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INTRINSIC GEOMETRY OF ALMOST CONTACT KÄHLERIAN MANIFOLDS

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Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. In this paper, the notion of an almost contact Kählerian structure is introduced. The intrinsic geometry of almost contact Kählerian spaces is investigated. On the zero-curvature distribution of an almost contact metric structure, as on the total space of a vector bundle, an almost contact Kählerian structure is obtained.

1. Introduction

Almost contact metric structures $(\varphi, \vec{\xi}, \eta, g)$ are odd-dimensional analogs of almost Hermitian structures. There are a lot of important interplays between these structures. The most of the works devoted to the investigation of the geometry of manifolds with almost contact metric structures, explicitly or not explicitly, either use these interplays or find their specifications. On the other hand, the presence of a smooth distribution D in the geometry of almost contact metric space gives the possibility to use the methods of the geometry of non-holonomic manifolds in the investigations of almost contact metric structures. Probably the possibility of the effective use of such approach to the investigation of almost contact metric spaces was stated for the first time in [5]. In the same time, the works, where per se an attempt to the attainment of the compromise on the way of the rapprochement of "holonomic" and "non-holonomic" points of view is done, appeared. An example of such works is [1]. The main result of [1] is the construction of a new linear connection ∇ on a contact metric space by using the Levi-Civita connection. The author of [1] called this connection a D-connection. This connection, in particular, satisfies the following property: a contact metric space is Sasakian if and only if $\nabla\varphi = 0$ [1, p. 1963]. The author of [1] writes: "As a conclusion we may say that the study of the contact distribution (D, φ, g) by using the D-connection ∇ is an

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alternative to the study of the contact metric manifold M via the Levi-Civita connection" [1, p. 1967]. The appearance of the contact distribution (D, φ, g) indicates the attempt to use the methods of the non-holonomic geometry for the investigation of almost contact metric structures. In the present paper we use the intrinsic connection introduced by Wagner [9] in order to investigate the almost contact metric structures. We develop the notion of the intrinsic connection (we call it a connection over a distribution), we introduce the notion of the extended connection and generalize these notions to the connections of the Finslerian type [4]. We show that if ∇ is an intrinsic metric connection and ∇^1 is the corresponding extended connection, then the following statement holds: an almost contact metric space is an almost contact Kählerian space if and only if $L_{\vec{\xi}}g = 0$ and $\nabla^1\varphi = 0$. The last statement is a theorem of the proper non-holonomic geometry.

Let $(\varphi, \vec{\xi}, \eta, g)$ be an almost contact metric structure (the main theses of the theory of almost contact metric structures can be found in the excellent books [7, 2]). By definition, an almost contact metric structure is Sasakian if it is normal, i.e.

$$N_\varphi + 2d\eta \otimes \vec{\xi} = 0,$$

where N_φ is the Nijenhuis torsion defined for the tensor φ and it holds $\Omega = d\eta$, where $\Omega(\vec{x}, \vec{y}) = g(\vec{x}, \varphi\vec{y})$ is the fundamental form of the structure. Thus with an almost contact metric space we associate two 2-forms, $\omega = d\eta$ and Ω. If these forms are equal, we get a contact metric space, which are more simple as the more general contact metric spaces. We will get a space with an almost contact Hermitian structure if we refuse the condition $\Omega = d\eta$, and the condition $N_\varphi + 2d\eta \otimes \vec{\xi} = 0$ change to the weaker one

$$N_\varphi + 2(d\eta \circ \varphi) \otimes \vec{\xi} = 0.$$

We also do not assume that the equality $\omega(\varphi\vec{x}, \varphi\vec{y}) = \omega(\vec{x}, \vec{y})$ holds. Almost contact Hermitian spaces preserve many important properties of Sasakian spaces and they remain to be analogs of Hermitian spaces. If we make some natural assumption about an almost contact Hermitian space, then we get an almost contact Káhlerian space, these spaces are analogs of Kählerian spaces.

Following the ideology developed in the works of Schouten and Wagner, we define the intrinsic geometry of an almost contact metric space X as the aggregate of the properties that possess the following objects: a smooth distribution D defined by a contact form η; an admissible field of endomorphisms φ of D (which we call an admissible almost complex structure) satisfying $\varphi^2 = -1$; an admissible Riemannian metric field g that is related to φ by $g(\varphi\vec{x}, \varphi\vec{y}) = g(\vec{x}, \vec{y})$, where $\vec{x}$ and $\vec{y}$ are admissible vector fields. To the objects of the intrinsic geometry of an almost contact metric space one should ascribe also the objects derived from the just mentioned: the 2-form $\omega = d\eta$; the vector field $\vec{\xi}$ (which is called the Reeb vector field) defining the closing $D^\perp$ of D, i.e. $\vec{\xi} \in D^\perp$, and given by the equalities $\eta(\vec{\xi}) = 1$, $\ker\omega = \mathrm{span}(\vec{\xi})$ in the case when the 2-form ω is of maximal rank; the intrinsic connection ∇ that defines the parallel transport of admissible vectors along admissible curves and is defined by the metric g; the connection ∇^1 that is a natural extension of the connection ∇ which accomplishes the parallel transport of admissible vectors along arbitrary curves of the manifold X.

Besides the introduction, the paper contains 4 sections. In Section 2 we introduce the notion of an almost contact Kählerian structure. In Section 3 we discuss connections over a distribution and the extended connections. The notion of the connection over a distribution was known before (see e.g. [10]). The connection over a distribution was used in the geometry of contact structures in [4, 3]. In these works also the notion of the extended connection was used. Per se, the extended connection was defined for the first time by Wagner in a little bit another context and in another terms in [9] in order to construct the curvature tensor of a non-holonomic manifold. In Section 4 we give the main result about the intrinsic geometry of almost contact Kählerian spaces. This section contains the main result of the paper: an almost contact Hermitian structure is an almost contact Kählerian structure if and only if $L_{\vec{\xi}}g = 0$ and $\nabla^1\varphi = 0$, where ∇^1 is the intrinsic metric torsion-free connection. The last section contains an example of an almost contact Kählerian space that is not a Sasakian space.

2. Almost contact Kählerian structure

Let X be a smooth manifold of an odd dimension n, $n \geq 3$. Denote by $\Xi(X)$ the $C^\infty(X)$-module of smooth vector fields on X. All manifolds, tensors and other geometric objects will be assumed to be smooth of the class C^∞. An almost contact metric structure on X is an aggregate $(\varphi, \vec{\xi}, \eta, g)$ of tensor fields on X, where φ is a tensor field of type $(1,1)$, which is called the structure endomorphism, $\vec{\xi}$ and η are a vector and a covector, which are called the structure vector and the contact form, respectively, and g is a (pseudo-)Riemannian metric. Moreover,

$$\eta(\vec{\xi}) = 1, \quad \varphi(\vec{\xi}) = 0, \quad \eta \circ \varphi = 0,$$
$$\varphi^2\vec{x} = -\vec{x} + \eta(\vec{x})\vec{\xi}, \quad g(\varphi\vec{x}, \varphi\vec{y}) = g(\vec{x}, \vec{y}) - \eta(\vec{x})\eta(\vec{y})$$

for all $\vec{x}, \vec{y} \in \Xi(X)$. The skew-symmetric tensor $\Omega(\vec{x}, \vec{y}) = g(\vec{x}, \varphi\vec{y})$ is called the fundamental tensor of the structure. A manifold with a fixed almost contact metric structure is called an almost contact metric manifold. If $\Omega = d\eta$ holds, then the almost contact metric structure is called a contact metric structure. An almost contact metric structure is called normal if

$$N_\varphi + 2d\eta \otimes \vec{\xi} = 0,$$

where N_φ is the Nijenhuis torsion defined for the tensor φ. A normal contact metric structure is called a Sasakian structure. A manifold with a given

Sasakian structure is called a Sasakian manifold. Let D be the smooth distribution of codimension 1 defined by the form η, and $D^{\perp} = \operatorname{span}(\vec{\xi})$ be the closing of D. If the restriction of the 2-form $\omega = d\eta$ to the distribution D is non-degenerate, then the vector $\vec{\xi}$ is uniquely defined by the condition

$$\eta(\vec{\xi}) = 1, \quad \ker \omega = \operatorname{span}(\vec{\xi}),$$

and it is called the Reeb vector field.

We say that an almost contact metric structure is almost normal, if it holds

$$N_{\varphi} + 2(d\eta \circ \varphi) \otimes \vec{\xi} = 0. \tag{1}$$

In what follows, an almost normal almost contact metric space will be called *an almost contact Hermitian space*. An almost contact Hermitian space is called *an almost contact Kählerian space*, if its fundamental form is closed. The following obvious theorem shows the difference between a normal almost contact metric structure and an almost contact Hermitian structure.

Theorem 1. *An almost contact Hermitian structure is normal if and only if it holds*

$$\omega(\varphi\vec{u}, \varphi\vec{v}) = \omega(\vec{u}, \vec{v}), \quad \vec{u}, \vec{v} \in \Gamma D.$$

It is obvious that an almost normal contact metric structure is a Sasakian structure. Sasakian manifolds are popular among the researchers of almost contact metric spaces by the following two reasons. On one hand, there exist a big number of interesting and deep examples of Sasakian structures (see e.g. [2]), on the other hand, the Sasakian manifolds have very important and natural properties.

We say that a coordinate map $K(x^{\alpha})$ $(\alpha, \beta, \gamma = 1, \ldots, n)$ $(a, b, c, e = 1, \ldots, n-1)$ on a manifold X is adapted to the non-holonomic manifold D if

$$D^{\perp} = \operatorname{span}\left(\frac{\partial}{\partial x^n}\right)$$

holds [5].

Let $P : TX \to D$ be the projection map defined by the decomposition $TX = D \oplus D^{\perp}$ and let $K(x^{\alpha})$ be an adapted coordinate map. Vector fields

$$P(\partial_a) = \vec{e}_a = \partial_a - \Gamma^n_a \partial_n$$

are linearly independent, and linearly generate the system D over the domain of the definition of the coordinate map:

$$D = \operatorname{span}(\vec{e}_a).$$

Thus we have on X the non-holonomic field of bases $(\vec{e}_a, \partial_n)$ and the corresponding field of cobases

$$(dx^a, \theta^n = dx^n + \Gamma^n_a dx^a).$$

It can be checked directly that

$$[\vec{e}_a, \vec{e}_b] = M^n_{ab} \partial_n,$$

where the components M^n_{ab} form the so-called tensor of non-holonomicity [9]. Under assumption that for all adapted coordinate systems it holds $\vec{\xi} = \partial_n$, the following equality takes place

$$[\vec{e}_a, \vec{e}_b] = 2\omega_{ba}\partial_n,$$

where $\omega = d\eta$. We say also that the basis

$$\vec{e}_a = \partial_a - \Gamma^n_a \partial_n$$

is adapted, as the basis defined by an adapted coordinate map. Note that $\partial_n \Gamma^n_a = 0$.

We call a tensor field defined on an almost contact metric manifold admissible (to the distribution D) if it vanishes whenever its vectorial argument belongs to the closing $D^\perp$ and its covectorial argument is proportional to the form η. The coordinate form of an admissible tensor field of type (p, q) in an adapted coordinate map looks like

$$t = t^{a_1,\ldots,a_p}_{b_1,\ldots,b_q} \vec{e}_{a_1} \otimes \cdots \otimes \vec{e}_{a_p} \otimes dx^{b_1} \otimes \cdots \otimes dx^{b_q}.$$

In particular, an admissible vector field is a vector field tangent to the distribution D, and an admissible 1-form is a 1-form that is zero on the closing $D^\perp$. It is clear that any tensor structure defined on the manifold X defines on it a unique admissible tensor structure of the same type. From the definition of an almost contact structure it follows that the field of endomorphisms φ is an admissible tensor field of type $(1, 1)$. The field of endomorphisms φ we call an admissible almost complex structure, taking into the account its properties. The 2-form $\omega = d\eta$ is also an admissible tensor field and it is natural to call it an admissible symplectic form.

All constructions done by Wagner in [9] are grounded on the usage of adapted coordinates (Wagner called such coordinates by gradient coordinates). Adapted coordinates are used in the foliation theory [6]. It seems that in the theory of almost contact metric spaces, the adapted coordinates were used in essence only in the works [5, 1, 4].

One of the main notions of this work is the notion of an admissible integrable tensor structure. In the definition of an admissible integrable tensor structure, the words "integrable" and "admissible" should be consider in the semantic union. We call an admissible tensor field *integrable* if there is an open neighborhood of each point of the manifold X and admissible coordinates on it such that the components of the tensor fields are constant with respect to these coordinates. The form $\omega = d\eta$ is an example of an admissible tensor structure. If the distribution D is integrable, then any admissible integrable structure is an integrable structure on the manifold X. The following facts show that the notion of an integrable admissible tensor structure is natural. As it is known, the integrable closing $D^\perp$ defines a foliation with one-dimensional lives. If one defines on this foliation a structure of a smooth manifold, then that any integrable tensor structure defines on this manifold an integrable tensor structure in the usual sense. Below we consider some important ideas of the development of the notion of the integrability in the geometry of almost contact metric structures. The following two theorems show the meaning of the notion of an integrable tensor structure in the context of our investigations.

Theorem 2. *The admissible almost complex structure φ is integrable if and only if $P(N_\varphi)=0$ holds.*

Proof. The expression of the non-zero components of the Nijenhuis torsion tensor

$$N_\varphi(\vec{x},\vec{y}) = [\varphi\vec{x},\varphi\vec{y}] + \varphi^2[\vec{x},\vec{y}] - \varphi[\varphi\vec{x},\vec{y}] - \varphi[\vec{x},\varphi\vec{y}]$$

of the tensor φ in adapted coordinates has the form:

$$N^e_{ab} = \varphi^c_a\vec{e}_c\varphi^e_b - \varphi^c_b\vec{e}_c\varphi^e_a + \varphi^e_c\vec{e}_b\varphi^c_a - \varphi^e_d\vec{e}_a\varphi^d_b, \tag{2}$$

$$N^n_{ab} = 2\varphi^c_a\varphi^d_b\omega_{dc}, \tag{3}$$

$$N^e_{na} = -\varphi^e_c\partial_n\varphi^c_a. \tag{4}$$

Thus the equality $P(N_\varphi)=0$ is equivalent to the condition that (2) and (4) are zero.

Conversely, suppose that $P(N_\varphi)=0$. Consider a sufficiently small neighborhood U of an arbitrary point of the manifold X. Assume that $U=U_1\times U_2$, $TU=\operatorname{span}(\partial_a)\oplus\operatorname{span}(\partial_n)$. We set the natural denotation $T(U_1)=\operatorname{span}(\partial_a)$. We define over the set U the isomorphism of bundles $\psi: D\to T(U_1)$ by the formula $\psi(\vec{e}_a)=\partial_a$. This endomorphism induces an almost complex structure on the manifold U_1. This complex structure is integrable due to the equality $P(N_\varphi)=0$. Indeed, from (4) it follows that the right hand side part of (2) coincides with the torsion of the almost complex structure induced on the manifold U_1. Choosing an appropriate coordinate system on U_1, and consequently, an appropriate adapted coordinate system on the manifold X, we get a coordinate map with respect to that the components of the endomorphism field φ are constant. □

Theorem 3. *An almost contact metric structure is is an almost contact Hermitian structure if and only if the admissible almost complex structure φ is integrable.*

Proof. Equality (1) written in adapted coordinates is equivalent to the condition that the right hand sides of (2) and (4) are zero. This and Theorem 2 gives the proof of the theorem. □

Note that we have de facto proved the equality

$$P(N_\varphi) = N_\varphi + 2(d\eta\circ\varphi)\otimes\vec{\xi}.$$

Using adapted coordinates we introduce the following admissible tensor fields:

$$h^a_b = \frac{1}{2}\partial_n\varphi^a_b,\quad C_{ab}=\frac{1}{2}\partial_n g_{ab},\quad C^a_b = g^{da}C_{db},\quad \psi^b_a = g^{db}\omega_{da}.$$

We denote by $\tilde{\nabla}$ and $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ the Levi-Civita connection and the Christoffel symbols of the metric g. The proof of the following theorem follows from direct computations.

Theorem 4. *The Christoffel symbols of the Levi-Civita connection of an almost contact metric space with respect to adapted coordinates are the following:*

$$\tilde{\Gamma}^c_{ab} = \Gamma^c_{ab}, \quad \tilde{\Gamma}^n_{ab} = \omega_{ba} - C_{ab}, \quad \tilde{\Gamma}^b_{an} = \tilde{\Gamma}^b_{na} = C^b_a - \psi^b_a, \quad \tilde{\Gamma}^n_{na} = \tilde{\Gamma}^a_{nn} = 0,$$

where

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\vec{e}_b g_{cd} - \vec{e}_c g_{bd} - \vec{e}_d g_{bc}).$$

In the case of a contact metric space the Christoffel symbols of the Levi-Civita connection are found in [1].

3. Connection over a distribution. The extended connection

An intrinsic linear connection on a non-holonomic manifold D is defined in [9] as a map

$$\nabla : \Gamma D \times \Gamma D \to \Gamma D$$

that satisfies the following conditions:

$$1) \quad \nabla_{f_1 \vec{u}_1 + f_2 \vec{u}_2} = f_1 \nabla_{\vec{u}_1} + f_2 \nabla_{\vec{u}_2};$$
$$2) \quad \nabla_{\vec{u}} f \vec{v} = f \nabla_{\vec{u}} \vec{v} + (\vec{u} f) \vec{v},$$

where ΓD is the module of admissible vector fields. The Christoffel symbols are defined by the relation

$$\nabla_{\vec{e}_a} \vec{e}_b = \Gamma^c_{ab} \vec{e}_c.$$

The torsion S of the intrinsic linear connection is defined by the formula

$$S(\vec{x}, \vec{y}) = \nabla_{\vec{x}} \vec{y} - \nabla_{\vec{y}} \vec{x} - p[\vec{x}, \vec{y}].$$

Thus with respect to an adapted coordinate system it holds

$$S^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba}.$$

The action of an intrinsic linear connection can be extended in a natural way to admissible tensor fields. An important example of an intrinsic linear connection is the intrinsic metric connection that is uniquely defined by the conditions $\nabla g = 0$ and $S = 0$ [4]. With respect to the adapted coordinates it holds

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\vec{e}_b g_{cd} - \vec{e}_c g_{bd} - \vec{e}_d g_{bc}). \tag{5}$$

In the same way as a linear connection on a smooth manifold, an intrinsic connection can be defined by giving a horizontal distribution over the total space of some vector bundle. The role of such bundle plays the distribution D. The notion of *a connection over a distribution* was applied to non-holonomic manifolds with admissible Finsler metrics in [4, 3]. One says that over a distribution D a connection is given if the distribution $\tilde{D} = \pi_*^{-1}(D)$, where

$\pi : D \to X$ is the natural projection, can be decomposed into a direct some of the form

$$\tilde{D} = HD \oplus VD,$$

where VD is the vertical distribution on the total space D.

Let us introduce a structure of a smooth manifold on D. This structure is defined in the following way. To each adapted coordinate map $K(x^\alpha)$ on the manifold X we put in correspondence the coordinate map $\tilde{K}(x^\alpha, x^{n+\alpha})$ on the manifold D, where $x^{n+\alpha}$ are the coordinates of an admissible vector with respect to the basis

$$\vec{e}_a = \partial_a - \Gamma^n_a \partial_n.$$

The constructed over-coordinate map will be called adapted. Thus the assignment of a connection over a distribution is equivalent to the assignment of the object $G^a_b(X^a, X^{n+a})$ such that

$$HD = \operatorname{span}(\vec{\epsilon}_a),$$

where $\vec{\epsilon}_a = \partial_a - \Gamma^n_a \partial_n - G^b_a \partial_{n+b}$. If it holds

$$G^a_b(x^a, x^{n+a}) = \Gamma^a_{bc}(x^a) x^{n+c},$$

then the connection over the distribution D is defined by the intrinsic linear connection. In [4] the notion of the prolonged connection was introduced. The prolonged connection can be obtained from an intrinsic connection by the equality

$$TD = \tilde{HD} \oplus VD,$$

where $HD \subset \tilde{HD}$. Essentially, the prolonged connection is a connection in a vector bundle. As it follows from the definition of the extended connection, for its assignment (under the condition that a connection on the distribution is already defined) it is enough to define a vector field on the manifold D that has the following coordinate form: $\vec{u} = \partial_n - G^a_n \partial_{n+a}$. The components of the object G^a_n are transformed as the components of a vector on the base. Setting $G^a_n = 0$, we get an extended connection denoted by ∇^1. In [9] the admissible tensor field

$$R(\vec{u}, \vec{v})\vec{w} = \nabla_{\vec{u}} \nabla_{\vec{v}} \vec{w} - \nabla_{\vec{v}} \nabla_{\vec{u}} \vec{w} - \nabla_{p[\vec{u},\vec{v}]} \vec{w} - p[q[\vec{u}, \vec{v}], \vec{w}]$$

is called by Wagner the first Schouten curvature tensor. With respect to the adapted coordinates it holds

$$R^a_{bcd} = 2\vec{e}_{[a} \Gamma^d_{b]c} + 2\Gamma^d_{[a||e||} \Gamma^e_{b]c}.$$

If the distribution D does not contain any integrable subdistribution of dimension $n-2$, then the Schouten curvature tensor is zero if and only if the parallel transport of admissible vectors does not depend on the curve [9]. We say that the Schouten tensor is the curvature tensor of the distribution D. If this tensor is zero, we say that the distribution D is a zero-curvature distribution. Note that the partial derivatives are components of an admissible tensor field [9].

4. Properties of almost contact Kählerian spaces related to the usage of the connection over a distribution

Let $(\varphi, \vec{\xi}, \eta, g)$ be an almost contact metric structure. In [5], the following theorem is proved:

Theorem 5. *Let ∇ be a torsion-free intrinsic linear connection on an almost contact metric space X. Then there exists on X a connection with the torsion*

$$S(\vec{x}, \vec{y}) = \frac{1}{4} P\left(N_\varphi\right)(\vec{x}, \vec{y}), \quad \vec{x}, \vec{y} \in \Gamma D$$

and compatible with φ.

The following theorem is the corollary of Theorem 5.

Theorem 6. *An almost contact metric space admits a torsion-free intrinsic connection ∇ such that $\nabla^1\varphi = 0$ if and only if the admissible structure φ is integrable.*

Proof. Let ∇ be a torsion-free connection such that $\nabla^1\varphi = 0$. Applying this ∇ to the proof of Theorem 5, we get

$$S(\vec{x}, \vec{y}) = \frac{1}{4} P\left(N_\varphi\right)(\vec{x}, \vec{y}) = 0, \quad \vec{x}, \vec{y} \in \Gamma D.$$

Adding to this condition the equality $\partial_n \varphi^a_b = 0$, we get

$$P\left(N_\varphi\right)(\vec{x}, \vec{y}) = 0, \quad \vec{x}, \vec{y} \in TX.$$

By Theorem 2, this is equivalent to the integrability of φ. The converse statement is obvious. □

Theorem 7. *An almost contact metric structure is an almost contact Kählerian structure if and only if it holds*

$$L_{\vec{\xi}} g = 0, \quad \nabla^1 \varphi = 0,$$

where ∇ is the intrinsic torsion-free metric connection.

Proof. According [1], any almost contact metric space satisfies the following equality:

$$\begin{aligned} (6) \quad 2g((\widetilde{\nabla}_{\vec{x}}\varphi)\vec{y}, \vec{z}) &= 3d\Omega\left(\vec{x}, \varphi\vec{y}, \varphi\vec{z}\right) - 3d\Omega\left(\vec{x}, \vec{y}, \vec{z}\right) + g\left(N^{(1)}\left(\vec{y}, \vec{z}\right), \varphi\vec{x}\right) \\ &\quad + N^{(2)}\left(\vec{y}, \vec{z}\right)\eta\left(\vec{x}\right) + 2d\eta\left(\varphi\vec{y}, \vec{x}\right)\eta\left(\vec{z}\right) - 2d\eta\left(\varphi\vec{z}, \vec{x}\right)\eta(\vec{y}), \end{aligned}$$

where

$$N^{(1)} = N_\varphi + 2d\eta \otimes \vec{\xi}, \quad N^{(2)}(\vec{x}, \vec{y}) = \left(L_{\varphi\vec{x}}\eta\right)\vec{y} - \left(L_{\varphi y}\eta\right)\vec{x}.$$

Theorem 6 and the definition of an almost contact Kählerian structure allow us to assume in what follows that the almost contact metric structure $(\varphi, \vec{\xi}, \eta, g)$ is almost normal. In this case,

$$P(N_\varphi) = N_\varphi + 2(d\eta \circ \varphi) \otimes \vec{\xi} = 0.$$

Thus,

$$N^{(1)} = 2(d\eta \otimes \vec{\xi} - (d\eta \circ \varphi) \otimes \vec{\xi}),$$

and the equality (6) takes the simpler form:

$$(7)\quad 2g((\widetilde{\nabla}_{\vec{x}}\varphi)\vec{y},\vec{z}) = 3d\Omega\,(\vec{x},\varphi\vec{y},\varphi\vec{z}) - 3d\Omega\,(\vec{x},\vec{y},\vec{z}) + N^{(2)}\,(\vec{y},\vec{z})\,\eta\,(\vec{x}) + 2d\eta\,(\varphi\vec{y},\vec{x})\,\eta\,(\vec{z}) - 2d\eta\,(\varphi\vec{z},\vec{x})\,\eta(\vec{y}).$$

Sufficiency. Substituting to (7) $\vec{x} = \vec{e}_a$, $\vec{y} = \partial_n$, $\vec{z} = \vec{e}_c$, we get $d\Omega_{abc} = 0$. Together with the equality $d\Omega_{abn} = 0$ this implies $d\Omega = 0$.

Necessity. Suppose that $d\Omega = 0$. We may rewrite (7) in the form

$$(8)\quad 2g((\widetilde{\nabla}_{\vec{x}}\varphi)\vec{y},\vec{z}) = N^{(2)}\,(\vec{y},\vec{z})\,\eta\,(\vec{x}) + 2d\eta\,(\varphi\vec{y},\vec{x})\,\eta\,(\vec{z}) - 2d\eta\,(\varphi\vec{z},\vec{x})\,\eta(\vec{y}).$$

Substituting $\vec{x} = \vec{e}_a$, $\vec{y} = \vec{e}_b$, $\vec{z} = \vec{e}_c$ to (8), we get $\nabla_a\varphi^b_c = 0$. □

In the rest of this section we formulate and prove a theorem generalizing the following classical result [1]: an almost contact metric space is a Sasakian space if and only if the following equality holds:

$$(\widetilde{\nabla}_{\vec{x}}\varphi)\vec{y} = g(\vec{x},\vec{y})\vec{\xi} - \eta\,(\vec{y})\,\vec{x}.$$

Theorem 8. *An almost contact metric structure is an almost contact Kählerian structure if and only if it holds*

$$(9)\quad (\widetilde{\nabla}_{\vec{x}}\varphi)\vec{y} = d\eta\,(\varphi\vec{y},\vec{x})\,\vec{\xi} + \eta\,(\vec{y})\,(\varphi\circ\psi)\,(\vec{x}) - \eta\,(\vec{x})\,(\varphi\circ\psi - \psi\circ\varphi)\,\vec{y}.$$

Proof. The equality (9) is equivalent to the following conditions:

$$L_{\vec{\xi}}g = 0, \quad \nabla^1\varphi = 0.$$

Together with Theorem 7 this proves the theorem. □

5. Almost contact metric structures over a zero-curvature distribution

Consider the vector bundle (D,π,X), where D is the distribution of the contact metric structure $(\varphi,\vec{\xi},\eta,g)$. If the distribution D is a zero-curvature distribution and it does not contain any involutive subdistribution of dimension $n-2$, then the equality $P^a_{bc} = 0$ holds [9]. In what follows we assume that $n > 3$. On the total space D of the vector bundle under the consideration we define an almost contact metric structure $(\tilde{D},\tilde{g},J,d(\pi^*\circ\eta),D)$ by setting

$$\tilde{g}(\vec{\epsilon}_a,\vec{\epsilon}_b) = \tilde{g}(\partial_{n+a},\partial_{n+b}) = \tilde{g}(\vec{e}_a,\vec{e}_b), \quad \tilde{g}(\vec{\epsilon}_a,\partial_n) = \tilde{g}(\partial_{n+a},\partial_n) = 0,$$

$$J(\vec{\epsilon}_a) = \partial_{n+a}, \quad J(\partial_{n+a}) = -\vec{\epsilon}_a, \quad J(\partial_n) = 0,$$

$\tilde{D} = \pi^{-1}_*(D)$,

$$\tilde{D} = HD \oplus VD,$$

VD is the vertical distribution on the total space D, and HD is the horizontal space defined by the intrinsic linear connection. The vector fields

$$\vec{\epsilon}_a = \partial_a - \Gamma^n_a\partial_n - \Gamma^b_{ac}x^{n+c}\partial_{n+b}, \quad \partial_n, \quad \partial_{n+a}$$

define on D a non-holonomic field of bases, and the forms

$$dx^a, \quad \Theta^n = dx^n + \Gamma^n_a dx^a, \quad \Theta^{n+a} = dx^{n+a} + \Gamma^a_{bc} x^{n+b} dx^c$$

define the corresponding field of cobases. The vector field ∂_n is the Reeb vector field of the almost contact metric structure $(\tilde{D}, \tilde{g}, J, d(\pi^* \circ \eta))$.

Theorem 9. *The almost contact metric structure $(\tilde{D}, \tilde{g}, J, d(\pi^* \circ \eta))$ is an almost contact Hermitian structure if and only if the distribution D is a zero-curvature distribution.*

Proof. It is easy to check that the following holds:

$$[\vec{\varepsilon}_a, \vec{\varepsilon}_b] = 2\omega_{ba}\partial_n + R^e_{abc} x^{n+c} \partial_{n+e}, \tag{10}$$

$$[\vec{\varepsilon}_a, \partial_n] = x^{n+c} P^b_{ac} \partial_{n+b}, \tag{11}$$

$$[\vec{\varepsilon}_a, \partial_{n+b}] = \Gamma^c_{ab} \partial_{n+c}. \tag{12}$$

These equalities directly imply

$$\begin{aligned} N_J(\vec{\varepsilon}_a, \vec{\varepsilon}_b) &= -R^e_{abc} x^{n+c} \partial_{n+e}, \\ N_J(\partial_{n+a}, \partial_{n+b}) &= 2\omega_{ba}\partial_n + R^e_{abc} x^{n+c} \partial_{n+e}, \\ N_J(\vec{\varepsilon}_a, \partial_{n+b}) &= 0, \\ N_J(\vec{\varepsilon}_a, \partial_n) &= N_J(\partial_{n+a}, \partial_n) = -x^{n+c} P^b_{ac} \partial_{n+b}. \end{aligned}$$

These equalities yield the proof of the theorem. □

Let us show that the structure $(\tilde{D}, \tilde{g}, J, d(\pi^* \circ \eta))$ is not normal. It holds

$$N_J(\partial_{n+a}, \partial_{n+b}) + 2d\tilde{\eta}(\partial_{n+a}, \partial_{n+b})\,\partial_n = 2\omega_{ba}\partial_n + R^e_{abc} x^{n+c} \partial_{n+e}.$$

It is clear that this expression can not be zero.

Theorem 10. *The almost contact metric structure $(\tilde{D}, \tilde{g}, J, d(\pi^* \circ \eta))$ is an almost contact Kählerian structure if and only if $(\varphi, \vec{\xi}, \eta, g)$ is a Sasakian structure with the zero-curvature distribution.*

Proof. It can be checked directly that $d\Omega = 0$ if and only if $d\tilde{\Omega} = 0$, where $\tilde{\Omega}(\vec{x}, \vec{y}) = g(\vec{x}, J\vec{y})$. This proves the theorem. □

Almost contact metric spaces of zero intrinsic curvature appear in mechanics and physics.

Wagner [9, 8] paid a big attention to non-holonomic manifolds of zero curvature. In particular, in [8], Wagner defines a non-holonomic manifold of zero curvature that is a geometric model of a solid body under non-holonomic constrains.

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