Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
31 (2015), 97–106
www.emis.de/journals
ISSN 1786-0091

FINSLER–BERWALD SPACE WITH VERY SPECIAL RELATIVITY

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Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. The symmetry of space time is described by using the so called isometric group. The generators of isometric group are directly connected with the Killing vectors [18]. In this paper, we present an explicit connection between the symmetries in the VSR and isometric group of Finsler space. The Killing vectors in Finsler space are constructed in a systematic way. Further, the solutions of Killing equations are present explicitly in the isometric symmetry of Finsler spaces. The Killing vectors of Finsler-Berwald space are given and we proved that the 4-dimensional Finsler-Berwald space with constant curvature has 15 independent Killing vectors.

1. Introduction

Finsler geometry as a natural generalization of Riemannian geometry could provide new sight on modern physics. The model of gravity and cosmology based on Finsler geometry is in good agreement with the recent astronomical observations. In the past few years, two interesting theories were proposed for investigating the violation of Lorentz invariance. One is the so called doubly special relativity(DSR), see [1, 2, 3, 17, 18], another one is the very special relativity(VSR) developed by Cohen and Glashow [7]. This theory suggested that the exact symmetry group of nature may be isomorphic to a subgroup SIM(2) of the Poincare group. Also the SIM(2) group semi direct product with the spacetime translation group gives an 8-dimensional subgroup of the Poincare group called ISIM(2) [13].

Recently, physicists found that both two theories mentioned above are related to Finsler geometry. Girelli, Liberati and Sindoni [10] showed that the Modified Dispersion Relation(MDR) in DSR can be studied through the frame work of Finsler geometry. Very recently, the authors Xin Li, Zhe Chang and

2010 *Mathematics Subject Classification.* 53B40, 53C60, 58J60.
Key words and phrases. Finsler–Berwald space, very special relativity, Killing vectors, Lorentz violation, isometric group.

Xiaohuon Mo [16], have studied Isometric group of (α, β) type Finsler space and the symmetry of very special relativity. They also found that the Killing vectors of Finsler-Funk space and proved that the 4 dimensional Finsler-Funk space with constant curvature has just 6 independent Killing vectors.

Thus, the symmetry of Finslerian space time is important for further study. Therefore, the way of describing spacetime symmetry in a covariant language i.e., the symmetry should not depend on any particular choice of coordinate system, involves the concept of isometric transformation. In fact, the symmetry of space time is described by the so called isometric group. The generators of isometric group is directly connected with the Killing vectors [12]. In this paper, we use solutions of the Killing equation to find the symmetry of a class of Finslerian spacetime. In particular, the Killing vectors of Finsler-Berwald space are given and further we showed that the 4-dimensional Finsler-Berwald space with constant curvature has 15 independent Killing vectors.

2. Killing vectors in Riemannian space

In this section, we give a brief introduction of the Killing vectors in Riemannian space. The terminology and notation are referred as in[14]. For a given coordinate transformation $x \to \bar{x}$, the Riemannian metric $g_{ij}(x)$ is defined as

$$\bar{g}_{ij}(\bar{x}) = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j} g_{kl}(x). \tag{1}$$

Any transformation $x \to \bar{x}$ is called isometry if and only if the transformation of the metric $g_{ij}(x)$ satisfies

$$g_{ij}(\bar{x}) = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j} g_{kl}(x), \text{ i.e., } \bar{g}_{ij}(\bar{x}) = g_{ij}(\bar{x}). \tag{2}$$

It is easy to check that the isometric transformations do form a group. Now it is convenient to investigate the isometric transformation under the infinitesimal coordinate transformation

$$\bar{x}^i = x^i + \epsilon V^i, \tag{3}$$

where $|\, \epsilon \,| \ll 1$. To first order in $|\, \epsilon \,|$, the equation (2) reads

$$V^m \frac{\partial g_{ij}}{\partial x^m} + g_{mi}\frac{\partial V^m}{\partial x^j} + g_{mj}\frac{\partial V^m}{\partial x^i} = 0. \tag{4}$$

By making use of the covariant derivatives with respect to Riemannian connection, we can write the above equation as

$$V_{i|j} + V_{j|i} = 0, \tag{5}$$

where "|" denotes the covariant derivative. Any vector field V_i satisfies equation (5) is called Killing vector. Thus, the problem of finding all isometries of a given metric $g_{ij}(x)$ is equivalent to find the dimension of the linear space formed by Killing vectors.

Ricci identities in Riemann geometry can be written as

$$V_{k|i|j} - V_{k|j|i} = -V_l R^l_{kji}, \tag{6}$$

where R^l_{kji} is the Riemannian curvature tensor. And the first Bianchi identity for the Riemannian curvature tensor gives

$$R^l_{kji} + R^l_{jik} + R^l_{ikj} = 0. \tag{7}$$

From equations (6) and (7), we obtain

$$V_{k|i|j} = V_l R^l_{jik}. \tag{8}$$

Thus, all the derivatives of V_i will be determined by the linear combinations of V_i and $V_{i|j}$. Once the V_i and $V_{i|j}$ at an arbitrary point of Riemannian space is given, then V_i and $V_{i|j}$ at any other point is determined by integration of the system of ordinary differential equations. Therefore, the dimension of linear space formed by Killing vector can be at most $n(n+1)/2$ in n dimensional Riemannian space. If a metric admits that the maximum number $n(n+1)/2$ of Killing vectors, its Riemannian space must be homogeneous and isotropic. Such space is called maximally symmetry space.

The best example of maximally symmetry space is the Minkowskian space. The Killing equation (5) of a given Minkowskian metric $\eta_{ij}(x)$ reduces to

$$\frac{\partial V_i}{\partial x^j} + \frac{\partial V_j}{\partial x^i} = 0. \tag{9}$$

The solution of (9) is

$$V^i = Q^i_j x^j + C^i, \tag{10}$$

where $Q_{ij} = \eta_{ki} Q^k_j$ is an arbitrary constant skew symmetric matrix and C^i is an arbitrary constant vector. Thus, substituting the solution (10) into the coordinate transformation (3), we obtain

$$\bar{x}^i = (\delta^i_j + \epsilon Q^i_j) x^j + \epsilon C^i. \tag{11}$$

The term $\delta^i_j + \epsilon Q^i_j$ in the above equation is just the Lorentz transformation matrix and the term ϵC^i is related to the spacetime translation. Expanding the matrix $\delta^i_j + \epsilon Q^i_j$ and the vector ϵC^i near identity, we obtain the famous Poincare algebra.

Other two types of maximally symmetry spaces are spherical and hyperbolic case. Without loss of generality, we set its constant sectional curvature to be ± 1 for spherical and hyperbolic case respectively. The length element of both the case is given in a unified form as

$$ds^2 = \frac{\sqrt{(1+k(x.x))(dx.dx) - k(x.dx)^2}}{1+k(x.x)}, \tag{12}$$

where the "." denotes the inner product with respect to Minkowskian metric and $k = \pm 1$ for spherical and hyperbolic case respectively. The metric is given as

$$g_{ij} = \left(\frac{\eta_{ij}}{1 + k(x.x)} - k \frac{x_i x_j}{(1 + k(x.x))^2} \right), \tag{13}$$

where $x_i \equiv \eta_{ij} x^j$. The Christoffel symbols of the above length element is given as

$$\gamma^k_{ij} = -k \frac{x_i \delta^k_j + x_j \delta^k_i}{1 + k(x.x)}. \tag{14}$$

Thus, the Killing equation (5), now reads as

$$\frac{\partial V_i}{\partial x^j} + \frac{\partial V_j}{\partial x^i} + \frac{2k}{1 + k(x.x)}(x_i V_j + x_j V_i) = 0. \tag{15}$$

The solution of the above equation is

$$V^i = g^{ij} V_j = Q^i_j x^j + C^i + k(x.c)x^i, \tag{16}$$

where the index of Q and C are raised and lowered by Minkowskian metric μ^{ij} and its inverse matrix μ_{ij}.

3. Killing vectors in Finsler space

In this section, we derive the Killing vectors in Finsler space. Now, we introduce the Finsler structure.

Let M be an n-dimensional manifold, let $T_x M$ denote the tangent space at $x \in M$, and by TM the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_x M$. The natural projection $\Pi \colon TM \to M$ is given by $\pi(x, y) = x$.

A Finsler space on the manifold M is a function $F \colon TM \to [0, \infty)$ with the following properties:

(i) Regularity: F is C^∞ on the entire tangent bundle $TM \backslash 0$.
(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.
(iii) Strong convexity: The $n \times n$ Hessain matrix

$$g_{ij} = \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j}(1/2F^2), \tag{17}$$

is positive definite at every point of $TM \backslash 0$, where $TM \backslash 0$ denotes the tangent vector y is non-empty in the tangent bundle TM.

Like Riemannian space, we shall now, find the Killing vectors in the Finsler space, for this we should construct the isometric transformation of Finsler structure.

Let us consider the coordinate transformation (3) together with the corresponding transformation for y.

$$\bar{y}^i = y^i + \epsilon \frac{\partial V^i}{\partial x^j} y^j, \tag{18}$$

Under the coordinate transformation (3) and (18), to first order in $|\ \epsilon\ |$, we obtain the expansion of the Finsler structure,

$$\bar{F}(\bar{x},\bar{y}) = \bar{F}(x,y) + \epsilon V^i \frac{\partial F}{\partial x^i} + \epsilon y^j \frac{\partial V^i}{\partial x^j}\frac{\partial F}{\partial y^i}, \tag{19}$$

where $\bar{F}(\bar{x},\bar{y})$ should be equal to $F(x,y)$. Under the transformation (3) and (18), a Finsler structure is called isometry if and only if

$$F(x,y) = \bar{F}(x,y). \tag{20}$$

Then, deducing from (19), we obtain Killing equation $K_V(F)$ in Finsler space

$$K_V(F) = V^i \frac{\partial F}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j}\frac{\partial F}{\partial y^i} = 0. \tag{21}$$

Searching the Killing vectors for general Finsler manifold is a difficult task. Here, we give the Killing vectors for a class of Finsler space-(α,β) space with metric defining as in [4]

$$F = \alpha\phi(s),\ s = \frac{\beta}{\alpha}, \tag{22}$$

where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differential one form, and $\phi(s)$ is a smooth function. Then, the Killing equation (21) in (α,β) space is given as follows

$$\begin{aligned} 0 &= K_V(\alpha)\phi(s) + \alpha K_V(\phi(s)),\\ &= \left(\phi(s) - s\frac{\partial\phi(s)}{\partial s}\right)K_V(\alpha) + \frac{\partial\phi(s)}{\partial s}K_V(\beta). \end{aligned} \tag{23}$$

By making use of the Killing equation (21), we obtain

$$K_V(\alpha) = \frac{1}{2\alpha}(V_{i|j} + V_{j|i})y^iy^j, \tag{24}$$

$$K_V(\beta) = (V^i\frac{\partial b_j}{\partial x^i} + b_i\frac{\partial V_i}{\partial x^j})y^j, \tag{25}$$

where " | " denotes the covariant derivative with respect to the Riemannian metric α. The solutions of the Killing equation (23) have been expressed in three cases:

Case-1: The first one is

$$\phi(s) - s\frac{\partial\phi(s)}{\partial s} = 0 \text{ and } K_V(\beta) = 0, \tag{26}$$

which implies $F = \lambda\beta$ for all $\lambda \in R$.

Case-2: If

$$\frac{\partial\phi(s)}{\partial s} = 0 \text{ and } K_V(\alpha) = 0, \tag{27}$$

which implies $F = \lambda\alpha$ for all $\lambda \in R$. The above two cases hold true for any trivial space. Next, we merely consider the following case:

Case-3: If $\phi(s) - s\frac{\partial\phi(s)}{\partial s} \neq 0$ and $\frac{\partial\phi(s)}{\partial s} \neq 0$, then we have solutions

$$V_{i|j} + V_{j|i} = 0, \tag{28}$$

$$V^i\frac{\partial b_j}{\partial x^j} + b_i\frac{\partial V_i}{\partial x^j} = 0. \tag{29}$$

The first equation (28) is none other than the Riemannian Killing equation (5). The second equation (29) can be regarded as the constraint for the Killing vectors that satisfy the Killing equation (28). Therefore, in general, the dimension of the linear space formed by Killing vectors of (α, β) metric is lower than the Riemannian one.

4. Symmetry of VSR

One important physical example of (α, β) space is VSR. When we take $\phi(s) = s^m$, where m is an arbitrary constant, the Finsler structure takes the form proposed by Gibbons et al. [9].

$$F = \alpha^{1-m}\beta^m = (\eta_{ij}y^iy^j)^{(1-m)/2}(b_ky^k)^m, \tag{30}$$

where η_{ij} is Minkowskian metric and b_k is a constant vector, the metric (30) is called VSR metric. One immediately obtain from the first Killing equation (28) of (α, β) space as,

$$V^i = Q^i_jx^j + C^i. \tag{31}$$

And the second Killing equation (29) gives the constraint for Killing vector V^i,

$$b_iQ^i_j = 0. \tag{32}$$

We use the following result proved by [13]:

Lemma 1. *The VSR metric is invariant under the group of two-dimensional Euclidean motion* $(E(2))$*.*

The above investigation and the Killing equations (28) and (29) obtained in section-3 are under the premise that the direction of y^i is arbitrary. It means that no preferred direction exists in spacetime. If the spacetime does have a special direction, the Killing equation (23) will have a special solution. The VSR metric is first suggested by Bogoslovsky[6]. Following the assumption and taking the null direction to be preferred direction, we deduce from Killing equation (23) that

$$\begin{aligned} 0 &= s^m\left(\frac{1-n}{2\alpha}\left(\frac{\partial V_i}{\partial x^j} + \frac{\partial V_j}{\partial x^i}\right)y^iy^j + ms^{-1}b_k\frac{\partial V^k}{\partial x^r}y^r\right), \\ &= s^m\frac{1}{\alpha\beta}\left(\frac{1-n}{2}\left(\frac{\partial V_i}{\partial x^j} + \frac{\partial V_j}{\partial x^i}\right)b_r + m\eta_{ij}b^k\frac{\partial V_k}{\partial x^r}\right)y^iy^jy^r. \end{aligned} \tag{33}$$

The above equation has a special solution

$$V_+ = (Q_{+-} + m\eta_{+-})x^- + C_+, \tag{34}$$

where Q_{+-} is not only an antisymmetrical matrix, but also satisfying the property

$$b_- = -b^+ Q_{+-}. \tag{35}$$

It implies that the Lorentz transformation for b^+ is

$$(\delta^+_+ + \epsilon(m\delta^+_+ + Q^+_+))b^+ = (1 + \epsilon(n+1))b^+, \tag{36}$$

which means the null direction b^+(or b_-) is invariant under the Lorentz transformation. Therefore, if the spacetime has a preferred direction in null direction, the symmetry corresponded to Q_{+-} is restored. In such case, the VSR metric is invariant under the transformations of the group DISIM$_b(2)$ proposed by Gibbons et al. [9]. Another important physical example of (α, β)- space is Randers space[19], where we set $\Phi(s) = 1 + s$, the Finsler structure takes the form

$$F = \alpha + \beta. \tag{37}$$

Then, in Randers space the Killing equation (23) leads to

$$K_V(\alpha) + K_V(\beta) = 0. \tag{38}$$

Since the $K_V(\alpha)$ contains irrational term of y^i and $K_V(\beta)$ only contains rational term of y^i, the equation (38) satisfies if and only if $K_V(\alpha) = 0$ and $K_V(\beta) = 0$. If Randers space is flat, its Killing vectors satisfies the same Killing equation with VSR metric.

5. Killing vectors in Finsler-Berwald space

In this section, we have investigate a special Berwald metric with constant flag curvature $K = 0$. Further, we find the number of independent Killing vectors for a Berwald metric. We prove the following main result.

Theorem 1. *The Finsler-Berwald metric with constant flag curvature K=0 has maximum 15 independent Killing vectors.*

Proof. Now, consider the special Berwald metric given by[20];

$$F = \frac{(\sqrt{(y.y)(1 - x.x) + (x.y)^2} + (x.y))^2}{(1 - (x.x))^2\sqrt{(y.y)(1 - x.x) + (x.y)^2}}, \tag{39}$$

where "." denotes the inner product with respect to Minkowskian metric. By using the equation (16) and the first Killing equation $K_V(\alpha) = 0$, (28) implies

$$V^i = Q^i_j x^j + C^i - (x.c)x^i. \tag{40}$$

The Funk metric θ and Berwald's metric B are related and they can be expressed in the form

$$\theta = \bar{\alpha} + \bar{\beta},\ B = \frac{(\tilde{\alpha} + \tilde{\beta})^2}{\tilde{\alpha}}.$$

Here,

$$\bar{\alpha} = \frac{\sqrt{(y.y)(1-x.x)+(x.y)^2}}{1-(x.x)^2},\ \bar{\beta} = \frac{(x.y)}{1-(x.x)^2}.$$

$\tilde{\alpha} = \lambda\bar{\alpha}$, $\tilde{\beta} = \lambda\bar{\beta}$, where $\lambda = \frac{1}{1-(x.x)^2}$.
And from Berwald metric, we have

$$b_i = \frac{x_i}{(1-(x.x))^2}.$$

Then, we obtain the partial derivative for $b_i(x)$,

$$\frac{\partial b_i}{\partial x^j} = \frac{\eta_{ij}}{(1-(x.x))^2} + \frac{4x_i x_j}{(1-x.x)^3}, \tag{41}$$

and the corresponding partial derivative for Killing vectors V^i is

$$\frac{\partial V^i}{\partial x^j} = Q^i_j - \delta^i_j(x.C) - x^i C_j. \tag{42}$$

By making use of the equation (40), (41) and (42), we derive the second Killing equation (29) of the form

$$\frac{4Q_{ji}x^i}{1-x.x} + C_j = 0. \tag{43}$$

Substituting the equation (43) into (40), we obtain

$$V^i = Q^i_j x^j \left(\frac{5-x.x}{1-x.x}\right). \tag{44}$$

Hence, the dimension of the linear space formed by the Killing vectors of Finsler-Berwald metric is 15. And the space time translation generators corresponded to C^i depends on the generators of Lorentz group corresponded to Q^i_j. □

6. Conclusion

Lorentz Invariance (LI) is one of the foundations of the standard models of particle physics. Of course, it is very interesting to test the fate of the LI both on experiments and theories. The theoretical approach of investigating the LI violation is studying the possible spacetime symmetry and some parts of special relativity. In this paper, we have presented an explicit relation between the isometric group of a specific Finsler space and symmetries of the VSR proposed by Cohen and Glashow [7]. We showed that the Killing vectors satisfy the same Killing equation of a Riemannian metric, and the major difference is the Killing vectors of (α, β) need to satisfy the constraints (29). Further, we consider the Finsler-Berwald metric with constant flag curvature $K = 0$ and we showed that the number of Killing vectors of Finsler-Berwald metric is 15. Finally, we conclude that, "the determination for the maximal number of independent Killing vectors of (α, β)-space or any general Finsler space is still a open problem. We hope, it could be solved in the future".

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Received December 3, 2013.

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