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A NOTE ON WEAKLY SYMMETRIC MANIFOLDS

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Abstract. The object of the present paper is to determine the nature of the associated 1-forms of a weakly symmetric manifold.

1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by Cartan [4], who, in particular, obtained a classification of those spaces.

Let $(M^n, g), (n = \dim M)$ be a Riemannian manifold, i.e., a manifold M with the Riemannian metric g and let ∇ be the Levi-Civita connection of (M^n, g). A Riemannian manifold is called locally symmetric [4] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [11]. The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature. During the last five decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as conformally symmetric manifolds by Chaki and Gupta [6], recurrent manifolds introduced by Walker [18], conformally recurrent manifolds by Adati and Miyazawa [1], conformally symmetric Ricci-recurrent spaces by Roter [15], pseudo symmetric manifolds introduced by Chaki [5] etc. The notion of recurrent manifolds have been generalized by various authors such as Ricci-recurrent manifolds by Patterson [12], 2-recurrent manifolds by Lichnerowicz [10], projective 2-recurrent manifolds by Ghosh [9] and others.

The notion of weakly symmetric and weakly projective symmetric manifolds were introduced by Tamassy and Binh [17] and later Binh [3] studied decomposable weakly symmetric manifolds. Weakly symmetric manifolds have been studied by several authors ([2, 7, 13, 14]) and many others. A Riemannian

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manifold is said to be semisymmetric [16] if its curvature tensor satisfies the condition

$$\nabla_l \nabla_m R^h_{ijk} - \nabla_m \nabla_l R^h_{ijk} = 0,$$

where ∇ denotes the Levi-Civita connection on (M^n, g).

A non-flat Riemannian manifold (M^n, g) is called weakly symmetric if the curvature tensor is not identically zero and satisfies the condition

$$\nabla_r R_{hijk} = A_r R_{hijk} + B_h R_{rijk} + C_i R_{hrjk} + D_j R_{hirk} + E_k R_{hijr}, \tag{1}$$

where A, B, C, D, E are 1-forms which are non-zero simultaneously. Such a manifold is denoted by $(WS)_n$. If $A = B = C = D = E = 0$, the manifold reduces to a symmetric manifold in the sense of Cartan. This justifies the name of weakly symmetric manifold. Prvanovic [13] proved that under certain condition the 1-forms are related as follows:

$$B_h = C_h = D_h = E_h,$$

for all h. That is, the weakly symmetric manifold is characterized by the condition

$$\nabla_r R_{hijk} = A_r R_{hijk} + B_h R_{rijk} + B_i R_{hrjk} + B_j R_{hirk} + B_k R_{hijr}.$$

The 1-forms A and B are called the associated 1-forms of a $(WS)_n$. The existence of a $(WS)_n$ was proved by Prvanovic [14] and a concrete example was given by De and Bandyopadhyay [7]. The present paper is concerned with certain investigations on a $(WS)_n$. The paper is organized as follows.

After introduction we prove that the associated 1-form A of a $(WS)_n$ is closed if and only if the 1-form B is closed. Further, we obtain a condition under which a $(WS)_n$ result to be a semisymmetric manifold. Finally, a condition under which a $(WS)_n$ reduces to a Pseudo-symmetric manifold in the sense of Deszcz [8] is obtained.

2. Nature of the associated 1-form of a weakly symmetric manifold

In this section we determine the nature of the associated 1-forms of a $(WS)_n$. Let

$$\nabla_r R_{hijk} = A_r R_{hijk} + B_h R_{rijk} + B_i R_{hrjk} + B_j R_{hirk} + B_k R_{hijr}.$$

Then

$$\begin{aligned} \nabla_s \nabla_r R_{hijk} - \nabla_r \nabla_s R_{hijk} = {} & (\nabla_s A_r - \nabla_r A_s) R_{hijk} \\ & + (\nabla_s B_h - B_s B_h) R_{rijk} + (\nabla_s B_i - B_s B_i) R_{hrjk} \\ & + (\nabla_s B_j - B_s B_j) R_{hirk} + (\nabla_s B_k - B_s B_k) R_{hijr} \\ & - (\nabla_r B_h - B_r B_h) R_{sijk} - (\nabla_r B_i - B_r B_i) R_{hsjk} \\ & - (\nabla_r B_j - B_r B_j) R_{hisk} - (\nabla_r B_k - B_r B_k) R_{hijs}, \end{aligned} \tag{2}$$

or,

$$\begin{aligned}\nabla_s\nabla_r R_{hijk} - \nabla_r\nabla_s R_{hijk} &= (\nabla_s A_r - \nabla_r A_s)R_{hijk}\\ &+ B_{sh}R_{rijk} + B_{si}R_{hrjk} + B_{sj}R_{hirk} + B_{sk}R_{hijr}\\ &- B_{rh}R_{sijk} - B_{ri}R_{hsjk} - B_{rj}R_{hisk} - B_{rk}R_{hijs},\end{aligned}$$

where

$$B_{sh} = \nabla_s B_h - B_s B_h.$$

Permuting cyclically h, j, r and i, k, s, we get

$$\begin{aligned}(3)\quad (\nabla_s A_r - \nabla_r A_s)R_{hijk} &+ (\nabla_i A_h - \nabla_h A_i)R_{jkrs} + (\nabla_k A_j - \nabla_j A_k)R_{rshi}\\ &+ B_{sh}R_{rijk} + B_{si}R_{hrjk} + B_{sj}R_{hirk} + B_{sk}R_{hijr}\\ &+ B_{ij}R_{hkrs} + B_{ik}R_{jhrs} + B_{ir}R_{jkhs} + B_{is}R_{jkrh}\\ &+ B_{kr}R_{jshi} + B_{ks}R_{rjhi} + B_{kh}R_{rsji} + B_{ki}R_{rshj}\\ &- B_{rh}R_{sijk} - B_{ri}R_{hsjk} - B_{rj}R_{hisk} - B_{rk}R_{hijs}\\ &- B_{hj}R_{ikrs} - B_{hk}R_{jirs} - B_{hr}R_{jkis} - B_{hs}R_{jkri}\\ &- B_{jr}R_{kshi} - B_{js}R_{rkhi} - B_{jh}R_{rski} - B_{ji}R_{rshk} = 0,\end{aligned}$$

because, in view of the Walker identity [18],

$$\begin{aligned}\nabla_s\nabla_r R_{hijk} - \nabla_r\nabla_s R_{hijk} &+ \nabla_i\nabla_h R_{jkrs}\\ &- \nabla_h\nabla_i R_{jkrs} + \nabla_k\nabla_j R_{rshi} - \nabla_j\nabla_k R_{rshi} = 0.\end{aligned}$$

Now, let us suppose

$$\nabla_s A_r = \nabla_r A_s.$$

Then

$$\begin{aligned}(4)\quad B_{sh}R_{rijk} &+ B_{si}R_{hrjk} + B_{sj}R_{hirk} + B_{sk}R_{hijr}\\ &+ B_{ij}R_{hkrs} + B_{ik}R_{jhrs} + B_{ir}R_{jkhs} + B_{is}R_{jkrh}\\ &+ B_{kr}R_{jshi} + B_{ks}R_{rjhi} + B_{kh}R_{rsji} + B_{ki}R_{rshj}\\ &- B_{rh}R_{sijk} - B_{ri}R_{hsjk} - B_{rj}R_{hisk} - B_{rk}R_{hijs}\\ &- B_{hj}R_{ikrs} - B_{hk}R_{jirs} - B_{hr}R_{jkis} - B_{hs}R_{jkri}\\ &- B_{jr}R_{kshi} - B_{js}R_{rkhi} - B_{jh}R_{rski} - B_{ji}R_{rshk} = 0.\end{aligned}$$

Let us suppose that $\nabla_r B_s \neq \nabla_s B_r$. Then there exists at least one $B_{rs} - B_{sr}$, say $B_{12} - B_{21}$, such that $B_{12} - B_{21} \neq 0$. Putting $h = j = r = 1$, $i = k = s = 2$ in (4) we find

$$6(B_{21} - B_{12})R_{1212} = 0,$$

that is,

$$R_{1212} = 0$$

Now, putting $h = r = 1$, $i = k = s = 2$ in (4), we have

$$4(B_{21} - B_{12})R_{12j2} = 0$$

that is,

$$R_{12j2} = 0 \text{ for all } j.$$

Putting $h = k = r = 1$, $i = s = 2$ in (4), we get

$$4(B_{21} - B_{12})R_{12j1} = 0,$$

that is,

$$R_{12j1} = 0 \text{ for all } j.$$

Now, putting $h = r = 1$, $i = s = 2$ in (4), we get

$$2(B_{21} - B_{12})R_{12jh} = 0.$$

that is,

$$R_{12jh} = 0 \text{ for all } j \text{ and all } h.$$

Further, putting $h = r = 1$, $s = 2$ in (4) we obtain

$$(B_{21} - B_{12})R_{1ijk} + (B_{2j} - B_{j2})R_{1i1k} + (B_{2k} - B_{k2})R_{1ij1} + (B_{j1} - B_{1j})R_{1i2k} + (B_{k1} - B_{1k})R_{1ij2} = 0, \tag{5}$$

from which, for $j = 1$, we get

$$R_{1i1k} = 0,$$

and for $j = 2$,

$$R_{1i2k} = 0.$$

Thus (5) reduces to

$$(B_{21} - B_{12})R_{1ijk} = 0,$$

that is,

$$R_{1ijk} = 0 \text{ for all } i, j, k.$$

Putting $h = s = 2$, $r = 1$ in (4), we get $R_{2i2k} = 0$ and $R_{2ijk} = 0$.

Finally, putting $h = 1$, $s = 2$ in (4) we get

$$(B_{21} - B_{12})R_{rijk} = 0$$

that is,

$$R_{rijk} = 0 \text{ for all } r, i, j, k.$$

Thus, if in (3) $\nabla_s A_r = \nabla_r A_s$, then $B_{ij} - B_{ji} = 0$ or the space is flat. Excluding this last possibility, we get $B_{ij} - B_{ji} = 0$, or $\nabla_i B_j = \nabla_j B_i$.

Now, let us suppose $\nabla_s B_r = \nabla_r B_s$. Then (3) reduces to

$$A_{sr}R_{hijk} + A_{ih}R_{jkrs} + A_{kj}R_{rshi} = 0, \tag{6}$$

where we have put

$$\nabla_s A_r - \nabla_r A_s = A_{sr}.$$

If $\nabla_s A_r \neq \nabla_r A_s$, then there exists at least one A_{rs}, say A_{12}, such that $A_{12} \neq 0$. Then, in the same way as before, we get $R_{ijhk} = 0$. Thus we obtain the following:

Theorem 1. *In a weakly symmetric manifold the associated 1-form A is closed if and only if the 1-form B is closed.*

Suppose the 1-form B is recurrent with the same vector of recurrence. That is, $\nabla_r B_s = B_r B_s$. Then obviously B is closed. Hence from Theorem 2.1 we get the 1-form A is also closed. Therefore from (2) we obtain

$$\nabla_r \nabla_s R_{hijk} - \nabla_s \nabla_r R_{hijk} = 0.$$

From the above discussion we obtain the following:

Theorem 2. *A weakly symmetric manifold reduces to a semisymmetric manifold provided the 1-form B is recurrent with the same vector of recurrence.*

Suppose now that the 1-form B is of the concircular form $\nabla_r B_s = B_r B_s + \gamma g_{rs}$, where γ is an arbitrary scalar function, then B is closed and again from Theorem 2.1 the 1-form A is also closed. From (2) we may infer:

$$(7)\quad (\nabla_s \nabla_r - \nabla_r \nabla_s) R_{hijk} = \gamma [g_{sh} R_{rijk} - g_{rh} R_{sijk} \\ + g_{is} R_{hrjk} - g_{ir} R_{hsjk} + g_{js} R_{hirk} - g_{jr} R_{hisk} + g_{ks} R_{hijr} - g_{kr} R_{hijs}].$$

We have thus obtained that the manifold is Pseudo-symmetric in the sense of Deszcz [8] and the following Theorem is true.

Theorem 3. *A weakly symmetric manifold having the 1-form B of the concircular form $\nabla_r B_s = B_r B_s + \gamma g_{rs}$ reduces to a Pseudo-symmetric manifold in the sense of Deszcz.*

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