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NONHOLONOMIC MANIFOLDS WITH BERWALD-MOOR METRIC

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Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. Distribution of codimension 1 with the Finsler metric type Berwald-Moor on a smooth five-dimensional manifold is considered. Interior connection associated with a given metric structure is determined.

1. Introduction

Commutative associative algebra (algebra polynumbers) consistent with the metric function Berwald-Moore (BM) is defined in a natural way in the n-dimensional vector space. The algebra of polynumbers P_n is a generalization of the algebra of double numbers. There is a basis $(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n)$ such that $\vec{e}_\alpha \vec{e}_\beta = \delta_{\alpha\beta}\vec{e}_\alpha$. If we consider the algebra polynumbers a smooth manifold X, then the corresponding Berwald-Moor metric is defined by function that is independent of the choice of the manifold X [6]. In general, the metric function BM is determined by a smooth field polyform. Apparently, such generalizations were considered for the first time in [2]. This article is a continuation of this work. The transition from algebra polynumbers to a manifold with a given field polyform in a way similar to the continuation of the special theory of relativity to general relativity. This article is a new step towards generalization of the geometry of spaces with metric BM. The ideas of Kaluza-Klein prompted researchers to build models of the physical space, based on the geometry of almost contact metric structures. The authors of [5] suggest that the space velocities of the particles is a four nonholonomic distribution on the manifold of higher dimension. This distribution is given 4-potential of the electromagnetic field. Equation of admissible (horizontal) geodesic for this distribution coincide with the equations of motion of a charged particle of the general theory of relativity. Metric tensor of the Lorentzian signature $(+,-,-,-)$ is defined

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on the distribution, which allows to determine causality, as in the general theory of relativity. The authors introduced the covariant derivative (linear connection) and the curvature tensor for distribution. However, connectivity in the distribution and its invariants have been studied by Professor Wagner [7]. Problem of constructing connection and its invariants with respect to the distribution of a Finsler metric considered in [3, 2]. In addition to the introduction paper is divided into three sections. The second section provides a summary of the work [1]. The third section discusses the concepts of interior connection. The fourth section provides an introduction to the geometry of the distribution of a Finsler metric BM.

2. A Berwald-Moor metric compatible with a poly-affine structure on a smooth manifold

Let X be a connected n-dimensional C^∞-manifold. All functions and geometric objects defined on X are assumed to be infinitely differentiable. For simplification, in what follows we call tensors fields simply be tensors. In the study of the spaces are generally used methods of Finsler geometry. Considering that the metric is based on multilinear form, we can use a linear covariant derivative and the corresponding differential invariants - curvature, torsion, etc. for learning spaces. In [1] on the manifold X with the Berwald-Moor metric, determined multilinear form g, was set poliaffinornaya algebra with affinors $(\varphi_1, \varphi_2, \dots, \varphi_n)$. Both object (polyform and algebra) were determined so that under certain conditions the space $(X, g, \varphi_1, \varphi_2, \dots, \varphi_n)$ would be reduced to the already known polynumbers space. In the future, triple $(X, g, \varphi_1, \varphi_2, \dots, \varphi_n)$ is called a manifold Berwald-Moore (BM). Affinors algebra is defined as follows [1]. On the manifold X-dimensional distributions of the field D_α is defined such that $TX = \oplus_{\alpha=1}^n D_\alpha$. A nonzero algebraic metric is called a Berwald-Moor metric if there is a field of bases $(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n)$ such that each basis vector $\vec{e}_\alpha$ defines a zero direction of g [1] and generates a corresponding one-dimensional distributions: $D_\alpha = \operatorname{span}(\vec{e}_\alpha)$. Such a field of bases is called an adapted field of bases of the form g, or simply an adapted basis. The form g has exactly one up to the permutation of indices non-zero component $g_{12\dots n}$ with respect to an adapted basis.

Consider the n distributions $D_{\hat{\alpha}}$ defined as follows:

$$D_{\hat{\alpha}} = D_1 \oplus \cdots D_{\alpha-1} \oplus D_{\alpha+1} \cdots \oplus D_n.$$

The tangent bundle can be decomposed (into the direct sum) as follows:

$$TX = D_{\hat{\alpha}} \oplus D_\alpha \tag{1}$$

Direct sum decomposition (1) defines the projector $\varphi_\alpha \colon TX \to D_\alpha$.

The set of the projectors φ_α with respect to the operation of composition is the n-dimensional algebra AH_n isomorphic to the algebra of polynumbers P_n. We say that such algebra AH_n is compatible with the metric g.

In the adapted basis the affinor φ_α has the form:

$$\begin{pmatrix} \ddots & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & \ddots \end{pmatrix}.$$

If there exists an atlas on X consisting of maps adapted to the metric g, i.e. defining an adapted basis: $\partial_\alpha = \vec{e}_\alpha$, then the algebra AH_n is integrable. In this case the manifold X can be regarded as a manifold over the algebra of polynumbers P_n [4].

Theorem 1 ([1])**.** *There exists a unique linear torsion-free connection compatible with the metric BM g on the manifold X whose coefficients are given by*

$$\Gamma^\alpha_{\alpha\alpha} = \frac{\partial_\alpha g_{12\ldots n}}{g_{12\ldots n}}. \tag{2}$$

(No summation over α in equation (2).)

A Tensor structure on a smooth manifold is a set of tensor fields. Thus, we consider the tensor structure which includes a poly-affinor structure compatible with a poly-linear form.

If there is an atlas on the manifold such that every tensor of the structure has constant components in any chart of this atlas, then the tensor structure is called integrable.

In his paper [4], Kruchkovich G.I. formulates the following proposition: "If the tensor structure admits a compatible torsion-free connection of zero curvature, then such a structure is integrable. Every integrable tensor structure admits a compatible torsion-free connection of zero curvature, at least locally."

Kruchkovich's proposition implies the following theorem:

Theorem 2 ([1])**.** *A tensor structure $(\varphi_1, \varphi_2, \ldots, \varphi_n, g)$ is integrable if and only if the curvature tensor of connection (2) is equal to zero.*

Using the expression in the coordinates of the curvature tensor R of the connection ∇, we find that the only nonzero components of the tensor R are

$$R^\gamma_{\alpha\gamma\gamma} = \partial_\alpha \frac{\partial_\gamma g_{12\ldots n}}{g_{12\ldots n}}$$

($\alpha \neq \gamma$, no summation over γ).

3. Interior connection

Let X be a smooth manifold of an odd dimension n, $n \geq 3$. Denote by $\Xi(X)$ the $C^\infty(X)$-module of smooth vector fields on X. All manifolds, tensors and other geometric objects will be assumed to be smooth of the class C^∞. An almost contact metric structure on X is an aggregate $(\varphi, \vec{\xi}, \eta, g)$ of tensor fields on X, where φ is a tensor field of type $(1, 1)$, which is called the structure endomorphism, $\vec{\xi}$ and η are a vector and a covector, which are called the structure vector and the contact form, respectively, and g is a (pseudo-)Riemannian metric. Moreover,

$$\eta(\vec{\xi}) = 1, \quad \varphi(\vec{\xi}) = 0, \quad \eta \circ \varphi = 0,$$

$$\varphi^2 \vec{X} = -\vec{X} + \eta(\vec{X})\vec{\xi}, \quad g(\varphi\vec{X}, \varphi\vec{Y}) = g(\vec{X}, \vec{Y}) - \eta(\vec{X})\eta(\vec{Y})$$

for all $\vec{X}, \vec{Y} \in \Xi(X)$. The skew-symmetric tensor $\Omega(\vec{X}, \vec{Y}) = g(\vec{X}\varphi\vec{Y})$ is called the fundamental tensor of the structure. A manifold with a fixed almost contact metric structure is called an almost contact metric manifold.

We say that a coordinate map $K(x^\alpha)$ $(\alpha, \beta, \gamma = 1, \dots, n)$, $(a, b, c, e = 1, \dots, n-1)$ on a manifold X is adapted to the non-holonomic manifold D if

$$D^\perp = \operatorname{span}\left(\frac{\partial}{\partial x^n}\right)$$

holds [6].

Let $P\colon TX \to D$ be the projection map defined by the decomposition $TX = D \oplus D^\perp$ and let $K(x^\alpha)$ be an adapted coordinate map. Vector fields

$$P(\partial_a) = \vec{e}_a = \partial_a - \Gamma^n_a \partial_n$$

are linearly independent, and linearly generate the system D over the domain of the definition of the coordinate map:

$$D = \operatorname{span}(\vec{e}_a).$$

Thus we have on X the non-holonomic field of bases $(\vec{e}_a, \partial_n)$ and the corresponding field of cobases

$$(dx^a, \theta^n = dx^n + \Gamma^n_a dx^a).$$

It can be checked directly that

$$[\vec{e}_a, \vec{e}_b] = M^n_{ab}\partial_n,$$

where the components M^n_{ab} form the so-called tensor of non-holonomicity [5]. Under assumption that for all adapted coordinate systems it holds $\vec{\xi} = \partial_n$, the following equality takes place

$$[\vec{e}_a, \vec{e}_b] = 2\omega_{ba}\partial_n,$$

where $\omega = d\eta$. We say also that the basis

$$\vec{e}_a = \partial_a - \Gamma^n_a \partial_n$$

is adapted, as the basis defined by an adapted coordinate map. Note that $\partial_n \Gamma^n_a = 0$.

We call a tensor field defined on an almost contact metric manifold admissible (to the distribution D) if it vanishes whenever its vectorial argument belongs to the closing $D^\perp$ and its covectorial argument is proportional to the form η. The coordinate form of an admissible tensor field of type (p, q) in an adapted coordinate map looks like

$$t = t^{a_1,\dots,a_p}_{b_1,\dots,b_q} \vec{e}_{a_1} \otimes \dots \otimes \vec{e}_{a_p} \otimes dx^{b_1} \otimes \dots \otimes dx^{b_q}.$$

In particular, an admissible vector field is a vector field tangent to the distribution D, and an admissible 1-form is a 1-form that is zero on the closing $D^\perp$.

We call an admissible tensor field *integrable* if there is an open neighborhood of each point of the manifold X and admissible coordinates on it such that the components of the tensor fields are constant with respect to these coordinates. The form $\omega = d\eta$ is an example of an admissible tensor structure. If the distribution D is integrable, then any admissible integrable structure is an integrable structure on the manifold X. The following facts show that the notion of an integrable admissible tensor structure is natural. As it is known, the integrable closing $D^\perp$ defines a foliation with one-dimensional lives. If one defines on this foliation a structure of a smooth manifold, then that any integrable tensor structure defines on this manifold an integrable tensor structure in the usual sense.

An intrinsic linear connection on a non-holonomic manifold D is defined in [5] as a map

$$\nabla \colon \Gamma D \times \Gamma D \to \Gamma D$$

that satisfies the following conditions:

$$\begin{aligned} &1) \quad \nabla_{f_1\vec{u}_1 + f_2\vec{u}_2} = f_1 \nabla_{\vec{u}_1} + f_2 \nabla_{\vec{u}_2};\\ &2) \quad \nabla_{\vec{u}} f\vec{v} = f\nabla_{\vec{u}}\vec{v} + (\vec{u}f)\vec{v}, \end{aligned}$$

where ΓD is the module of admissible vector fields. The Christoffel symbols are defined by the relation

$$\nabla_{\vec{e}_a}\vec{e}_b = \Gamma^c_{ab}\vec{e}_c.$$

The torsion S of the intrinsic linear connection is defined by the formula

$$S(\vec{X}, \vec{Y}) = \nabla_{\vec{X}}\vec{Y} - \nabla_{\vec{Y}}\vec{X} - p[\vec{X}, \vec{Y}].$$

Thus with respect to an adapted coordinate system it holds

$$S^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba}.$$

In the same way as a linear connection on a smooth manifold, an intrinsic connection can be defined by giving a horizontal distribution over a total space of some vector bundle. The role of such bundle plays the distribution D.

In order to define a connection over the distribution D, it is necessary first to introduce a structure of a smooth manifold on D. This structure is defined in the following way. To each adapted coordinate map $K(x^\alpha)$ on the manifold X we put in correspondence the coordinate map $\tilde{K}(x^\alpha, x^{n+\alpha})$ on the manifold D, where $x^{n+\alpha}$ are the coordinates of an admissible vector with respect to the basis $\vec{e}_a = \partial_a - \Gamma^n_a \partial_n$.

One says that over a distribution D a connection is given if the distribution $\tilde{D} = \pi_*^{-1}(D)$, where $\pi\colon D \to X$ is the natural projection, can be decomposed into a direct some of the form

$$\tilde{D} = HD \oplus VD,$$

where VD is the vertical distribution on the total space D. Thus the assignment of a connection over a distribution is equivalent to the assignment of the object $G^a_b(x^a, x^{n+a})$ such that

$$HD = \operatorname{span}(\vec{\varepsilon}_a),$$

where $\vec{\varepsilon}_a = \partial_a - \Gamma^n_a \partial_n - G^b_a \partial_{n+b}$.

The extended connection can be obtained from an intrinsic one by the equality

$$TD = \widetilde{HD} \oplus VD, \qquad HD \subset \widetilde{HD}.$$

Essentially, the extended connection is a connection in a vector bundle. For its assignment (under the condition that a connection on the distribution is already defined) it is enough to define a vector field on the manifold D that has the following coordinate form: $\vec{u} = \partial_n - G^a_n \partial_{n+a}$. The components of the object G^a_n are transformed as the components of a vector on the base. Setting $G^a_n = 0$, we get an extended connection denoted by ∇^1. The admissible tensor field

$$R(\vec{u}, \vec{v})\vec{w} = \nabla_{\vec{u}} \nabla_{\vec{v}} \vec{w} - \nabla_{\vec{v}} \nabla_{\vec{u}} \vec{w} - \nabla_{p[\vec{u},\vec{v}]} \vec{w} - p[q[\vec{u}, \vec{v}], \vec{w}],$$

where $q = 1-p$, is called by Wagner the first Schouten curvature tensor. With respect to the adapted coordinates it holds

$$R^a_{bcd} = 2\vec{e}_{[a}\Gamma^d_{b]c} + 2\Gamma^d_{[a||e||}\Gamma^e_{b]c}.$$

Suppose now that on the manifold D is defined a function $L(x^\alpha, x^{n+a})$ that satisfies the following conditions:

1) L is smooth at least on $D\backslash\{0\}$;
2) L is homogeneous of degree 1 with respect to the coordinates of an admissible vector, i.e. $L(x^\alpha, \lambda x^{n+a}) = \lambda L(x^\alpha, x^{n+a}), \quad \lambda > 0$;
3) $L(x^\alpha, x^{n+a})$ is positive if not all x^{n+a} are zero simultaneously;
4) the quadric form

$$L^2_{\cdot a \cdot b}\xi^a \xi^b = \frac{\partial^2 L^2}{\partial x^{n+a} \partial x^{n+b}} \xi^a \xi^b$$

is positive definite.

We call the triple (X, D, F), where $F = L^2$, a contact sub-Finslerian manifold.

If the pair $(D,\, L)$ defined on the manifold X, then D, an intrinsic connection generated by the distribution

$$HD = \operatorname{span}(\vec{\varepsilon}_\alpha),$$

where

$$
\begin{aligned}
\vec{\varepsilon}_a &= \partial_a - \Gamma^n_a \partial_n - G^b_{ac} x^{n+c} \partial_{n+b}, \\
G^a_{bc} &= G^a_{\cdot b \cdot c} = \partial_{n+b}\partial_{n+c} G^a, \\
G^a &= \frac{1}{4} g^{ab} (\vec{e}_c L^2_{\cdot b} x^{n+c} - \vec{e}_b L^2), \\
g_{ab} &= \frac{1}{2} L^2_{\cdot a \cdot b}.
\end{aligned}
$$

4. Basic notions of geometry distribution with Finsler metric type Berwald-Moor

Suppose, now, X is a smooth manifold of dimension 5. As in the previous section, we assume that $TX = D \oplus D^{\perp}$ where D is distribution of codimension 1. Consider the case of $X = R^5$ and D distribution generated by the vector fields

$$\vec{e}_1 = \partial_1 - x^2 \partial_5, \quad \vec{e}_2 = \partial_2, \quad \vec{e}_3 = \partial_3 - x^4 \partial_5, \quad \vec{e}_4 = \partial_4.$$

We define a family of admissible affinors (φ_a) such that

$$\varphi_a(\vec{e}_b) = \begin{cases} \vec{0}, & a \neq b, \\ \vec{e}_a, & a = b. \end{cases}$$

Let g be a field of symmetric forms. The form g has exactly one up to the permutation of indices non-zero component $g_{1234} = g$ where $g_{1234} = g(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$. We call the quadruple $(X, D, g, \varphi_1, \dots, \varphi_4)$ a nonholonomic manifold of Berwald-Moor. We have

Theorem 3. *There exists a unique interior torsion-free connection ∇ such that $\nabla g = 0$.*

Proof. Uniqueness. For all a from $\nabla g = 0$ it follows that $\Gamma^b_{ab} = \frac{\vec{e}_a g_{1234}}{g_{1234}}$. If we demand that the torsion tensor vanishes, then nonzero components of this connection will only $\Gamma^a_{aa} = \frac{\vec{e}_a g_{1234}}{g_{1234}}$ (no summation over a). The existence of a connection directly verified. □

Making the necessary calculations, we obtain the following expression for the non-zero components of the curvature tensor Schouten $R^c_{acc} = \vec{e}_a \frac{\vec{e}_c g_{1234}}{g_{1234}}$ ($a \neq c$, no summation over c). The integrability condition of g is equivalent to the vanishing of the curvature tensor.

Equating to zero the right components, we get partial differential equations of the form $\partial_a \frac{\partial_c g_{1234}}{g_{1234}} = 0$, $(a \neq c)$.

Functions of the form $g_{1234} = e^{\alpha_1 x^1 + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4}$ are among the solutions of the system.

In fact, in the case of the vanishing of the curvature tensor, each point of X can be set in two coordinates. One of them - a real number, the other - polynumbers.

Geodesic equation admissible connection can be written as

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0,$$
$$\dot{x}^n = -\dot{x}^a\Gamma^n_a\partial_n.$$

Or, with the above calculations,

$$\ddot{x}^a + \frac{\vec{e}_a g_{1234}}{g_{1234}}(\dot{x}^a)^2 = 0,$$
$$\dot{x}^n = -\dot{x}^a\Gamma^n_a\partial_n.$$

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