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A SHORT REVIEW ON AVERAGING PROCESSES IN FINSLER GEOMETRY

CSABA VINCZE

Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. Averaging processes are widely used in mathematics. It is a relatively new trend in Finsler geometry. I would like to present the theoretical and historical background together with some typical applications.

1. Introduction

Finsler geometry is a non-Riemannian geometry in a finite number of dimensions. The differentiable structure is the same as the Riemannian one but distance is not uniform in all directions. Instead of the Euclidean spheres in the tangent spaces, the unit vectors form the boundary of general convex sets containing the origin in their interiors. (M. Berger). The metric feature of a Finsler manifold is based on a smoothly varying family of compact convex bodies in the tangent spaces. They are working as unit balls under some regularity conditions. Having unit balls we can measure the length of tangent vectors with the help of the induced Minkowski functionals. Manifolds equipped with a smoothly varying family of Minkowski functionals are called Finsler manifolds [5], see also [13]. Let M be a connected differentiable manifold with local coordinates $u^1, \ldots, u^n$ on $U \subset M$. The induced coordinate system on the tangent manifold consists of the functions

$$x^1 := u^1 \circ \pi, \ldots, x^n = u^n \circ \pi \quad \text{and} \quad y^1 := du^1, \ldots, y^n = du^n,$$

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where $\pi\colon TM \to M$ is the canonical projection. The Finsler structure is given by a fundamental function $F\colon TM \to \mathbb{R}$ satisfying the following conditions:

- each non-zero element $v \in TM$ has an open neighbourhood such that the restricted function is of class at least $\mathcal{C}^4$ in all of its variables $x^1, \ldots, x^n$ and $y^1, \ldots, y^n$,
- F is positively homogeneous of degree one: $F(rv) = rF(v)$ for any positive real number r and $F(v) = 0$ if and only if v is the zero element of the tangent space,
- (regularity condition) the Hessian matrix
$$g_{ij} := \frac{\partial^2 E}{\partial y^j \partial y^i}$$
of the energy function $E := (1/2)F^2$ with respect to the variables $y^1, \ldots, y^n$ is positive definite.

The components g_{ij} of the so-called *Riemann-Finsler metric* is defined only on the punctured tangent spaces because the second order partial differentiability of the energy function at the origin does not follow automatically. Further canonical objects are
$$d\mu = \sqrt{\det g_{ij}}\; dy^1 \wedge \ldots \wedge dy^n,$$
the *Liouville vector field*
$$C := y^1\partial/\partial y^1 + \ldots y^n\partial/\partial y^n$$
and the *induced volume form*
$$\mu = \frac{1}{F}\iota_C d\mu = \sqrt{\det g_{ij}}\; \sum_{i=1}^{n}(-1)^{i-1}\frac{y^i}{F}dy^1 \wedge \ldots \wedge dy^{i-1} \wedge dy^{i+1} \ldots \wedge dy^n$$
on the indicatrix hypersurfaces $\partial K_p := F^{-1}(1) \cap T_pM \quad (p \in M)$.

2. Averaged Riemannian metrics

The primary aim of averaging in Finsler geometry is to construct associated Riemannian structures on the base manifold. A typical example is the Riemannian metric given by integration of the Riemann-Finsler metric on the indicatrix hypersurfaces: let $f\colon TM \to \mathbb{R}$ be a zero homogeneous function and let us define the average-valued function [10]
$$A_f(p) := \int_{\partial K_p} f\,\mu;$$
especially the averaged Riemannian metric is defined by
$$\gamma_p(v,w) := \int_{\partial K_p} g(v,w)\,\mu. \tag{1}$$
It is probably hard to detect the first appearance of the construction (1). It can be found (among others) in Z. Shen [18], L. Tamássy [21], Cs. Vincze

[26] and R. G. Torromé [22]. In general *these averaging processes kill some important geometric data of a Minkowski functional* (Z. Shen). On the other hand lots of quadrics can be associated with a convex body which means that lots of Riemannian metrics can be introduced on a Finsler manifold [17], see also [12] and [10]. Instead of a kind of general theory our aim is to present successful and consequent applications of averaging processes based on the metric (1) and its relatives. It is a relatively new and popular trend in Finsler geometry together with a rapidly increasing numbers of the papers. Without completeness the most important reference works related to the achievements of this new trend are due to Cs. Vincze [26], [27], [31] and [23], M. Crampin [10] and [9], R. G. Torromé [22], V. S. Matveev et al. [15], [16] and [17], T. Aikou [2].

3. Generalized Berwald manifolds

Canonical differential geometric objects of a Finsler manifold are living on the tangent manifold in general. The averaged metric (1) provides a Riemannian environment for the investigations. The connection between the Riemannian and Finslerian levels can be provided by an additional structure on the base manifold.

Definition 1. A linear connection on the base manifold is compatible with the Finslerian structure if the parallel transports preserve the Finslerian norm of tangent vectors. Finsler manifolds admitting compatible linear connections are called generalized Berwald manifolds. If the compatible connection is torsion-free then we have the classical Berwald manifolds.

The basic result related to the theory of the generalized Berwald manifolds is an observation on the Riemann metrizability of the compatible connections which is a direct generalization of Szabó's well-known result [20] for classical Berwald manifolds.

Theorem 1 ([26])**.** *If a linear connection on the base manifold is compatible with the Finslerian structure then it must be metrical with respect to the averaged Riemannian metric (1).*

As mentioned above lots of Riemannian metrics can be introduced on a Finsler manifold in general. They correspond to the varieties of quadrics related to the indicatrix bodies. The following theorem shows that the averaged metric (1) is a reasonable choice in case of generalized Berwald manifolds.

Theorem 2. *Let ∇ be a a linear connection which is compatible with the Finslerian structure and suppose that ∇ is metrical with respect to the Riemannian metric $\tilde{\gamma}$. If ∇ is locally irreducible then $\tilde{\gamma} = c\gamma$, where c is a positive real constant.*

Proof. Let a point $p \in M$ be given. Since the unit component of the holonomy group of the compatible linear connection ∇ at p belongs to the orthogonal groups with respect to both $\tilde{\gamma}$ and γ, the irreducibility implies that they are proportional to each other. The proportional term depends on the position in general. Using the metric property of ∇ it follows that

$$0 = (\nabla_X \tilde{\gamma})(Y, Z) = (Xc)\gamma(Y, Z) + c(\nabla_X \gamma)(Y, Z) \overset{\text{Thm. 1}}{=} (Xc)\gamma(Y, Z)$$

for any vector fields X, Y and Z. Therefore c must be independent of the position. □

It is well-known that metrical linear connections are uniquely determined by the torsion tensor. Consider the decomposition

$$T(X, Y) := T_1(X, Y) + T_2(X, Y),$$

where

$$T_1(X, Y) := T(X, Y) - \frac{1}{n-1}\big(\tilde{T}(X)Y - \tilde{T}(Y)X\big),$$
$$T_2(X, Y) := \frac{1}{n-1}\left(\tilde{T}(X)Y - \tilde{T}(Y)X\right)$$

and $\tilde{T}$ is the trace tensor of the torsion [1]. The trace-less part T_1 is automatically zero in case of $n = 2$. In case of $n \geq 3$ the trace-less part can be divided into two further components A_1 and S_1 by separating the axial (or totally anti-symmetric) part A_1. This means that its lowered tensor with respect to the Riemannian metric is totally anti-symmetric:

$$T(X, Y) = A_1(X, Y) + S_1(X, Y) + T_2(X, Y).$$

Then we have eight classes of metrical linear connections depending on the surviving terms among A_1, S_1 and T_2. At the same time we have eight classes of generalized Berwald manifolds. They correspond to the elements of $(\mathbb{Z}_2)^3$ in such a way that we use the term 1 if the corresponding component is not identically zero. Generalized Berwald manifolds of type

- $(0, 0, 0)$ are the classical Berwald manifolds admitting torsion-free compatible linear connections on the base manifold [20].
- $(0, 0, 1)$ are Finsler manifolds admitting compatible linear connections with vanishing trace-less part in the torsion (the only surviving term is T_2). In an equivalent terminology, such a metrical linear connection is called semi-symmetric [31].
- $(1, 0, 0)$ are Finsler manifolds admitting compatible linear connections with totally anti-symmetric torsion tensor (the only surviving term is A_1). It is a well-known [1] that metric connections with totally anti-symmetric torsion have the same geodesics as the Lévi-Civita connection, i.e. all of these connections have an associated spray in common (the spray of the Lévi-Civita connection of the averaged Riemannian metric); for some preliminary results see [32].

etc.

3.1. **Generalized Berwald manifolds admitting semi-symmetric compatible linear connections.** Suppose that we have a Finsler manifold admitting a linear connection on the base manifold such that the parallel transports preserve the Finslerian length of tangent vectors. The basic questions are the *unicity* of such a compatible linear connection and its *expression* in terms of the canonical data of the Finsler manifold (intrinsic characterization). The first essential results were formulated for compatible connections with torsion of the form

$$T = \frac{1}{2}\left(1 \otimes d\alpha - d\alpha \otimes 1\right), \tag{2}$$

where $\alpha\colon M \to \mathbb{R}$ is a smooth function. The so-called Hashiguchi-Ichijyō's theorem [11] states that a Finsler manifold admits a compatible linear connection with torsion (2) if and only if it is conformal to a Berwald manifold: the conformal change $E_\alpha = e^{\alpha\circ\pi}E$ results in a Berwald manifold, i.e. the proportional term is just the exponent of the vertically lifted function $\alpha^v = \alpha \circ \pi$. Conformally Berwald Finsler manifolds and Finsler manifolds admitting compatible semi-symmetric linear connections with exact 1-forms in the torsion mean the same type of spaces. They are called exact Wagner manifolds. The unicity-problem of the compatible linear connection with torsion (2) is equivalent to the so-called Matsumoto problem in [14]: are there non-homothetic conformally equivalent Berwald manifolds?

Theorem 3 ([27]). *The scale function between conformally equivalent Berwald manifolds must be constant unless they are Riemannian.*

For the solution of the unicity problem see also [25], [29] and [17]. The solution of the unicity problem allows us to take some steps forward. Linear connections with torsion (2) belong to the more general class of semi-symmetric linear connections.

Definition 2. A linear connection is said to be semi-symmetric if the torsion tensor is of the form

$$T(X,Y) = \frac{1}{2}\left(\beta(Y)X - \beta(X)Y\right) \tag{3}$$

where β is a 1-form on the manifold.

An intermediate level is to consider closed 1-forms in the torsion[1]: $d\beta = 0$, i.e. for any point $p \in M$ there exists an open neighbourhood around p and a smooth function $\alpha_p\colon U_p \to \mathbb{R}$ such that

$$d\alpha_p = \beta \quad \text{and} \quad T = \frac{1}{2}\left(1 \otimes \beta - \beta \otimes 1\right). \tag{4}$$

[1]Metric linear connections with closed 1-forms in formula (3) for the torsion are very important in differential geometry: if β is closed then all the classical curvature properties are satisfied which is crucial for the classification of the holonomy groups and Simon's theory of holonomy systems [19].

Let us introduce the notion of closed Wagner manifolds for Finsler spaces admitting compatible semi-symmetric linear connections with closed 1-forms in the torsion. The main question is how to generalize Hashiguchi-Ichijyō's theorem. It is clear from the earlier version of the theorem that for any point of a closed Wagner manifold has a neighbourhood over which it is conformal equivalent to a Berwald manifold. This means that closed Wagner manifolds are locally conformal to Berwald manifolds. What about the converse? The exterior derivatives of the local scale functions constitute a globally well-defined 1-form if and only if they coincide on the intersection of overlapping neighbourhoods. This gives the question that how many essentially different ways there are for a Finsler manifold to be conformal to a Berwald manifold. Alternatively: are there non-homothetic conformally equivalent Berwald spaces? This is just the Matsumoto's problem again.

Definition 3. Two Finsler structures over the same base manifold are conformally related if the corresponding Riemann-Finsler metrics are conformally related.

Using that

$$E = g(C, C)$$

conformally related Riemann-Finsler metrics $\tilde{g} = e^{\varphi} g$ results in conformally related energy functions $\tilde{E} = e^{\varphi} E$. If we rebuilt the Riemann-Finsler metrics as the second order partial derivatives of the energy functions with respect to the variables $y^1, \dots, y^n$ we have the so-called Knebelman's theorem: the scale function e^{φ} depends only on the position, i.e. the conformal equivalence of Finsler manifolds can be written into the form $\tilde{g} = e^{\alpha \circ \pi} g$ for some smooth function $\alpha \colon M \to \mathbb{R}$ on the common base manifold.

Definition 4. A Finsler manifold is called locally conformally Berwald manifold if for any point has a neighbourhood over which it is conformally Berwald by a locally given scale function.

The following figure shows the neighbourhoods U_p and U_q of the points p and q of a locally conformally Berwald manifold such that the restricted Finslerian structures are conformally equivalent to the Berwald manifolds M_1 and M_2, respectively. Therefore the Finsler manifold $U_p \cap U_q$ is simultaneously conformally equivalent to M_1 and M_2 which are conformally equivalent Berwald manifolds.

As Theorem 3 says $d\alpha_1 = d\alpha_2$ and we have the following generalization of the classical Hashiguchi-Ichyjiō's theorem[2].

[2]Aikou's paper [2] also deals with the problem of locally conformally Berwald manifolds but the starting point is essentially different. The definition of locally conformal Berwald manifolds involves a strange condition of the existence of a globally defined torsion-free linear connection, see Definition 6.1 : "A Finsler manifold (M, L) is said to be a locally conformal Berwald manifold, if there exists an open covering $(\mathcal{U}_\alpha)_{\alpha \in A}$ of M, a family $(\sigma_\alpha)_{\alpha \in A}$ of

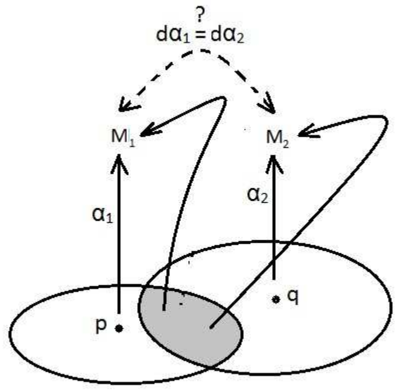

FIGURE 1

Theorem 4 ([27])**.** *A non-Riemannian Finsler manifold is a closed Wagner manifold if and only if it is a locally conformally Berwald manifold.*

The advantage of the paper [27] is that the unicity problem is solved by the solution of the intrinsic characterization of the compatible linear connection which is possibly the first place where a consequent and successful application of averaging was realized. The intrinsic characterization of the compatible linear connection with torsion (4) is given in terms of averaged objects. In order to avoid unnecessary repetitions we formulate the main result without any requirements of the exactness or closedness of the 1-form in the torsion. The intrinsic characterization of compatible semi-symmetric linear connections in general is a recent result [31], see also Theorem 2, Remark 3 (iii) and (iv) in [27].

Theorem 5 ([31])**.** *A non-Riemannian Finsler manifold is a generalized Berwald manifold admitting a semi-symmetric compatible linear connection if and only if $\sigma(p) > 0$ for any $p \in M$ and the linear connection*

$$\nabla_X Y = \nabla^*_X Y + \frac{1}{2\sigma}\left(\eta^*(Y)X - \gamma(X,Y)\eta^{*\sharp}\right) \tag{5}$$

is compatible with the Finslerian structure, where ∇^* is the Lévi-Civita connection of the averaged Riemannian metric,

$$\rho^* := \frac{d_{h^*}E}{E} - \frac{1}{2}\frac{S^*E}{E}\frac{d_J E^*}{E^*} \quad \textit{and} \quad f := \log\frac{E^*}{E},$$

---

smooth functions $\sigma_\alpha : \mathcal{U}_\alpha \to \mathbb{R}$ and a torsion-free covariant derivative ∇ on M such that

$$d_{\mathbf{h}^\nabla}((e^{\sigma_\alpha})^v L) = 0 \quad \text{for all} \quad \alpha \in A,$$

where $\mathbf{h}^\nabla$ denotes the horizontal endomorphism induced by ∇." Therefore the Matsumoto's problem is simplified because the condition says that for any locally given Finsler (especially Berwald) manifold $(\mathcal{U}_\alpha, (e^{\sigma_\alpha})^v L)$ has the same canonical connection ∇. The conformal invariance of the canonical connections obviously implies that we have homothetic changes between the members of the family with overlapping neighbourhoods.

where J is the canonical vertical endomorphism/almost tangent structure on the tangent manifold, h^* is the induced horizontal endomorphism with associated spray S^* and E^* denotes the Riemannian energy. Furthermore

$$\eta^*(X_p) := \int_{\partial K_p^*} d_J\rho^*(\Theta, X^h) + \frac{1}{2}\frac{S^*E}{E}X_i^v f\, \mu^* \tag{6}$$

and

$$\sigma(p) := \int_{\partial K_p^*} \frac{1}{2E^*}\|J\Theta\|^2\, \mu^*, \quad \textit{where} \;\; \Theta = E^*\frac{\partial f}{\partial y^i}\gamma^{ij}\circ\pi\frac{\partial}{\partial x^j} \tag{7}$$

is a gradient-type vector field and the norm is taken with respect to the vertical lift of the averaged metric.

Note that one can use any horizontal lifting process $X \mapsto X^h$ in (6) because the integrand is semibasic. From equation (5)

$$T(X,Y) = \frac{1}{2}\left(\frac{\eta^*(Y)}{\sigma}X - \frac{\eta^*(X)}{\sigma}Y\right).$$

The theorem says that the compatible semi-symmetric linear connection can be explicitly expressed in terms of the averaged Riemannian metric and associated objects. Such an intrinsic characterization implies immediately the solution of the Matsumoto's problem for conformally equivalent Berwald manifolds. The problem of special conformal relationships forms a kind of starting point of the theory of averaging. To complete the panoramic view we note that both the exterior derivative

$$\theta := \frac{1}{\sigma}\left(d\eta^* - \frac{1}{\sigma}d\sigma \wedge \eta^*\right) \tag{8}$$

of η^*/σ and

$$d_h \log E \tag{9}$$

are conformally invariant, where h is the horizontal endomorphism induced by ∇ (see formula (5)). Therefore the theory can be presented in terms of conformal invariants too, for the details see [31]. This means a partial contribution to Shen's open problem: *find all conformal invariants of a Finsler metric* (Problem 30 in www.math.iupui.edu/˜zshen.)

4. Randers manifolds

Conformal (and lots of other kind of) Finsler geometry has extremely complicated formulas in general. Explicit computations are possible only in case of special Finsler manifolds. An important computable case is the class of Randers manifolds. The family of the unit balls of a Randers manifold is given by translations of Riemannian unit balls. Analytically the Minkowski functionals are coming from a Riemannian metric tensor by using one-form perturbation in the tangent spaces. This important type of Finsler manifolds was introduced by G. Randers in 1941. Randers manifolds occur naturally in physical applications related to electron optics, navigation problems [6] or the Lagrangian of relativistic electrons [3]. According to the importance of these applications Randers manifolds are a prosperous subject of the investigations up to this day. They form an excellent and well-motivated intermediate level between Riemannian and Finsler geometry - see e.g. [8]. Randers manifolds will also come to the front as associated objects with Finsler manifolds in the series of differential geometric structures given by averaging (see section 4.1).

Let (M, α) be a connected Riemannian manifold and suppose that the one-form β in $\wedge^1(M)$ satisfies the condition

$$\sup_{\alpha(v,v)=1} \beta(v) < 1 \tag{10}$$

for any point p in M and $v \in T_pM$. The Randers functional on the manifold M is defined as

$$F(v) = \sqrt{\alpha(v,v)} + \beta(v) \tag{11}$$

and the pair (M, F) is called a *Randers manifold* with perturbating term β. The general treatment of the generalized Berwald manifolds admitting semi-symmetric compatible linear connections is based on the averaged Riemannian metric (1). In case of a Randers manifold the Riemannian environment is directly given by the Riemannian part α of the initial data. The following theorem shows that it is a natural choice instead of the averaged metric.

Theorem 6. *A Randers manifold is a generalized Berwald manifold if and only if there exists a linear connection ∇ on the manifold M such that $\nabla\alpha = 0$ and $\nabla\beta = 0$.*

As the next step we are going to formulate a necessary and sufficient condition for a Randers manifold to be a generalized Berwald manifold in terms of the dual vector field

$$\alpha(\beta^\sharp, X) = \beta(X) \tag{12}$$

of the perturbating term.

Theorem 7 ([24])**.** *A Randers manifold is a generalized Berwald manifold if and only if $\beta^\sharp$ is of constant Riemannian length.*

Proof. Suppose that the functional F is invariant under the parallel transport with respect to the linear connection ∇. By Theorem 6 we can easily conclude that the sharp operator (12) gives a vector field of constant Riemannian length:

$$\alpha(\beta^\sharp, \beta^\sharp) = \text{constant}. \tag{13}$$

Conversely suppose that $\beta^\sharp$ is of constant length K and let ∇^* be the Lvi-Civita connection of α. In what follows we are going to construct a linear connection ∇ such that $\nabla\alpha = 0$ and $\nabla\beta = 0$. If

$$\nabla_X Y = \nabla^*_X Y + A(X, Y), \tag{14}$$

where

$$A(X,Y) = \frac{\alpha(\nabla^*_X\beta^\sharp, Y)\beta^\sharp - \alpha(Y,\beta^\sharp)\nabla_X\beta^\sharp}{K^2} \tag{15}$$

then

$$\alpha(A(X,Y),Z) = -\alpha(A(X,Z),Y)$$

which means that ∇ is metrical with respect to α, i.e. $\nabla\alpha = 0$. On the other hand

$$\nabla\beta(X,Y) =$$

$$X\beta(Y) - \beta(\nabla^*_X Y) - \frac{1}{K^2}\left(\alpha(\nabla^*_X\beta^\sharp, Y)\beta(\beta^\sharp) - \alpha(Y,\beta^\sharp)\beta(\nabla_X\beta^\sharp)\right).$$

Here

$$\beta(\beta^\sharp) = K^2 \quad \text{and} \quad \beta(\nabla_X\beta^\sharp) = \alpha(\beta^\sharp, \nabla_X\beta^\sharp) = \frac{1}{2}X\|\beta^\sharp\|^2 = 0.$$

Therefore

$$\nabla\beta(X,Y) = X\beta(Y) - \beta(\nabla^*_X Y) - \alpha(\nabla^*_X\beta^\sharp, Y) = 0$$

because of the metrical property of the Lvi-Civita connection. □

Remark 1. The discussion of Randers manifolds admitting semi-symmetric compatible linear connections can be found in [24]. They are characterized up to local isometries. Special questions including the so-called existence theorem of Wagner manifolds [4] were clarified in [27] and [28], see also [30].

The question is natural: what can be the analogue of the one-form β in case of a generic Finsler manifolds?

4.1. **Randers metrics given by averaging.** The motivation of using averaging processes in Finsler geometry is that we can simplify the solution of general problems by restricting the investigations to more transparent associated structures. The first step was the introduction of the averaged metric (1) to provide a Riemannian environment for the investigations. Unfortunately the most of different theories of Finsler spaces have extremely complicated formulas. Explicit computations are possible only in case of special Finsler manifolds. But how can special Finsler manifolds (Riemannian or Randers manifolds etc.) play a significant role in the study of Finsler manifolds in general. To give a kind of answer to the question we can use "averaging" again. Our way is to develop the theory of averaging by creating a "Randers manifold environment" for Finsler manifolds [23]. The perturbating term is given by the integration of the contracted-normalized Riemann-Finsler metric on the indicatrix hypersurfaces:

$$\beta_p(v) := \int_{\partial K_p} VF\,\mu, \text{ where } V = v^1\frac{\partial}{\partial y^1} + \cdots + v^n\frac{\partial}{\partial y^n}. \tag{16}$$

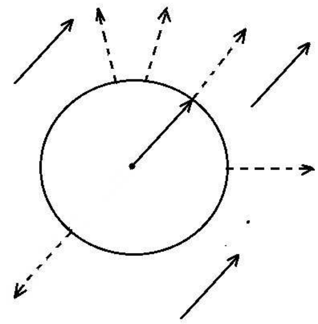

FIGURE 2

It is just the integral of the contracted-normalized Riemann-Finsler metric because of $VF = g(C,V)/F$. Using Cauchy-Buniakowsky-Schwarz's inequality

$$\beta_p^2(v) = \left(\int_{\partial K_p} VF\,\mu\right)^2 \leq \int_{\partial K_p} \mu \int_{\partial K_p} (VF)^2\,\mu =$$

$$\text{Area }(\partial K_p) \int_{\partial K_p} (VF)^2\,\mu.$$

On the other hand

$$(VF)^2 = \frac{1}{F^2}\, g^2(C,V) \leq \frac{1}{F^2}\, g(C,C)\, g(V,V) = g(V,V).$$

Using integration

$$\beta_p^2(v) \leq \text{Area }(\partial K_p)\ \gamma_p(v,v),$$

where the inequality is strict because C and V are linearly independent away from the points v and $-v$ as Figure 2 shows.
Introducing the weighted inner product

$$\Gamma_p(v,w) = \frac{1}{\text{Area }(\partial K_p)}\gamma_p(v,w)$$

we can write that

$$\left(\frac{\beta_p}{\text{Area }(\partial K_p)}\right)^2(v) < \Gamma_p(v,v) \quad \Leftrightarrow$$

$$\text{the sup. norm of the weighted linear functional } \frac{\beta_p}{\text{Area }(\partial K_p)} < 1.$$

Definition 5. [23] The associated Randers functional of the Finsler manifold is defined by the formula

$$F_p^{**}(v) := \sqrt{\Gamma_p(v,v)} + \frac{\beta_p(v)}{\text{Area }(\partial K_p)}.$$

We are going to motivate the construction by some applications.

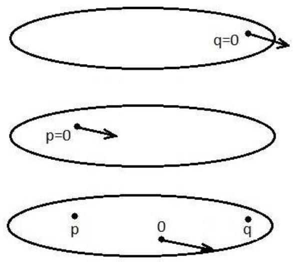

Figure 3

5. Funk manifolds

Let $K \subset \mathbb{R}^n$ be a convex body containing the origin in its interior. Changing the origin in the interior of K we have a smoothly varying family of Finsler-Minkowski functionals parametrized by the interior points of K:

$$p + \frac{v}{F_p(v)} \in \partial K. \tag{17}$$

The manifold $U :=$ the interior of K equipped with the fundamental function $F\colon TU \to \mathbb{R}$ is called a Funk manifold. Figure 3 shows how the origin is changing in the interior of K to constitute a smoothly varying family of convex bodies as indicatrices in different tangent spaces.

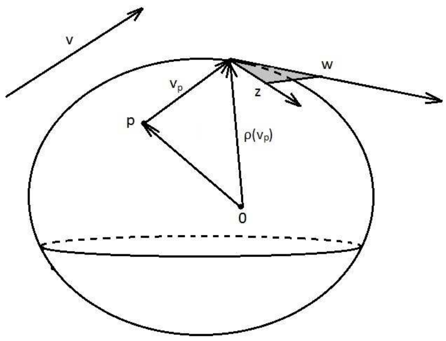

Figure 4

Theorem 8 ([23]). *The projection*

$$\rho(v_p) := p + \frac{v}{F_p(v)} \in \partial K_{\mathbf{0}}$$

is a conform mapping between the indicatrices ∂K_p and $\partial K_{\mathbf{0}} = \partial K$. Especially

$$g_{\rho(v_p)}(w, z) = \left(1 - p^k \frac{\partial L}{\partial u^k}_{\rho(v_p)}\right) g_{v_p}(w, z),$$

where w and z are tangential to ∂K_p at v_p, i.e. they are tangential to ∂K at $\rho(v_p)$. (See Figure 4.)

The conformality between the indicatrices (as Riemannian submanifolds in the tangent spaces) motivates us to use the so-called angular metric tensor to be averaged. The angular metric tensor is given by the action of the Riemann-Finsler metric on the tangential components of vector fields to the indicatrices. The construction appears in Crampin's list [10] among some other candidates of metrics for averaging. The case of Funk manifolds gives its geometric inspiration. Taking the area function

$$r\colon U \to \mathbb{R}, \quad p \mapsto r(p) := A_1(p) = \int_{\partial K_p} \mu_p \tag{18}$$

the following expressions can be given for the first and second order partial derivatives [23]:

$$\frac{\partial r}{\partial u_p^i} = \frac{n-1}{2} \int_{\partial K_p} \frac{\partial F}{\partial y^i}\, \mu_p \tag{19}$$

and

$$\frac{\partial^2 r}{\partial u^j \partial u_p^i} = \frac{n^2-1}{2} \int_{\partial K_p} \frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j}\, \mu_p. \tag{20}$$

The positive definiteness of the Hessian matrix by (20) means that the area function is strictly convex. Moreover it is another candidate of Riemannian metrics given by averaging - see [23]. Equation (19) implies that

$$\mathrm{d}r = \frac{n-1}{2}\beta.$$

Corollary 1. In case of a Funk manifold the associated Randers functional can be written into the form

$$F_p^{**}(v) := \sqrt{\Gamma_p(v,v)} + \frac{2}{n-1}\mathrm{d}\log r.$$

Corollary 2. The associated Randers functional is projectively equivalent to the weighted Riemannian metric tensor Γ.

6. An application: Brickell's theorem and its generalization

Now we are in the position to prove Brickell's conjecture for Finsler manifolds with balanced indicatrices [23]. Brickell proved the result under the condition of absolute homogenity. In an equivalent way, the indicatrices were assumed to be symmetric with respect to the origin in the tangent spaces [7].

Definition 6. The indicatrix body at p is called balanced if $\beta_p = 0$.

Corollary 3. If the Finslerian indicatrices are balanced then the associated Randers functional and the associated (weighted) Riemannian fundamental function coincide.

Corollary 4. The domain of the Funk metric, i.e. the indicatrix at $\mathbf{0}$ is balanced if and only if the function r has a global minimum at the origin.

Remark 2. Note that the vanishing of the first order partial derivatives is enough to conclude that the function has a global minimizer because of the convexity.

Theorem 9 ([23]). *Let M be a Finsler manifold with balanced indicatrices in the tangent spaces. If dim $M \geq 3$ and the Cartan connection has a vanishing vv-curvature tensor then the manifold is Riemannian.*

The steps of the proof are

1. Let a point of the base manifold be given and consider the interior of the indicatrix body as a Funk manifold.
2. The indicatrix is balanced $\Rightarrow$ the area function has a global minimum at the origin.
3. The minimum is just the area of the Euclidean sphere of the same dimension because of the vanishing of the curvature.
4. The area function attains a greater (or the same) value at the centroid of the indicatrix body - recall that the indicatrix is considered as a Funk manifold and the origin is moving in its interior. If the origin is positioned at the centroid of then the generalized Santaló's inequality (for bodies with center at the origin) says that the indicatrix must be the translate of an ellipsoid.
5. It is true at each point of the manifold.
6. We have a Randers manifold with vanishing vv-curvature tensor.
7. The manifold is Riemannian.

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Institute of Mathematics,
University of Debrecen,
H-4010 Debrecen, P.O.Box 12 Hungary
E-mail address: csvincze@science.unideb.hu