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ON MODAL BE–ALGEBRAS

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Abstract. In this paper, we introduce modal BE-algebra and study some structural properties of modal BE-algebra. The notions of modal upper set, modal BE-filter, $BE-\Box$-tautology filter, dual modal BE-algebra and quotient modal BE-algebraare introduced and their basic properties are investigated. We will prove that every self- distributive BE-algebra, induce a dual modal BE-algebra. Finally, we will prove that every dual modal BE-algebra is a modal BE-algebras under special conditions.

1. Introduction and Preliminaries

Modal logic is a theoretical field that is important not only in philosophy, where logic in general is commonly studied, but also in mathematics, linguistics, computer and information sciences as well. Classical modal logics have been a matter of growing interest in the last decades due to their role in the formalization of several aspects of computer science. The earliest paper on a many-valued modal logic appears to have been Segerberg (1967), which specifies some 3-valued modal logics.

Modal logics and many-valued logics were both historically introduced in order to free oneself from the rigidity of propositional logic. With many-valued logics, the logician can choose the truth values of the propositions in a set with more than two elements. With modal logics, the logician introduce a new connector whose aim is, for instance, to model the possibility. Many systems with various kind of modal operators have been constructed in order to provide effective formalisms for talking about time, space, knowledge, beliefs, actions, obligations, temporal, spatial, epistemic, dynamic, deontic, and so forth. However, modern applications often require rather complex formal models and corresponding languages that are capable of reflecting different features of the application domain [1, 9, 10].

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Furthermore, the study of BCK/BCI–algebras was initiated by K. Iséki in 1966 as a generalization of propositional logic. There exist several generalization of BCK/BCI–algebras, such as BCH–algebras, d–algebras, B–algebras, BH–algebras, etc. Especially, the notion of BE–algebras was introduced by H. S. Kim and Y. H. Kim [13], in which was deeply studied by S. S. Ahn and et al. in [2, 3, 4], Walendziak in [18], A. Rezaei and et al. in [7, 8, 15, 16, 17].

The idea of introducing modal operators in residuated lattices and other algebraic structures has been adapted by some researchers, for several purpose: Belohlavek and Vychodil [6] defined a so-called "truth stresser" ν for a residuated lattice $(L, \cup, \cap, *, \to, 0, 1)$ as a unary operator on L such that

- $\nu x \leq x$,
- $\nu 1 = 1$,
- $\nu(x \to y) \leq \nu x \to \nu y$, for all $x, y \in L$.

Ono [14] defined modal structures $(L, \cup, \cap, *, \to, \nu, 0, 1)$ in which $(L, \cup, \cap, *, \to, 0, 1)$ is a residuated lattice and ν is a unary operator on L such that:

- $\nu x \leq x$,
- $\nu x \leq \nu\nu x$,
- $\nu 1 = 1$,
- $\nu(x \cap y) \leq \nu x$
- $\nu x * \nu y \leq \nu(x * y)$, for all $x, y \in L$.

Hajek [11] used a unary operator $\triangle$ on the BL–algebra $\mathcal{L}$ to get the algebra $BL_\triangle$ such that axioms of $BL_\triangle$ are those of BL plus:

- $\triangle\phi \vee \neg\triangle\phi$,
- $\triangle(\phi \vee \psi) \implies (\triangle\phi \vee \triangle\psi)$,
- $\triangle\phi \implies \phi$,
- $\triangle\phi \implies \triangle\triangle\phi$,
- $\triangle(\phi \implies \psi) \implies (\triangle\phi \implies \triangle\psi)$.

The axioms evidently resemble modal logic with $\triangle$ as necessity; but in the axiom on $\triangle(\phi \vee \psi)$, $\triangle$ be haves as possibility rather than necessity.

Magdalena and Rachunek [12] defined a unary operator f on an MV–algebra $\mathscr{A}$ as follows: If $\mathscr{A} = (A, \oplus, \neg, 0)$ is an MV–algebra where $x \odot y = \neg(\neg x \oplus \neg y)$, then $f\colon A \to A$ is called a modal operator on $\mathscr{A}$ satisfying:

- $x \leq f(x)$,
- $f(f(x)) = f(x)$,
- $f(x \odot y) = f(x) \odot f(y)$, for all $x, y \in A$.

In fact the modal operator f be haves as possibility $\Diamond$ in modal logics. All above motivates us to introduce a modal operator on BE–algebra to get a modal BE–algebra as an algebraic structure.

This paper has been organized in three sections. In section 1, we give some definitions and some previous results. In section 2 we define modal BE–algebras and modal BE–filters. Finally, in section 3 we construct quotient modal BE–algebra via the modal normal BE–filter.

Definition 1.1 ([13])**.** An algebra $(X;*,1)$ of type $(2,0)$ is called a BE–algebra if following axioms hold:

(BE1) $x*x=1$,
(BE2) $x*1=1$,
(BE3) $1*x=x$,
(BE4) $x*(y*z)=y*(x*z)$, for all $x,y,z\in X$.

We introduce a relation "$\leq$" on X by $x\leq y$ if and only if $x*y=1$.

Definition 1.2 ([13])**.** A BE–algebra X is said to be self distributive if

$$x*(y*z)=(x*y)*(x*z),\ \text{for all } x,y,z\in X.$$

Proposition 1.3 ([16])**.** *Let X be a self distributive. If $x\leq y$, then*

(i) $z*x\leq z*y$, *and* $y*z\leq x*z$,
(ii) $y*z\leq(z*x)*(y*x)$, *for all* $x,y,z\in X$.

Definition 1.4 ([15, 18])**.** A BE–algebra X is said to be commutative if

$$(x*y)*y=(y*x)*x,\ \text{for all } x,y\in X.$$

Proposition 1.5 ([18])**.** *If X is a commutative BE–algebra, then for all $x,y\in X$, $x*y=1$ and $y*x=1$ imply $x=y$.*

Proposition 1.6 ([13])**.** *Let X be a BE–algebra. Then*

(i) $x*(y*x)=1$,
(ii) $y*((y*x)*x)=1$, *for all* $x,y\in X$.

Definition 1.7 ([13])**.** A subset F of X is called a filter of X if

(F1) $1\in F$,
(F2) $x\in F$ and $x*y\in F$ imply $y\in F$, for all $x,y\in X$.

Definition 1.8 ([19])**.** A filter F is said to be normal if it satisfies the following condition:

$$(NF)\qquad x*y\in F\Rightarrow[(z*x)*(z*y)\in F \text{ and } (y*z)*(x*z)\in F],$$

for all $x,y,z\in X$.

2. Modal BE-algebras

Definition 2.1. An algebra $(X;*,\Box,1)$ of type $(2,1,0)$ is called a modal BE–algebra if it satisfies the following:

(BE) $(X;*,1)$ is a BE–algebra,
$(MBE1)$ $\Box 1=1$,
$(MBE2)$ $\Box x\leq x$,
$(MBE3)$ $\Box x=\Box\Box x$,
$(MBE4)$ $\Box(x*y)=\Box x*\Box y$,

From now on, for simply in this section X is a modal BE–algebra, unless otherwise is stated.

Example 2.2. (i). Let $X = \{1, a, b, c\}$. Define the operations "$*$" and "$\Box$" on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	c
c	1	b	1	1

x	1	a	b	c
$\Box$	1	a	c	c

Then, $(X; *, \Box, 1)$ is a modal BE–algebra.

(ii). Let $X = \mathbb{N}$ and "$*$" be the binary operation on X defined by

$$x * y = \begin{cases} y, & \text{if } x = 1, \\ 1, & \text{if } x \neq 1. \end{cases}$$

Then, $(X; *, 1)$ is a BE–algebra. Now, if we define the unary operation "$\Box$" such as:

$$\overbrace{(\Box \cdots \Box)}^{n-\text{times}} x = \begin{cases} 1, & \text{if } x = 1, \\ 2, & \text{if } x = 2, \\ (x-n)+(n-1), & \text{if } x \neq 1, 2. \end{cases}$$

Then, $(X; *, \Box, 1)$ is a modal BE–algebra.

Proposition 2.3. *Let X be a modal BE–algebra. Then*

(i) *if $x \leq y$, then $\Box x \leq \Box y$,*
(ii) $\Box x * \Box x = 1$,
(iii) $\Box x * 1 = 1$,
(iv) $1 * \Box x = \Box x$,
(v) $\Box x * (\Box y * \Box z) = \Box y * (\Box x * \Box z)$, *for all $x, y, z \in X$.*

For every modal BE–algebra X, put $\Box X = \{\Box x : x \in X\}$.

If X is a modal BE–algebra, then $\Box X = X$ does not hold, necessary. Indeed, in Example 2.2 (i) we have $\Box X = \{1, a, c\} \neq X$.

Theorem 2.4. *Let $(X; *, 1)$ be a BE–algebra. Then $(\Box X; *, 1)$ is a BE–algebra.*

Proof. By using Proposition 2.3, the proof is clear. $\Box$

Definition 2.5. Let $(X; *, \Box, 1)$ be a modal BE–algebra and $x, y \in X$. Modal upper set of x, y is denoted by $mA(x, y)$ and defined as follows:

$$mA(x, y) = \{z \in X : x * (y * \Box z) = 1\}.$$

Obviously, it is a non empty set. Because $1 \in mA(x, y)$.

Remark 2.6. The upper set $A(x, y)$ does not equal to modal upper set $mA(x, y)$. Indeed, in the Example 2.2(i), $mA(1, b) = \{1, a\} \neq \{1, a, b\} = A(1, b)$.

Proposition 2.7. *If $\Box y = y$, then $A(1,y) = mA(1,y)$, for all $x \in X$.*

Proof. Let $y \in X$. Then we have

$$\begin{aligned} A(1,y) &= \{z \in X : y * z = 1\} \\ &= \{z \in X : \Box(y * z) = 1\} \\ &= \{z \in X : \Box y * \Box z = 1\} \\ &= \{z \in X : y * \Box z = 1\} \\ &= mA(1,y). \end{aligned}$$

□

Proposition 2.8. *$mA(x,1) \subseteq mA(x,y)$, for all $x, y \in X$.*

Proof. Let $z \in mA(x,1)$. Then $1 = x * (1 * \Box z) = x * \Box z$. Now, we get that

$$x * (y * \Box z) = y * (x * \Box z) = 1.$$

Therefore, $z \in mA(x,y)$. □

Theorem 2.9. *Let X be a modal BE–algebra and $x, y \in X$. Then*

(i) *$mA(\Box x, 1) \subseteq mA(\Box x, y)$,*

(ii) *if $mA(\Box x, 1)$ is a filter of X and $y \in mA(\Box x, 1)$, then*

$$mA(\Box x, y) \subseteq mA(\Box x, 1).$$

Proof. (i). Let $z \in mA(\Box x, 1)$. Then $\Box x * (1 * \Box z) = 1$, i.e. $\Box x * \Box z = 1$. Hence $\Box x * (y * \Box z) = y * (\Box x * \Box z) = y * 1 = 1$, i.e. $z \in mA(\Box x, y)$.

(ii). Since $\Box x * (1 * \Box x) = 1$, we can see that $\Box x \in mA(\Box x, 1)$. Now, let $y \in mA(\Box x, 1)$, then we have $1 = \Box x * \Box y = \Box x * (1 * \Box y) \in mA(\Box x, 1)$. Thus

$$\Box y \in mA(\Box x, 1).$$

Let $z \in mA(\Box x, y)$. Then by using $(BE4)$

$$1 = \Box x * (y * \Box z) = y * (\Box x * \Box z).$$

Now, by $(MBE1)$, $(MBE3)$ and $(MBE4)$ we get that

$$\begin{aligned} 1 = \Box 1 = \Box(y * (\Box x * \Box z)) &= \Box y * \Box(\Box x * \Box z) \\ &= \Box y * (\Box x * \Box z) \in mA(\Box x, 1). \end{aligned}$$

Hence $\Box x * \Box z \in mA(\Box x, 1)$. Thus $\Box z \in mA(\Box x, 1)$ and so

$$1 = \Box x * (1 * \Box\Box z) = \Box x * (1 * \Box z).$$

Therefore, $z \in mA(\Box x, 1)$. □

Proposition 2.10. *Let F be a filter of X. Then $\Box mA(x,y) \subseteq F$, for all $x, y \in F$.*

Proof. Let $z \in \Box mA(x,y)$, then there exists a $c \in mA(x,y)$ such that $z = \Box c$. Hence $x * (y * \Box c) = 1 \in F$. Thus $y * \Box c \in F$. Therefore, $z = \Box c \in F$. □

Theorem 2.11. *Let F be a subset of X containing* 1. *$\Box F$ is a modal filter if and only if $x \leq y * z$ imply $z \in \Box F$, for all $x, y \in \Box F$.*

Proof. Let $\Box F$ be a modal filter and $x \leq y * z$, for all $x, y \in \Box F$. Since $x, y \in \Box F$ and $\Box F$ is a modal filter, we have $y * z \in \Box F$ and so $z \in \Box F$.

Conversely, $1 \in \Box F$, since $1 \in F$. If $x, x * y \in \Box F$, since $x * y \leq x * y$, we can see that by hypothesis $y \in \Box F$. Then there is a $z \in F$ such that $y = \Box z$. Therefore, $\Box y = \Box(\Box z) = \Box z = y \in \Box F$. $\Box$

Theorem 2.12. *Let F be a subset of X containing* 1. *$\Box F$ is a modal filter of X if and only if $x \in \Box F$, $y \in X \setminus \Box F$, then $x * y \in X \setminus \Box F$.*

Proof. Assume that $\Box F$ is a modal filter of X and let $x, y \in X$ be such that $x \in \Box F$ and $y \in X \setminus \Box F$. If $x * y \notin X \setminus \Box F$. Then $x * y \in \Box F$, i.e. $y \in \Box F$. which is a contradiction. Hence $x * y \in X \setminus \Box F$.

Conversely, $1 \in F$ by hypothesis. Let $x, x * y \in \Box F$. Let $y \notin \Box F$. By assumption $x * y \in X \setminus F$. This is a contradiction. Hence $y \in \Box F$. Thus there is a $z \in F$ such that $y = \Box z$. Therefore, $\Box y = \Box(\Box z) = \Box z = y \in \Box F$. $\Box$

Theorem 2.13. *Let F be a modal filter. Then*

$$\Box F = \bigcup_{x,y \in F} \Box mA(\Box x, y).$$

Proof. Let F be a modal filter of X and consider $\Box z$, for $z \in F$. Since

$$\Box z * (1 * \Box z) = \Box z * (1 * \Box\Box z) = 1 \text{ by } (MBE3),$$

we have $\Box z \in mA(\Box z, 1)$. Now, by Proposition 2.9, we have

$$\Box z \in mA(\Box z, 1) \subseteq mA(\Box z, y).$$

Thus $\Box z = \Box\Box z \in \Box mA(\Box z, 1) \subseteq \Box mA(\Box z, y)$. Therefore,

$$\Box F \subseteq \Box mA(\Box z, y) \subseteq \bigcup_{y \in F} \Box mA(\Box z, y).$$

Now, by Theorem 2.10, $\Box mA(x, y) \subseteq F$, for all $x, y \in F$. Thus $\Box mA(\Box x, y) \subseteq \Box F$, for all $x, y \in F$. Therefore, $\bigcup_{x,y \in F} \Box mA(\Box x, y) \subseteq \Box F$. $\Box$

Definition 2.14. A (normal)filter F of a modal BE–algebra X is called a modal (normal)BE–filter if it closed under $\Box$ (i.e. if $x \in F$, then $\Box x \in F$, for all $x \in X$).

Example 2.15. In Example 2.2(i), $F_1 = \{1, a\}$ is a modal BE–filter of X and $F_2 = \{1, b\}$ is a filter but it is not a modal BE–filter.

Theorem 2.16. *If $\{F_i\}_{i \in I}$ is a family of modal BE–filters of X, then $\bigcap_{i \in I} F_i$ is a modal BE–filter of X, too.*

Proposition 2.17. *Let X be a modal BE–algebra and* $\ker(\Box) := \{x \in X : \Box x = 1\}$*. Then*

(i) $\ker(\Box)$ *is a filter of X,*
(ii) $\ker(\Box)$ *is closed under $\Box$.*

Proof. (i). Since $\Box 1 = 1$, we have $1 \in \ker(\Box)$. Hence $\ker(\Box)$ is a non-empty set. Now, let $x * y \in \ker(\Box)$ and $x \in \ker(\Box)$. Thus $\Box(x * y) = \Box(x) = 1$. By using $(MBE4)$ and $(BE3)$ we have

$$1 = \Box(x * y) = \Box x * \Box y = 1 * \Box y = \Box y.$$

Therefore, $y \in \ker(\Box)$.

(ii). Let $x \in \ker(\Box)$. Then $\Box x = 1$. Using $(MBE1)$ and $(MBE3)$ we have

$$1 = \Box 1 = \Box(\Box x).$$

Therefore, $\Box x \in \ker(\Box)$. $\Box$

Definition 2.18. The $\ker(\Box)$ is called the $\Box$–tautology filter related to BE–algebra X or is called a BE–$\Box$–tautology filter.

Example 2.19. In Example 2.2(i), $F_1 = \{1\}$ is a BE–$\Box$–tautology filter.

Proposition 2.20. *Let $[\alpha, 1] = \{x \in X : \alpha \leq x \leq 1\}$, where X is a commutative self distributive BE–algebra and $\alpha \in X$. Then* $\ker(\Box_\alpha) = [\alpha, 1]$, *where* $\Box_\alpha(x) = \alpha * x$.

Proof. Let $x \in [\alpha, 1]$. Then, $\alpha \leq x \leq 1$. Hence by using self distributivity and commutativity $1 = \alpha * \alpha \leq \alpha * x \leq \alpha * 1 = 1$ and so $\Box_\alpha(x) = \alpha * x = 1$. Therefore, $x \in \ker(\Box_\alpha)$.

Conversely, let $x \in \ker(\Box_\alpha)$. Then $\Box_\alpha(x) = 1$, i.e. $\alpha * x = 1$. Hence $\alpha \leq x$ and so $x \in [\alpha, 1]$. Therefore, $[\alpha, 1]$ is a BE–$\Box$–tautology filter. $\Box$

Definition 2.21. An algebra $(X; *, \Box, 1)$ of type $(2, 1, 0)$ is called a dual modal BE–algebra if it satisfies the following:

(BE) $(X; *, 1)$ is a BE–algebra,
$(MBE1)$ $\Box 1 = 1$,
$(dMBE2)$ $x \leq \Box x$,
$(MBE3)$ $\Box x = \Box\Box x$,
$(MBE4)$ $\Box(x * y) = \Box x * \Box y$, for all $x, y \in X$.

Example 2.22. (i). Let $X = \{1, a, b, c\}$. Define the operations " $*$ " and "$\Box$" on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	c
c	1	b	1	1

x	1	a	b	c
$\Box$	1	a	1	1

Then, $(X; *, \Box, 1)$ is a dual modal BE–algebra.

(ii). Let $X = \mathbb{N}$ and " $*$ " be the binary operation on X defined by

$$x * y = \begin{cases} y, & \text{if } x = 1 \\ 1, & \text{if } x \neq 1 \end{cases}$$

Then, $(X; *, 1)$ is a BE–algebra. Now, we define the unary operation "$\Box$" on X as:

$$\overbrace{(\Box \cdots \Box)}^{n-\text{times}} x = \begin{cases} 1, & \text{if } x = 1 \\ (x+n) - (n-1), & \text{if } x \neq 1 \end{cases}$$

Therefore, $(X; *, \Box, 1)$ is a dual modal BE–algebra.

Proposition 2.23. *Let X be a self distributive BE–algebra. Define $\Box_\alpha(x) = \alpha * x$, for all $x \in X$. Then $(X; *, \Box_\alpha, 1)$ is a dual modal BE–algebra.*

Proof. Clearly, $(X; *, 1)$ is a BE–algebra. For $(MBE1)$, we have $\Box_\alpha(1) = \alpha * 1 = 1$, by $(BE3)$. Since $x * (\alpha * x) = \alpha * (x * x) = \alpha * 1 = 1$, we have $x \leq \alpha * x$, i.e. $x \leq \Box_\alpha(x)$. Hence $(dMBE3)$ is valid. For $(MBE4)$,

$$\Box_\alpha(\Box_\alpha(x)) = \alpha * (\alpha * x)) = (\alpha * \alpha) * (\alpha * x) = 1 * (\alpha * x) = \alpha * x = \Box_\alpha(x).$$

Also, $\Box_\alpha(x * y) = \alpha * (x * y) = (\alpha * x) * (\alpha * y) = \Box_\alpha(x) * \Box_\alpha(y)$. $\Box$

3. Quotient modal BE–algebra

For a modal normal BE–filter F of X we define the binary relation $\sim_F$ in the following way:

$$x \sim_F y \;\Leftrightarrow\; x * y \in F \text{ and } y * x \in F.$$

Clearly $\sim_F$ is reflexive and symmetry. Now, let $x \sim_F y$ and $y \sim_F z$. Then $x * y, y * x, y * z, z * y \in F$. Since F is a normal filter, $(y * z) * (x * z) \in F$. Hence $x * z \in F$. By a similar way, $z * x \in F$. Consequently, $x \sim_F z$. So, $\sim_F$ is a transitive relation. Thus $\sim_F$ is an equivalence relation on X.

Theorem 3.1. [19] *Let F be a normal filter of a BE–algebra X. Then $\sim_F$ is a congruence relation on X.*

We have

$$F_x = \{y \in X : x \sim_F y\}$$

Also, define the operations "$\Box$" and " $*$ " on congruence classes as follows:

$$\Box F_x = F_{\Box x} \text{ and } F_x * F_y = F_{x*y}.$$

We show that $\Box$ and $*$ on congruence classes are well defined. Let $F_x = F_y$. Then $x \sim_F y$ and $y \sim_F x$, i.e. $x * y, y * x \in F$. We get that $\Box x * \Box y = \Box(x * y) \in F$ and $\Box y * \Box x = \Box(y * x) \in F$, since F is a modal filter. Therefore, $\Box x \sim_F \Box y$, i.e. $F_{\Box x} = F_{\Box y}$. Equivalently, $\Box F_x = \Box F_y$. Also, let $F_x = F_y$ and $F_u = F_v$, i.e. $x \sim_F y$, $y \sim_F x$ and $u \sim_F v$, $v \sim_F u$. Hence $x * y, y * x \in F$ and $u * v, v * u \in F$. Since F is a normal filter, $(z * x) * (z * y), (z * y) * (z * x) \in F$. Therefore, $z * x \sim_F z * y$. By a similar way $x * z \sim_F y * z$. Now, $x * u \sim_F y * u$

and $y * u \sim_F y * v$. Since $\sim_F$ is transitive, we have $x * u \sim_F y * v$. Therefore, $F_{x*u} = F_{y*v}$, i.e. $F_x * F_u = F_y * F_v$. Set $\frac{X}{\sim_F} := \{[x] : x \in X\} = \{F_x : x \in X\}$. It can be easily seen that $F_1 = F$. Since:

$$\begin{aligned} x \in F_1 &\Leftrightarrow x \sim_F 1 \\ &\Leftrightarrow x * 1 = 1, 1 * x = x \in F \\ &\Leftrightarrow x \in F, \end{aligned}$$

we define a binary operation " $*$ " on $\frac{X}{\sim_F}$ as follows:

$$F_x * F_y = F_{x*y} \text{ and } \Box F_x = F_{\Box x}.$$

We saw in above, this binary operation is well-defined.

We can define an order such as " $\leq$ " on $\frac{X}{\sim_F}$ as follows:

$$F_x \leq F_y \Leftrightarrow x * y = 1.$$

Theorem 3.2. $(\frac{X}{\sim_F}; *, \Box, F)$ *is a modal BE–algebra.*

Proof. By Proposition 3.11 of [19], $(\frac{X}{\sim_F}; *, F_1)$ is a BE–algebra,

$(MBE1)$ $\Box F_1 = F_{\Box 1} = F_1 = F = 1_{\frac{X}{\sim_F}}$,
$(MBE2)$ $\Box F_x = F_{\Box x} \leq F_x$, since $\Box x * x = 1$,
$(MBE3)$ $\Box F_x = F_{\Box x} = F_{\Box\Box x} = \Box F_{\Box x} = \Box\Box F_x$,
$(MBE4)$ $\Box(F_x * F_y) = \Box F_{x*y} = F_{\Box(x*y)} = F_{\Box x * \Box y} = F_{\Box x} * F_{\Box y} = \Box F_x * \Box F_y$. □

Theorem 3.3. *Let F be a modal normal BE–filter of a commutative modal BE–algebra X. Then $(\frac{X}{\sim_F}; *, \Box, F)$ is a commutative modal BE–algebra.*

Proof. Let $F_x, F_y \in \frac{X}{\sim_F}$. Then

$$\begin{aligned} (F_x * F_y) * F_y &= (F_{x*y}) * F_y \\ &= F_{(x*y)*y} \\ &= F_{(y*x)*x} \\ &= F_{y*x} * F_x \\ &= (F_y * F_x) * F_x. \end{aligned}$$ □

Example 3.4. Let $X = \{1, a, b, c\}$. Define the operations " $*$ " and "$\Box$" on X as follow:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	c
c	1	b	1	1

x	1	a	b	c
$\Box$	1	a	b	a

Then $(X; *, \Box, 1)$ is a modal BE–algebra, $F = \{1, b\}$ is a modal normal BE–filter, $F_1 = \{b, 1\} = F$, $F_a = \{a, c\}$, $F_b = \{b, c, 1\}$ and $F_c = \{a, c\}$. Hence

$(\frac{X}{\sim_F};*,\Box,F_1=F)$ is a modal BE–algebra, where $\frac{X}{\sim_F}=\{\{b,1\},\{a,c\},\{b,c,1\}\}$ with the following table:

$*$	F_1	F_a	F_b
F_1	F_1	F_a	F_b
F_a	F_1	F_1	F_1
F_b	F_1	F_a	F_1

Let $(X;*,\Box,1)$ be a dual modal BE–algebra. We define

$$F_x=\{y\in X: x\sim_F y\}$$

Also, define the operations "$\Box$" and "$*$" on congruence classes as follows:

$$\Box F_x=F_{\Box x} \text{ and } F_x*F_y=F_{x*y}.$$

Then by a similar way $(\frac{X}{\sim_F};*,\Box,F)$ is a dual modal BE–algebra. Because, it remains only to prove the condition $(dMBE2)$. Since $x*\Box x=1$, we get

$$F_x\leq\Box F_x=F_{\Box x}.$$

Theorem 3.5. *Let X be a modal BE–algebra. Then $(\frac{X}{\sim_F};*,\Box,F)$ is a dual modal BE–algebra if and only if the relation $\leq$ has been defined as follows:*

$$F_x\leq F_y \iff y*x=1.$$

Proof. Since X is a modal BE–algebra, we have $\Box x*x=1$. Thus $F_x\leq F_{\Box x}=\Box F_x$. Therefore, the condition $(dMBE2)$ is valid. $\Box$

Theorem 3.6. *Let $(X;*,\Box,1)$ be a dual modal BE–algebra. Let the relation $\leq$ has been defined as*

$$F_x\leq F_y \iff y*x=1.$$

Then $(\frac{X}{\sim_F};*,\Box,F)$ is a modal BE–algebra. In particular, if $(X;*,\Box,1)$ is a commutative dual modal BE–algebra and the operator $\Box$ is one-to-one, then the structure $(X;*,\Box,1)$ is a modal BE–algebra.

Proof. Clearly $(\frac{X}{\sim_F};*,\Box,F)$ is a modal BE–algebra by Theorem 3.2. Since X is a commutative BE-algebra then every filter is a normal filter. Hence, $F_1=\{1\}$ is a normal filter. Now, let

$$F_x=\{y\in X: x\sim_F y \text{ and } \Box x=\Box y\}.$$

Hence the equivalence class $F_x=\{x\}$ (in particular $F_1=\{1\}$), since $\Box$ is one-to-one. Thus the natural map $\pi\colon X\to\frac{X}{\sim_{F_1}}$ where $\pi(x)=[x]_{F_1}$, is an isomorphism. Now, we can easily see that $(X;*,\Box,1)$ is a modal BE–algebra. $\Box$

4. Conclusion and future research

In this paper, we introduced the notion of modal BE–algebras and get some results.

In the future work, we try assemble of calculus relative to different kinds of modal algebraic structure.

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