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MAPS PRESERVING THE SPECTRUM OF CERTAIN PRODUCTS OF OPERATORS

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Abstract. Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{B}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H})$ be the spaces of all bounded operators and all self-adjoint operators on $\mathcal{H}$, respectively. We give the concrete forms of the maps on $\mathcal{B}(\mathcal{H})$ and also $\mathcal{S}(\mathcal{H})$ which preserve the spectrum of certain products of operators.

1. Introduction And Statement of the Results

The study of spectrum-preserving linear maps between Banach algebras goes back to Frobenius [3] who studied linear maps on matrix algebras preserving the determinant. The following conjecture seems to be still open: Any spectrum-preserving linear map from a unital Banach algebra onto a unital semi-simple Banach algebra that preserves the unit is a Jordan morphism. There are many other papers concerning this type of linear preservers; for example, see [1, 6, 7, 10, 12].

Without assuming linearity, spectrum-preserving maps are almost arbitrary; see [2, 5, 6, 9, 11]. In [9], Molnár considered multiplicatively spectrum-preserving surjective maps on Banach algebras in the sense that the spectrum of the product of the image of any two elements is equal to the spectrum of the product of those two elements, and proved that the maps are almost isomorphisms in the sense that isomorphisms multiplied by a signum function for the Banach algebra of all complex-valued continuous functions on a first countable compact Hausdorff space. In [5], the authors considered the same problem for triple Jordan products of operators and proved that such maps must be Jordan isomorphisms multiplied by a cubic root of unity. Moreover, they extended the results of [6].

The main aim of the present paper is to consider the spectrum preserving of certain products of operators on the spaces of all self-adjoint operators and also all bounded operators on a Hilbert space.

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We recall some notations. Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space, $\mathcal{B}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H})$ be the spaces of all bounded operators and self-adjoint operators on $\mathcal{H}$, respectively. I denotes the identity operator and for any operator A in $\mathcal{B}(\mathcal{H})$, $\sigma(A)$ and $|A|$ denote the spectrum of A and the absolute value of A that is equal to $(A^*A)^{1/2}$, respectively. If $A \in \mathcal{B}(\mathcal{H})$, then $f(t) = t^r$ is continuous and nonnegative on $\sigma(|A|)$ for any positive rational number r. Hence f belongs to $\mathbf{C}(\sigma(|A|))$. By the continuous functional calculus, $f(A) = |A|^r$ belongs to $\mathcal{B}(\mathcal{H})$. If $P \in \mathcal{B}(\mathcal{H})$ is self-adjoint and $P^2 = P$, then P is called projection. The set of all projections on $\mathcal{H}$ is denoted by $\mathcal{P}(\mathcal{H})$. If $x, y \in \mathcal{H}$, then $x \otimes y$ stands for the operator of rank at most one defined by

$$(x \otimes y)z = < z, y > x \quad (z \in \mathcal{H}).$$

The set of all rank-one operators on $\mathcal{H}$ is denoted by $\mathcal{F}_1(\mathcal{H})$. The set of all rank-one projections on $\mathcal{H}$ is denoted by $\mathcal{P}_1(\mathcal{H})$.

Let $A \in \mathcal{B}(\mathcal{H})$ be a finite-rank operator. On the set of all finite-rank operators on $\mathcal{H}$, one can define the trace functional tr by

$$\mathrm{tr}A = \sum_{i=1}^{n} < x_i, y_i >,$$

where $A = \sum_{i=1}^{n} x_i \otimes y_i$. Then tr is a well-defined linear functional.

Our main results are the follows.

Theorem 1. *Let $\mathcal{H}$ be a complex Hilbert space, r and s positive rational numbers such that $r + s > 1$ and $\phi\colon \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ a surjective function which satisfies*

$$(*) \qquad \sigma(|A|^r B|A|^s) = \sigma(|\phi(A)|^r \phi(B)|\phi(A)|^s)$$

for all A in $\mathcal{P}_1(\mathcal{H}) \cup \{I\}$ and B in $\mathcal{S}(\mathcal{H})$. Then there exists a bounded linear or conjugate linear bijection $T\colon \mathcal{H} \to \mathcal{H}$ satisfying $T^* = T^{-1}$ such that

$$\phi(A) = TAT^*$$

for all $A \in \mathcal{S}(\mathcal{H})$.

Theorem 2. *Let $\mathcal{H}$ be a complex Hilbert space, r a positive rational number such that $r > 1$ and $\phi\colon \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ a surjective function which satisfies*

$$\sigma(|A|^r B) = \sigma(|\phi(A)|^r \phi(B))$$

for all A in $\mathcal{P}_1(\mathcal{H}) \cup \{I\}$ and B in $\mathcal{S}(\mathcal{H})$. Then there exists a bounded linear or conjugate linear bijection $T\colon \mathcal{H} \to \mathcal{H}$ satisfying $T^* = T^{-1}$ such that

$$\phi(A) = TAT^*$$

for all $A \in \mathcal{S}(\mathcal{H})$.

Theorem 3. *Let $\mathcal{H}$ be a complex Hilbert space, r a positive rational number such that $r > 1$ and $\phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ a surjective function which satisfies*

$$\sigma(|A|^r B) = \sigma(|\phi(A)|^r \phi(B))$$

for all $A, B \in \mathcal{B}(\mathcal{H})$. Then there exists a bounded linear or conjugate linear bijection $T\colon \mathcal{H} \to \mathcal{H}$ satisfying $T^* = T^{-1}$ such that

$$\phi(A) = TAT^*$$

for all $A \in \mathcal{B}(\mathcal{H})$.

Theorem 4. *Let r and s be positive rational numbers such that $r + s > 1$ and $\phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ a surjective function which satisfies*

$$(*) \qquad \sigma(|A|^r B|A|^s) = \sigma(|\phi(A)|^r \phi(B)|\phi(A)|^s)$$

for all $A, B \in \mathcal{B}(\mathcal{H})$. Then there exists a bounded linear or conjugate linear bijection $T\colon \mathcal{H} \to \mathcal{H}$ satisfying $T^* = T^{-1}$ such that

$$\phi(A) = TAT^*$$

for all $A \in \mathcal{B}(\mathcal{H})$.

2. Proofs

We use the following propositions and theorem to prove our results.

The following proposition is useful criteria for characterizing rank-one projections.

Proposition 1. *Let $A \in \mathcal{P}(\mathcal{H})$. Then the following statements are equivalent.*
(a) A is rank-one.
(b) $A \neq 0$ and $\operatorname{card}(\sigma(AT) \setminus \{0\}) \leq 1$ for all $T \in \mathcal{S}(\mathcal{H})$.

Proof. In order to complete the proof, it is sufficient to prove that (b) implies (a). Assume on the contrary that there exist y_1, y_2 in the range of A such that are linearly independent. Without loss of generality, suppose that $\|y_1\| = 1$. If

$$z_1 = \frac{y_2 - < y_2, y_1 > y_1}{\|y_2 - < y_2, y_1 > y_1\|}$$

then $\|z_1\| = 1$ and $< y_1, z_1 >= 0$. Let $S = ay_1 \otimes y_1 + bz_1 \otimes z_1$ for some nonzero and different reals a and b. It is easy to check that $S \in \mathcal{S}(\mathcal{H})$, $ASy_1 = ay_1$ and $ASz_1 = bz_1$, because $Ay_1 = y_1$ and $Ay_2 = y_2$. Hence $a, b \in \sigma(AS)$. This is a contradiction and so the proof is completed. $\square$

Proposition 2. *Let $A \in \mathcal{B}(\mathcal{H})$. Then the following statements hold.*

(a) *Let A be a positive operator and r a positive rational number. A is rank-one if and only if A^r is.*

(b) *A is rank-one if and only if $|A|$ is.*

Proof. (a) If A is rank-one, it is clear that A^r is rank-one. Let m and n be natural numbers such that $r = \frac{m}{n}$. It is enough to prove that A is rank-one when A^m is rank-one, because if $A^{\frac{m}{n}}$ is rank-one, then it is clear that A^m is rank-one. Assume that A^m is rank-one but A^{m-1} isn't. Then there are vectors $y_1, y_2 \in \operatorname{Ran} A^{m-1}$ such that y_1 and y_2 are linearly independent. Since A^m is rank-one, Ay_1 and Ay_2 are linearly dependent. So there exist scalars $a, b \in \mathbb{C}$

with $ab \neq 0$ such that $aAy_1 + bAy_2 = 0$ which implies $ay_1 + by_2 \in \ker A$. Since A is positive, $ay_1 + by_2 \in (\mathrm{Ran}A)^{\perp}$. On the other hand, $ay_1 + by_2 \in \mathrm{Ran}A$. So we obtain $ay_1 + by_2 = 0$ which implies the linear independence of y_1 and y_2 that is contradiction. Therefore A^{m-1} is rank-one. Inductively, it can be concluded that A is rank-one.

(b) The statement "A is rank-one if and only if A^*A is" is proved similar to the previous part. Now since $|A| = (A^*A)^{1/2}$, the assertion can be concluded from (a). □

Proposition 3. *Let $A \in \mathcal{S}(\mathcal{H})$ and $x \in \mathcal{H}$. Then $Ax = 0$ if and only if $|A|x = 0$.*

Proof. Assertion follows easily from the following equalities

$$< Ax, Ax >=< A^2x, x >=< |A|^2x, x >=< |A|x, |A|x > . \qquad \square$$

We use the following theorem to give the general forms of maps satisfying the hypothesis of the mentioned main results.

Theorem 5 ([15])**.** *Let $\mathcal{H}$ be a complex Hilbert space, and let $L\colon \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ be an $\mathbb{R}$-linear and weakly continuous operator. If $L(\mathcal{P}_1(\mathcal{H})) = \mathcal{P}_1(\mathcal{H})$, then there exists a bounded linear or conjugate linear bijection $T\colon \mathcal{H} \to \mathcal{H}$ satisfying $T^* = T^{-1}$ such that $L(A) = TAT^*$ for all $A \in \mathcal{S}(\mathcal{H})$.*

Proof of Theorem 1. We prove that ϕ is a continuous additive bijection of $\mathcal{S}(\mathcal{H})$ that preserves rank-one projections in both directions. In order to prove this assertion we divide the proof into several steps.

Step 1. Let $B \in \mathcal{S}(\mathcal{H})$. $B = 0$ if and only if $\phi(B) = 0$. Since ϕ is surjective, there exists $B \in \mathcal{S}(\mathcal{H})$ such that $\phi(B) = 0$. So by $(*)$ we have

$$\sigma(|A|^rB|A|^s) = \{0\}$$

for all $A \in \mathcal{P}_1(\mathcal{H})$. This implies

$$\{0, < Bx, x >\} = \{0\}$$

for all x from unit ball, which yields $B = 0$.

Step 2. $\phi(I) = I$. First we assert that $|\phi(I)|^{\frac{r+s}{2}}\phi(I)|\phi(I)|^{\frac{r+s}{2}} = I$. The condition $(*)$ yields

$$\begin{aligned}\sigma(|\phi(I)|^{\frac{r+s}{2}}\phi(I)|\phi(I)|^{\frac{r+s}{2}}) \cup \{0\} &= \sigma(|\phi(I)|^{r+s}|\phi(I)) \cup \{0\}\\ &= \sigma(|\phi(I)|^{r}\phi(I)|\phi(I)|^{s}) \cup \{0\}\\ &= \sigma(I) \cup \{0\}\\ &= \{0, 1\}.\end{aligned}$$

Hence $|\phi(I)|^{\frac{r+s}{2}}\phi(I)|\phi(I)|^{\frac{r+s}{2}}$ is a projection. So in order to complete the proof of assertion, it is enough to show that $|\phi(I)|^{\frac{r+s}{2}}\phi(I)|\phi(I)|^{\frac{r+s}{2}}$ is injective. First we show that $|\phi(I)|^{r+s}$ is injective. Because if $x \in \mathcal{H}$ such that $|\phi(I)|^{r+s}x = 0$, then we have

$$< |\phi(I)|^{r+s}x, x >= 0$$

and hence

$$0 = \sigma(|\phi(I)|^{r+s} x \otimes x) = \sigma(|\phi(I)|^r x \otimes x |\phi(I)|^s).$$

From $(*)$ we obtain $\sigma(A) = 0$ where A is an operator such that $\phi(A) = x \otimes x$. This implies that $A = 0$. So by Step 1, $x = 0$ and therefore $|\phi(I)|^{r+s}$ is injective.

Now let $x \in \mathcal{H}$ such that $|\phi(I)|^{\frac{r+s}{2}} \phi(I) |\phi(I)|^{\frac{r+s}{2}} x = 0$. The injectivity of $|\phi(I)|^{r+s}$ yields the injectivity of $|\phi(I)|^{\frac{r+s}{2}}$. So $\phi(I)|\phi(I)|^{\frac{r+s}{2}} x = 0$. By Proposition 3, We have $|\phi(I)|^{\frac{r+s}{2}+1} x = 0$. The condition $r + s > 1$ together with the injectivity of $|\phi(I)|^{r+s}$ and also $|\phi(I)|^{\frac{r+s}{2}}$ yields $x = 0$ and this completes the proof of assertion.

From assertion we can conclude that $|\phi(I)|^{\frac{r+s}{2}}$ is invertible which multiplying by $|\phi(I)|^{\frac{r+s}{2}}$ from right and then by $|\phi(I)|^{-\frac{r+s}{2}}$ from left follows

$$\phi(I)|\phi(I)|^{r+s} = I$$

and hence

$$\phi(I) = \phi(I)|\phi(I)|^{r+s}\phi(I).$$

So $\phi(I) \geq 0$, because we have

$$\phi(I) = [\phi(I)|\phi(I)|^{-\frac{r+s}{2}}][|\phi(I)|^{-\frac{r+s}{2}}\phi(I)]^*.$$

$\phi(I) \geq 0$ and condition $(*)$ imply

$$1 = \sigma(I) = \sigma(|\phi(I)|^r \phi(I) |\phi(I)|^s) = \sigma(\phi(I)^{r+s+1}).$$

Therefore $\phi(I) = I$ and this completes the proof.

Step 3. ϕ is injective. Let $\phi(B) = \phi(B')$. By $(*)$ we obtain

$$\sigma(|A|^r B |A|^s) = \sigma(|A|^r B' |A|^s)$$

for all $A \in \mathcal{P}_1(\mathcal{H})$. This implies

$$\{0, < Bx, x >\} = \{0, < B'x, x >\}$$

and so

$$< (B - B')x, x >= 0$$

for all x from unit ball, which yields $B = B'$ and thus ϕ is injective.

Step 4. ϕ preserves the projections and also the rank-one projections in both directions. $(*)$ and Step 2 yield that $\sigma(B) = \sigma(\phi(B))$ for all $B \in \mathcal{S}(\mathcal{H})$. So it is clear that B is a projection if and only if $\phi(B)$ is. Now let $P \in \mathcal{P}_1(\mathcal{H})$. Since $\phi(P)$ is idempotent, by $(*)$ we obtain

$$\sigma(PB) \setminus \{0\} = \sigma(\phi(P)\phi(B)) \setminus \{0\} \tag{1}$$

for all $B \in \mathcal{S}(\mathcal{H})$. On the other hand, by Proposition 1. we have $P \neq 0$ and $\mathrm{card}(\sigma(PB) \setminus \{0\}) \leq 1$ for all $B \in \mathcal{S}(\mathcal{H})$. This together with (1), surjectivity of ϕ and Step 1 follows that $\phi(P) \neq 0$ and $\mathrm{card}(\sigma(\phi(P)B') \setminus \{0\}) \leq 1$ for all $B' \in \mathcal{S}(\mathcal{H})$. Again by Proposition 1. we infer that $\phi(P)$ is rank-one. Hence ϕ preserves the rank-one projections. Since ϕ is injective and ϕ^{-1} has the same properties of ϕ, the rank-one projections are preserved by ϕ in both directions and this completes the proof.

Step 5. ϕ is additive. Step 4 and $(*)$ imply that

$$\mathrm{tr}(ABA) = \mathrm{tr}(\phi(A)\phi(B)\phi(A))$$

for all $A \in \mathcal{P}_1(\mathcal{H})$ and $B \in \mathcal{S}(\mathcal{H})$. So the assertion can be proved in a very similar way as the discussion in [9].

Step 6. ϕ is continuous. We know that $\sigma(A) = \sigma(\phi(A))$ for all $A \in \mathcal{S}(\mathcal{H})$. By this fact and Step 5, we can conclude that ϕ is continuous.

So by above steps, ϕ is a continuous additive bijection of $\mathcal{S}(\mathcal{H})$ preserving the rank-one projections in both directions. The forms of such transformations is given in Theorem 5. So there exists a bounded linear or conjugate linear bijection $T\colon \mathcal{H} \to \mathcal{H}$ satisfying $T^* = T^{-1}$ such that

$$\phi(A) = TAT^*$$

for any $A \in \mathcal{S}(\mathcal{H})$. Therefore the proof of Theorem 1. is complete. □

Proof of Theorem 2. is similar to the proof of Theorem 1.

Proof of Theorem 1.3. We prove that ϕ is a bijective linear map that preserves self-adjoint operators in both directions. In order to prove this assertion we divide the proof into several steps.

Step 1. ϕ is injective. Let $\phi(B) = \phi(B')$. By $(**)$ we obtain

$$\sigma(|A|^r B) = \sigma(|A|^r B')$$

for all $A \in \mathcal{B}(\mathcal{H})$ and so for all $A \in \mathcal{P}_1(\mathcal{H})$. This implies

$$\{0, < Bx, x >\} = \{0, < B'x, x >\}$$

and so

$$< (B - B')x, x >= 0$$

for all x from unit ball, which yields $B = B'$ and thus ϕ is injective.

Step 2. Let $A \in \mathcal{B}(\mathcal{H})$. $|A|^r = 0$ if and only if $|\phi(A)|^r = 0$. If $|A|^r = 0$, then by $(**)$ we have

$$\sigma(|\phi(A)|^r \phi(B)) = \{0\}$$

for all $B \in \mathcal{B}(\mathcal{H})$. This and surjectivity of ϕ follow

$$\{0, < |\phi(A)|^r x, x >\} = \{0\}$$

for all $x \in \mathcal{H}$, which yields $|\phi(A)|^r = 0$. The converse is proved similarly.

Step 3. ϕ preserves rank-one operators in both directions. By Definition 2.2 in [4] we have the following statement: A is rank-one if and only if $A \neq 0$ and $\mathrm{card}(\sigma(AB) \setminus \{0\}) \leq 1$ for all $B \in \mathcal{B}(\mathcal{H})$.

Let A be rank-one. By Proposition 2.2, $|A|^r$ is rank-one and so $|A|^r \neq 0$ and $\mathrm{card}(\sigma(|A|^r B) \setminus \{0\}) \leq 1$ for all $B \in \mathcal{B}(\mathcal{H})$. By Step 2, $(**)$ and the surjectivity of ϕ we obtain $|\phi(A)|^r \neq 0$ and $\mathrm{card}(\sigma(|\phi(A)|^r B') \setminus \{0\}) \leq 1$ for all $B' \in \mathcal{B}(\mathcal{H})$. This implies that $|\phi(A)|^r$ is rank-one. Again using Proposition 2. yields that $\phi(A)$ is rank-one. The converse is proved similarly.

Step 4. ϕ is linear. Step 3 and $(**)$ yield

$$\mathrm{tr}(|\phi(A)|^r \phi(B)) = \mathrm{tr}(|A|^r B)$$

for all rank-one operator A and all $B \in \mathcal{B}(\mathcal{H})$. So the assertion can be proved in a very similar way as the discussion in [9].

Step 5. ϕ preserves self-adjoint operators in both directions. Let y be an arbitrary element from unit ball. By Step 3, there exists a rank-one operator $A \in \mathcal{B}(\mathcal{H})$ such that $\phi(A) = y \otimes y$. Thus $|A|^r = x \otimes x$ for some $x \in \mathcal{H}$. Now let B be a self-adjoint operator. So by $(**)$ we obtain

$$\sigma(x \otimes B^* x) = \sigma(y \otimes \phi(B)^* y)$$

which implies

$$< Bx, x > = < \phi(B)y, y > .$$

Since $< Bx, x >$ is real, $< \phi(B)y, y >$ is real. This together with the arbitrariness of y implies that $\phi(B)$ is self-adjoint. The converse is proved in a similar way.

So by above steps, ϕ is a bijective linear map such that preserves self-adjoint operators in both directions and satisfies in $(**)$. The forms of such transformations on $\mathcal{S}(\mathcal{H})$ is given in Theorem 2. So there exists a bounded linear or conjugate linear bijection $T\colon \mathcal{H} \to \mathcal{H}$ satisfying $T^* = T^{-1}$ such that

$$\phi(A) = TAT^*$$

for all $A \in \mathcal{S}(\mathcal{H})$. Now let $A \in \mathcal{B}(\mathcal{H})$ be arbitrary. Then there exist $A_1, A_2 \in \mathcal{S}(\mathcal{H})$ such that $A = A_1 + iA_2$. So we have

$$\phi(A) = \phi(A_1 + iA_2) = \phi(A_1) + i\phi(A_2) = TA_1T^* + iTA_2T^* = TAT^*.$$

The proof is complete. □

Proof of Theorem 1.4. is similar to the proof of Theorem 1.3.

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