\mu\}_{n \in \mathbb{N}}$ is fundamental in measure $\mu$, i.e., if $\lim_{m,n \to \infty} \mu(\|(F_n - F_m)P_\mu\| > \varepsilon) = 0$ for all $\varepsilon > 0$. It *converges in measure $M$ to $F \in \widetilde{\Phi}_s(M)$* if $\{F_n P_\mu\}_{n \in \mathbb{N}}$ converges in measure $\mu$ to $FP_\mu$, i.e., if $\lim_{n \to \infty} \mu(\|(F_n - F)P_\mu\| > \varepsilon) = 0$ for all $\varepsilon > 0$. It *converges to $F \in \widetilde{\Phi}_s(M)$ $M$-a.e.* if $\{F_n P_\mu\}_{n \in \mathbb{N}}$ converges to $FP_\mu$ $\mu$-a.e.

Since all norms on the finite-dimensional space $\mathbf{M}_{s,t}$ are equivalent, Definition 2.8 does not depend on the choice of the norm $\|\cdot\|$. The independence on the choice of $\mu \in \Delta_M$ is established by the following lemma, which can be obtained with the aid of the chain rule similarly to the proofs of Lemmas 2.1 or 2.5.

**Lemma 2.9.** *If $\mu, \nu \in \Delta_M$, $\{F_n\}_{n\in\mathbb{N}}$ is a sequence of elements of $\widetilde{\Phi}_s(M)$, and $F \in \widetilde{\Phi}_s(M)$, then the following assertions hold:*

(a) *The sequence $\{F_nP_\mu\}_{n\in\mathbb{N}}$ is fundamental in measure $\mu$ if and only if $\{F_nP_\nu\}$ is fundamental in measure $\nu$.*
(b) *The sequence $\{F_nP_\mu\}_{n\in\mathbb{N}}$ converges in measure $\mu$ or $\mu$-a.e. to $FP_\mu$ if and only if $\{F_nP_\nu\}_{n\in\mathbb{N}}$ converges in measure $\nu$ or $\nu$-a.e., resp., to $FP_\nu$.*

*Remark* 2.10. Let $t, r, u \in \mathbb{N}$. Note that for an $\mathbf{M}_{t,r}$-valued measure $M$ and an $\mathbf{M}_{t,u}$-valued measure $N$ on $\mathfrak{A}$, the notions of convergence in measure coincide if and only if $\mathcal{R}(\frac{dM}{d\mu}) = \mathcal{R}(\frac{dN}{d\mu})$ $\mu$-a.e., $\mu \in \Delta_M \cap \Delta_N$. This is a generalization of the fact that for arbitrary finite non-negative measures $\sigma$ and $\tau$, convergence in measure $\sigma$ is equivalent to convergence in measure $\tau$ if and only if $\sigma$ and $\tau$ are equivalent, i.e., if and only if $\sigma$ and $\tau$ have the same sets of measure 0. An analogous remark on $M$-a.e. convergence could be made.

Since a sequence $\{F_n\}_{n\in\mathbb{N}}$ of elements of $\widetilde{\Phi}_s(M)$ converges in measure $M$ or $M$-a.e. if and only if $\{F_nP_\mu\}_{n\in\mathbb{N}}$ converges in measure $\mu$ or $\mu$-a.e., resp., $\mu \in \Delta_M$, basic properties of convergence in measure $M$ or convergence $M$-a.e. can be derived from corresponding properties of convergence in measure $\mu$ or convergence $\mu$-a.e., resp. For future use we only mention the following facts.

**Theorem 2.11.** *A sequence converges in measure $M$ if and only if it is fundamental in measure $M$. The limit of a sequence converging in $M$ is unique (within to $M$-equivalence) and there exists a subsequence converging $M$-a.e. to the same limit. If a sequence converges $M$-a.e., it converges in measure $M$.*

We conclude the present section by giving equivalent conditions for convergence in measure $M$, which sometimes are simpler to apply.

**Lemma 2.12.** *Let $\mu \in \Delta_M$ and let $H$ be a measurable $\mathbf{M}_t$-valued function such that $\mathcal{R}(H) \subseteq \mathcal{R}(P_\mu)$ $\mu$-a.e. If $F$ and $G$ are $M$-equivalent functions of $\Phi_s(M)$, then $FH$ and $GH$ are measurable $M$-equivalent functions.*

*Proof.* Since from the conditions of the lemma it follows $FP_\mu H = FH = GH$ $\mu$-a.e., the result is obvious. □

**Lemma 2.13.** *Let $\mu \in \Delta_M$ and $H$ be a measurable $\mathbf{M}_t$-valued function satisfying*

$$\mathcal{R}(H) = \mathcal{R}(H^*) = \mathcal{R}(P_\mu) \quad \mu\text{-a.e.} \tag{2.2}$$

*Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of functions of $\Phi_s(M)$. Then the following assertions are equivalent:*

(i) $\lim_{n\to\infty} \mu(\|F_nP_\mu\| > \varepsilon) = 0$ *for all* $\varepsilon > 0$,
(ii) $\lim_{n\to\infty} \mu(\|F_nH\| > \varepsilon) = 0$ *for all* $\varepsilon > 0$.

*Proof.* We can assume that $\|\cdot\|$ is the spectral norm.

(i)$\Rightarrow$(ii): Note first that $F_nH$, $n \in \mathbb{N}$, is measurable according to Lemma 2.12. For $\delta > 0$, choose $c > 0$ satisfying $\mu(\|H\| > c) < \delta$. From (i) it follows that for $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(\|F_nP_\mu\| > \varepsilon/c) < \delta$ if $n \geq n_0$. The inequality $\|(F_nH)(\omega)\| = \|(F_nP_\mu H)(\omega)\| \leq \|(F_nP_\mu)(\omega)\|\|H(\omega)\|$ implies that $\|(F_nH)(\omega)\| \leq \varepsilon$ if $\|(F_nP_\mu)(\omega)\| \leq \varepsilon/c$ and $\|H(\omega)\| \leq c$, $\omega \in \Omega$. Therefore, if $n \geq n_0$, we have $\mu(\|F_nH\| > \varepsilon) \leq \mu(\|F_nP_\mu\| > \varepsilon/c) + \mu(\|H\| > c) < 2\delta$, which yields (ii).

(ii)$\Rightarrow$(i): Since $\|F_nHP_\mu\| \leq \|F_nH\|$ $\mu$-a.e., from (ii) we get

$$\lim_{n\to\infty} \mu\left(\|F_nHP_\mu\| > \varepsilon\right) = 0 \tag{2.3}$$

for any $\varepsilon > 0$. Denote by $H(\omega)^+$ the Moore-Penrose inverse of $H(\omega)$, $\omega \in \Omega$, and recall that $\mathcal{R}(H(\omega)^+) = \mathcal{R}(H(\omega)^*)$ and that $H^+$ is measurable if $H$ is measurable. Therefore (2.2) and (2.3) show that we can apply the conclusion (i)$\Rightarrow$(ii) to the sequence $\{F_nH\}_{n\in\mathbb{N}}$ and the function $H^+$. Taking into account that $H(\omega)H(\omega)^+$ is the orthoprojection onto $\mathcal{R}(H(\omega))$, $\omega \in \Omega$, we obtain $\lim_{n\to\infty} \mu(\|F_nP_\mu\| > \varepsilon) = \lim_{n\to\infty} \mu(\|F_nHH^+\| > \varepsilon) = 0$ for all $\varepsilon > 0$. $\square$

Applying Lemmas 2.12 and 2.13 to the function $H := \frac{dM}{d\mu}$ we can formulate the following equivalent condition for convergence in measure $M$.

**Proposition 2.14.** *Let $\mu \in \Delta_M$. A sequence $\{F_n\}_{n\in\mathbb{N}}$ of elements of $\widetilde{\Phi}_s(M)$ converges in measure $M$ to $F \in \widetilde{\Phi}_s(M)$ if and only if for all $\varepsilon > 0$ one has $\lim_{n\to\infty} \mu(\|(F_n - F)\frac{dM}{d\mu}\| > \varepsilon) = 0$.*

## 3. The metric space $L_0(M)$

Let $M$ be an $\mathbf{M}_{t,r}$-valued measure on $\mathfrak{A}$. We denote by $L_{0,s}(M)$ the left $\mathbf{M}_s$-module $\widetilde{\Phi}_s(M)$, equipped with the topology of convergence in measure $M$. To simplify the notation we shall omit the dependence on $s$ in the notation of $L_{0,s}(M)$ and set $L_{0,s}(M) =: L_0(M)$. For $\mu \in \Delta_M$ and a norm $\|\cdot\|$ on $\mathbf{M}_{s,t}$, one can define a metric $d$:

$$d(F,G) := \int_\Omega \frac{\|(F-G)P_\mu\|}{1 + \|(F-G)P_\mu\|} d\mu, \quad F, G \in \widetilde{\Phi}_s(M), \tag{3.1}$$

on $\widetilde{\Phi}_s(M)$, which is invariant, i.e. $d(F,G) := d(F-H, G-H)$, $F, G, H \in \widetilde{\Phi}_s(M)$. It is not hard to see (or follows from a well known result on convergence in measure $\mu$) that a sequence converges with respect to the metric $d$ if and only if it converges in measure $M$. Taking into account the first assertion of Theorem 2.11, we obtain the following result.

**Theorem 3.1.** *The space $L_0(M)$ is an $F$-space, i.e. a complete topological vector space, whose topology is generated by an invariant metric.*

Thomasian [17] characterized those finite non-negative measures $\sigma$, for which convergence in measure $\sigma$ and $\sigma$-a.e. convergence coincide as well as those, for which the space $L_0(\sigma)$ can be normed, see also [14, 5]. To generalize Thomasian's results to matrix-valued measures recall that a set $A \in \mathfrak{A}$ is called an *atom* of a finite non-negative measure $\mu$ on $\mathfrak{A}$ if $\mu(A) > 0$ and $\mu(B) = 0$ or $\mu(B) = \mu(A)$ for every subset $B$ of $A$, $B \in \mathfrak{A}$.

**Theorem 3.2.** *Let $\mu_M$ be a measure defined by* (2.1)*. The following assertions are equivalent:*

(i) *The set $\Omega$ is a union of atoms and a set of measure* 0 *of the measure $\mu_M$.*
(ii) *A sequence $\{F_n\}_{n\in\mathbb{N}}$ of elements of $L_0(M)$ converges $M$-a.e. if and only if it converges in measure $M$.*
(iii) *The space $L_0(M)$ is locally convex.*

*Proof.* (i)⇒(ii): Note that if $F \in L_0(M)$, then the function $FP_{\mu_M}$ is $\mu_M$-a.e. constant on any atom of $\mu_M$. However, the set of atoms of $\mu_M$ is an at most countable set. Therefore, assertion (i) implies that from convergence in measure $M$ it follows convergence $M$-a.e. To complete the proof use Theorem 2.11.

(ii)⇒(i): Assume that (i) is not satisfied. Then there exists a set $A \in \mathfrak{A}$ of positive measure $\mu_M$, which does not contain any atom of $\mu_M$. It follows that for every $n \in \mathbb{N}$ there exists a finite sequence $\{A_{n,j}\}_{j=1,\dots,n}$ of pairwise disjoint sets of $\mathfrak{A}$ such that $\bigcup_{j=1}^n A_{n,j} = A$ and $\mu(A_{n,j}) = \frac{1}{n}\mu(A)$, $j = 1,\dots,n$. Set $F_{n,j} = \mathbf{1}_{A_{n,j}} P_{\mu_M}$ for some $P_{\mu_M} \in \mathcal{P}_{\mu_M}$ and note that $F_{n,j} \neq 0$ $\mu$-a.e. on $A_{n,j}$, $j = 1,\dots,n$, $n \in \mathbb{N}$. Obviously, the sequence $\{F_{n,j}\}_{j=1,\dots,n,n\in\mathbb{N}}$ converges in measure $M$ but does not converge $M$-a.e.

(i)⇒(iii): Let $\Omega = B \cup (\bigcup_{j\in J} A_j)$, where $\mu_M(B) = 0$, $A_j$ are atoms of $\mu_M$, and $J$ is at most countable. For $F \in L_0(M)$, choose $X_j \in \mathbf{M}_{s,t}$ satisfying $FP_{\mu_M} = X_j$ $\mu_M$-a.e. on $A_j$ and set $\|F\|_j := \|X_j\|$, $j \in J$. Therefore, the topology of $L_0(M)$ can be defined by an at most countable set of semi-norms $\|\cdot\|_j$, $j \in J$, which implies that $L_0(M)$ is locally convex.

(iii)⇒(i): Assume that (i) is not satisfied and define $A$ and $A_{n,j}$, $j = 1,\dots,n$, $n \in \mathbb{N}$, as in the proof of the conclusion (ii)⇒(i). Let $V$ be a convex neighbourhood of 0. Let $F \in L_0(M)$ and define $F_{n,j} := n\mathbf{1}_{A_{n,j}}F$, $j = 1,\dots,n$, $n \in \mathbb{N}$. Setting $\mu := \mu_M$ in (3.1), we obtain $d(F_{n,j}, 0) < \frac{1}{n}\mu_M(A)$, $j = 1,\dots,n$, $n \in \mathbb{N}$. Since there exists $c > 0$ such that $\{G \in L_0(M)\colon d(G, 0) < c\}$ is a subset of $V$, we can conclude that for $n$ large enough, $F_{n,j} \in V$, $j = 1,\dots,n$, hence $F := \frac{1}{n}\sum_{j=1}^n F_{n,j} \in V$ by convexity of $V$. Since $F \in L_0(M)$ was arbitrary, it follows $V = L_0(M)$, which shows that $L_0(M)$ does not have non-trivial convex neighbourhoods of 0. In particular, $L_0(M)$ is not locally convex □

**Theorem 3.3.** *Let $\mu_M$ be a measure defined by* (2.1)*. The following assertions are equivalent:*

(i) *The set $\Omega$ is a finite union of atoms and a set of measure* 0 *of $\mu_M$.*
(ii) *The space $L_0(M)$ can be normed.*

*Proof.* (i)⇒(ii): For $F \in L_0(M)$, define $A_j$ and $X_j \in \mathbf{M}_{s,t}$, $j \in J$, as in the proof of the conclusion (i)⇒(iii) of Theorem 3.2, where the set $J$ is finite now. By $\|F\|_{L_0(M)} := \sum_{j\in J}\|X_j\|\mu_M(A_j)$, $F \in L_0(M)$, a norm on $L_0(M)$ is defined, and $\|\cdot\|_{L_0(M)}$ and the metric $d$ generate equivalent topologies.

(ii)⇒(i): Assume that (i) is not satisfied. Then there exist an infinite set of atoms of $\mu_M$ or a set $A \in \mathfrak{A}$ of positive measure $\mu_M$, which does not contain any atom. In either case we can find a sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets of $\mathfrak{A}$ such that $\mu_M(A_n) > 0$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} \mu_M(A_n) = 0$. If $\|\cdot\|_M$ denotes an arbitrary norm on $\widetilde{\Phi}_s(M)$, we have $\|\mathbf{1}_{A_n}P_{\mu_M}\|_M \neq 0$ and can define $F_n := \|\mathbf{1}_{A_n}P_{\mu_M}\|_M^{-1}\mathbf{1}_{A_n}P_{\mu_M}$, $n \in \mathbb{N}$. Obviously, $\{F_n\}_{n\in\mathbb{N}}$ converges in measure $M$ to 0, however $\|F_n\|_M = 1$, $n \in \mathbb{N}$. Therefore, the norm $\|\cdot\|_M$ and the metric $d$ do not generate equivalent topologies. □

## 4. Density results

From Definition 2.8 it follows that a sequence of elements of $\widetilde{\Phi}_s(M)$ converges in measure $M$ if and only if it converges in measure $P_\mu d\mu$, $\mu \in \Delta_M$. Therefore, it is enough to study $\mathbf{M}_t^{\geq}$-valued measures, where $\mathbf{M}_t^{\geq}$ denotes the cone of non-negative hermitian $t \times t$ matrices. From now on we shall assume that $M$ is an $\mathbf{M}_t^{\geq}$-valued measure on $\mathfrak{A}$. In this case $\frac{dM}{d\mu}$ can assumed to be an $\mathbf{M}_t^{\geq}$-valued function and we can define $(\frac{dM}{d\mu}(\omega))^{1/p}$, $\omega \in \Omega$, $p > 0$, according to the functional calculus of normal matrices. Recall that the function $(\frac{dM}{d\mu})^{1/p}$ is measurable, cf. [1]. Moreover, for simplification of the notation and in accordance with the papers [3, 10] we shall assume that the norm $\|\cdot\|$ is the Frobenius norm.

Let $p \in (0, \infty)$. By $L_p(M)$ we denote the space of all $F \in \widetilde{\Phi}_s(M)$ such that

$$\|F\|_{M,p} := \left(\int_\Omega \left\|F\left(\frac{dM}{d\mu}\right)^{1/p}\right\|^p d\mu\right)^{1/p} < \infty,$$

where $\mu \in \Delta_M$, cf. [3, 10], see also [9, 11] for infinite-dimensional generalizations. Again one can derive from the chain rule that the definition of $L_p(M)$ does not depend on the choice of the measure $\mu \in \Delta_M$. The space $L_p(M)$ is a left $\mathbf{M}_s$-module. For every $p \geq 1$, it is a Banach space under the norm $\|\cdot\|_{M,p}$. For every $p \in (0, 1)$, it is an $F$-space under the invariant metric $\|F - G\|_{M,p}^p$, $F, G \in L_p(M)$. If $0 \leq p_1 \leq p_2 < \infty$, then $L_{p_2}(M) \subseteq L_{p_1}(M)$.

**Lemma 4.1.** *Let $p \in (0,\infty)$ and $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $L_p(M)$ tending to 0 in $L_p(M)$. Then $\lim_{n\to\infty} F_n = 0$ with respect to the metric of $L_0(M)$.*

*Proof.* Let $\mu \in \Delta_\mu$. For $\varepsilon > 0$, set $A_n = \{\omega \in \Omega \colon \|F_n(\omega)(\frac{dM}{d\mu}(\omega))^{1/p}\| > \varepsilon\}$. Since $\lim_{n\to\infty} \mu(A_n) \leq \lim_{n\to\infty} \frac{1}{\varepsilon^p}\int_{A_n}\|F_n(\frac{dM}{d\mu})^{1/p}\|^p d\mu \leq \lim_{n\to\infty} \frac{1}{\varepsilon^p}\|F_n\|_{M,p}^p = 0$, the sequence $\{F_n(\frac{dM}{d\mu})^{1/p}\}_{n\in\mathbb{N}}$ tends to 0 in measure $\mu$. From Lemma 2.13 it follows that $\lim_{n\to\infty} F_n = 0$ in $L_0(M)$. □

A function $S$ of the form $S = \sum_{j=1}^{k} \mathbf{1}_{A_j} X_j$, $A_j \in \mathfrak{A}$, $X_j \in \mathbf{M}_{s,t}$, $j = 1, \dots, k$, $k \in \mathbb{N}$, is called a *simple function*. The left $\mathbf{M}_s$-module of simple functions is denoted by $\mathcal{S}$.

**Proposition 4.2.** *The set $\mathcal{S}$ is dense in $L_0(M)$.*

*Proof.* If $F \in L_0(M)$ and $\mu \in \Delta_M$, we can assume that $FP_\mu$ is measurable. Thus, there exists a sequence $\{S_n\}_{n\in\mathbb{N}}$ of simple functions tending to $FP_\mu$ $\mu$-a.e. Since $\|FP_\mu - S_n P_\mu\| \le \sqrt{t}\|FP_\mu - S_n\|$ $\mu$-a.e., Theorem 2.11 yields the result. $\square$

**Proposition 4.3.** *Let $p \in (0, \infty)$ and let $\mathcal{D}$ be a dense subset of $L_p(M)$. Then $\mathcal{D}$ is a dense subset of $L_0(M)$.*

*Proof.* Since $\mathcal{S} \subseteq L_p(M)$, the closure of $\mathcal{D}$ with respect to the metric of $L_p(M)$ includes $\mathcal{S}$. From Lemma 4.1 it follows that $\mathcal{S}$ is also contained in the closure of $\mathcal{D}$ with respect to the metric of $L_0(M)$. An application of Proposition 4.2 gives the result. $\square$

We recall the following definition of strong absolute continuity of $\mathbf{M}_t^{\geq}$-valued measures, which generalizes the notion of absolute continuity for non-negative measures, see [15, Section 5].

**Definition 4.4.** Let $M$ and $N$ be $\mathbf{M}_t^{\geq}$-valued measures on $\mathfrak{A}$. If $\mathcal{R}(\frac{dN}{d\mu}) \subseteq \mathcal{R}(\frac{dM}{d\mu})$ $\mu$-a.e. for some $\mu \in (\Delta_M \cap \Delta_N)$, we shall call $N$ *strongly absolutely continuous with respect to* $M$ and write $N \lll M$.

Note that Definition 4.4 does not depend on the choice of $\mu \in (\Delta_M \cap \Delta_N)$, see [15, Section 5].

Let $N \lll M$. From the preceding definition it follows that $M(A) = 0$ yields $N(A) = 0$, $A \in \mathfrak{A}$. One obtains $\Delta_M \subseteq \Delta_N$, hence, $\Delta_M \cap \Delta_N = \Delta_M$. Moreover, if $\mu \in \Delta_M$ and $Q_\mu(\omega)$ denotes the orthoprojector in $\mathbb{C}^t$ onto $\mathcal{R}(\frac{dN}{d\mu}(\omega))$, $\omega \in \Omega$, we have

$$Q_\mu \le P_\mu \quad \mu\text{-a.e.,} \tag{4.1}$$

which implies that

$$FQ_\mu = FQ_\mu P_\mu = FP_\mu Q_\mu \quad \mu\text{-a.e.,} \quad F \in \Phi_s(M). \tag{4.2}$$

Relation (4.2) yields $\Phi_s(M) \subseteq \Phi_s(N)$ and $\|FQ_\mu\| \le \sqrt{t}\|FP_\mu\|$ $\mu$-a.e., $F \in \Phi_s(M)$. From (4.1) one can conclude that if two functions $F$ and $G$ of $\Phi_s(M)$ are $M$-equivalent, they are $N$-equivalent. Summarizing we obtain the following result.

**Proposition 4.5.** *Let $M$ and $N$ be $\mathbf{M}_t^{\geq}$-valued measures on $\mathfrak{A}$ and $N \lll M$. There exists a continuous map $\mathbf{j}$ from $L_0(M)$ onto $L_0(N)$ such that $\mathbf{j}F = F$, $F \in L_0(M)$. If $\mathcal{D}$ is a dense subset of $L_0(M)$, then $\mathbf{j}\mathcal{D}$ is dense in $L_0(N)$.*

In what follows a certain converse of the density assertion of the preceding proposition will be proved. Let $M$ be an $\mathbf{M}_t^{\geq}$-valued measure on $\mathfrak{A}$, $\mu \in \Delta_M$, and $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{A}$ be a sequence satisfying $\lim_{n\to\infty} \mu(\Omega \setminus A_n) = 0$. Define a measure $M_n$ by $M_n(A) := M(A \cap A_n)$, $A \in \mathfrak{A}$, $n \in \mathbb{N}$. Obviously, $M_n \lll M$ and we can introduce a map $\mathbf{j}_n$ from $L_0(M)$ onto $L_0(M_n)$ according to Proposition 4.5.

**Proposition 4.6.** *Let $\mathcal{D}$ be a subset of $L_0(M)$. If for every $n \in \mathbb{N}$, the set $\mathbf{j}_n\mathcal{D}$ is dense in $L_0(M_n)$, then $\mathcal{D}$ is dense in $L_0(M)$.*

*Proof.* Let $\mu \in \Delta_M$ and $F \in L_0(M)$. For $n \in \mathbb{N}$ choose $k \in \mathbb{N}$ such that $\mu(\Omega \setminus A_k) < \frac{1}{2n}$. By density of $\mathbf{j}_k\mathcal{D}$ in $L_0(M_k)$ there exists a function $F_n \in \mathbf{j}_k\mathcal{D}$ satisfying $\mu(\{\|FP_\mu - F_nP_\mu\| > \frac{1}{n}\} \cap A_k) < \frac{1}{2n}$. For $\varepsilon > 0$, let $n_0 \in \mathbb{N}$ be such that $\frac{1}{n_0} < \varepsilon$. If $n \geq n_0$, we obtain that $\mu(\|FP_\mu - F_nP_\mu\| > \varepsilon) \leq \mu(\|FP_\mu - F_nP_\mu\| > \frac{1}{n}) \leq \mu(\{\|FP_\mu - F_nP_\mu\| > \frac{1}{n}\} \cap A_k) + \mu(\Omega \setminus A_k) < \frac{1}{n}$. It follows that the sequence $\{F_n\}_{n\in\mathbb{N}} \in \mathcal{D}$ converges to $F$ with respect to the metric of $L_0(M)$. □

Using the preceding proposition we can derive a density result for a particular $L_0(M)$-space, which can be applied to the description of shift invariant sub-modules.

Let $\Omega$ be the interval $(-\pi, \pi]$, $\mathfrak{A} =: \mathfrak{B}$ the $\sigma$-algebra of Borel subsets of $(-\pi, \pi]$, and $M$ be an $\mathbf{M}_t^{\geq}$-valued measure on $\mathfrak{B}$. Denote by $\mathcal{T}$ the set of all $\mathbf{M}_{s,t}$-valued analytic trigonometric polynomials, i.e. the set of all functions of the form

$$\sum_{j=0}^{k} X_j \mathrm{e}^{\mathrm{i}j\cdot}, \quad X_j \in \mathbf{M}_{s,t}, \quad j = 0, \ldots, k, \quad k \in \mathbb{N},$$

on $(-\pi, \pi]$.

**Proposition 4.7.** *The set $\mathcal{T}$ is dense in $L_0(M)$.*

*Proof.* For $n \in \mathbb{N}$, let $M_n$ be an $\mathbf{M}_t^{\geq}$-valued measure, which is defined by $M_n(B) = M(B \cap (-\pi, \pi - 1/n])$, $B \in \mathfrak{B}$. From the theory of vector-valued stationary processes it follows that $\mathcal{T}$ is dense in $L_2(M_n)$, $n \in \mathbb{N}$, cf. [18, Section 7, Main Lemma I]. Thus, the result is a consequence of Propositions 4.3 and 4.6. □

Let $M$ be an $\mathbf{M}_t^{\geq}$-valued measure on $\mathfrak{B}$. A closed left $\mathbf{M}_s$-sub-module $\mathcal{I}$ of $L_0(M)$ is called *invariant*, if it is invariant under the shift operator, i.e., if $\mathrm{e}^{\mathrm{i}\cdot}\mathcal{I} \subseteq \mathcal{I}$. In accordance with a definition by Helson [7] we call an invariant sub-module *doubly invariant* if $\mathrm{e}^{-\mathrm{i}\cdot}\mathcal{I} \subseteq \mathcal{I}$.

**Proposition 4.8.** *Every invariant sub-module of $L_0(M)$ is doubly invariant.*

*Proof.* Let $\mathcal{I}$ be an invariant sub-module of $L_0(M)$ and $F \in \mathcal{I}$. Consider an $\mathbf{M}_{s,t}$-valued measure $FdM := F\frac{dM}{d\mu}d\mu$, $\mu \in \Delta_M$. From Proposition 4.7 it follows that the set $\mathcal{T}$ of $\mathbf{M}_s$-valued analytic trigonometric polynomials is dense in $L_0(FdM)$. Therefore, Proposition 2.14 implies that there exists a sequence $\{T_n\}_{n\in\mathbb{N}}$ of functions of $\mathcal{T}$ satisfying $\lim_{n\to\infty}\mu(\|\mathrm{e}^{-\mathrm{i}\cdot}F\frac{dM}{d\mu}-T_nF\frac{dM}{d\mu}\|>\varepsilon)=0$ for all $\varepsilon>0$. Since $T_nF\in\mathcal{I}$, $n\in\mathbb{N}$, and $\mathcal{I}$ is a closed subset of $L_0(M)$, another application of Proposition 2.14 yields $\mathrm{e}^{-\mathrm{i}\cdot}F\in\mathcal{I}$. □

For $p\in(0,\infty)$, all doubly invariant sub-modules of $L_p(M)$ were described in [13]. It is not hard to see that the method used there can also be applied to $L_0(M)$. We omit the details and only mention the result.

**Theorem 4.9.** *Let $\mathcal{I}$ be a closed left $\mathbf{M}_s$-sub-module of $L_0(M)$. The following assertions are equivalent:*

(i) *$\mathcal{I}$ is invariant.*
(ii) *$\mathcal{I}$ is doubly invariant.*
(iii) *For $\mu\in\Delta_M$, there exists a measurable orthoprojection-valued function $P\colon(-\pi,\pi]\to\mathbf{M}_t^{\geq}$ such that $\mathcal{R}(P)\subseteq\mathcal{R}(P_\mu)$ $\mu$-a.e. and $\mathcal{I}=L_0(M)P:=\{FP\colon F\in L_0(M)\}$.*

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