Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
31 (2015), 227–232
www.emis.de/journals
ISSN 1786-0091

2-PRIMAL ORE EXTENSIONS OVER WEAK σ-RIGID RINGS

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Abstract. Let R be a ring, σ an automorphism of R and δ a σ-derivation of R. We recall that a ring R is said to be a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, where $P(R)$ denotes the prime radical of R.

It is known that, if R is a Noetherian ring, σ an automorphism of R such that $a\sigma(a) \in P(R)$ implies $a \in P(R)$ and δ a σ-derivation of R such that R is a δ-ring with $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$, then $R[x;\sigma,\delta]$ is a 2-primal Noetherian ring. We investigate this result if $P(R)$ is replaced with $N(R)$ and prove that if R is a Noetherian ring, which is also an algebra over $\mathbb{Q}$, σ an automorphism of R such that $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$, where $N(R)$ denotes the set of nilpotent elements of R and δ a σ-derivation of R such that R is a δ-ring with $\delta(P(R)) \subseteq P(R)$, then $R[x;\sigma,\delta]$ is 2-primal.

1. Introduction

In this article a ring R always means an associative ring. The field of complex numbers is denoted by $\mathbb{C}$, the field of rational numbers is denoted by $\mathbb{Q}$, and the set of positive integers is denoted by $\mathbb{N}$. $\mathrm{Spec}(R)$ denotes the set of prime ideals of R. $\mathrm{MinSpec}(R)$ denotes the set of minimal prime ideals of R. For a ring R the prime radical and the set of nilpotent elements are denoted by $P(R)$ and $N(R)$ respectively. We note that for a commutative ring $P(R)$ and $N(R)$ coincide.

Now let R be a ring, σ an endomorphism of R and δ a σ-derivation of R. Recall that the skew polynomial ring $R[x;\sigma,\delta]$ is the set of polynomials

$$\left\{\sum_{i=0}^{n} x^i a_i,\ a_i \in R,\ n \in \mathbb{N}\right\}$$

with usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x;\sigma,\delta]$ to be of the

2010 *Mathematics Subject Classification.* 16S36, 16N40, 16P40, 16S32, 16W20, 16W25.
Key words and phrases. 2-primal, minimal prime, automorphism, derivation, Ore extensions.

form $f(x) = \Sigma_{i=0}^{n} x^i a_i$, $a_i \in R$ as in McConnell and Robson [7]. This definition of non commutative polynomial rings was first introduced by Ore 1933. Skew polynomial rings were one of the earliest examples in non commutative algebra also known as Ore extensions. We denote $R[x;\sigma,\delta]$ by $O(R)$. In case δ is the zero map, we denote $R[x;\sigma]$ by $S(R)$.

We recall the following definitions:

Definition 1.1 (Krempa [5])**.** Let R be a ring and σ an endomorphism of R. Then σ is said to be a rigid endomorphism if $a\sigma(a) = 0$ implies that $a = 0$, for $a \in R$. The ring R is said to be a σ-rigid ring if there exists a σ-rigid endomorphism R.

For example let $R = \mathbb{C}$, and $\sigma\colon \mathbb{C} \to \mathbb{C}$ be the map defined by $\sigma(a+ib) = a - ib$, a, $b \in R$. Then it can be seen that σ is a rigid endomorphism of R.

Definition 1.2 (Kwak [6])**.** Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 1.3 (Example (2) of [6]). Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field.

Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $\sigma\colon R \to R$ defined by

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.$$

Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.

We note that the above ring is not σ-rigid. For let $0 \neq a \in F$. Then

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

but

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Another related area of interest since recent past has been the study of 2-primal rings. This involves the notions of prime radical and the set of nilpotent elements of a ring R. Further more the concept of completely prime ideals and the completely semi-prime ideals are also studied in this area. Recall that a ring R is 2-primal if $N(R) = P(R)$; i.e. if the prime radical is a completely semi-prime ideal. An ideal I of a ring R is called completely semi-prime if $a^2 \in I$ implies $a \in I$ for $a \in R$. We note that a reduced ring (i.e. a ring with no non-zero nilpotent elements) is 2-primal and a commutative ring is also 2-primal.

Kwak in [6] also establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. The property is also extended to the skew polynomial ring $S(R)$.

Remark 1.4 (Proposition 2 of [3]). If R is a ring and σ an automorphism of R such that R is a $\sigma(*)$-ring, then R is 2-primal.

Proof. We will show that $P(R)$ is a completely semi-prime ideal of R. Let $a \in R$ be such that $a^2 \in P(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R)$. Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$. □

Now, let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$, σ an automorphism of R such that $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ and δ a σ-derivation of R such that R is a δ-ring with $\delta(P(R)) \subseteq P(R)$, then $R[x;\sigma,\delta]$ is 2-primal. This is proved in Theorem (3.5).

2. Ore extensions over weak σ-rigid 2-primal rings

Ouyang in [8] introduced weak σ-rigid rings, where σ is an endomorphism of a ring R. These rings are related to 2-primal rings.

Definition 2.1 (Ouyang [8])**.** Let R be a ring and σ an endomorphism of R such that $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$. Then R is called a weak σ-rigid ring.

Example 2.2 (Example (2.1) of Ouyang [8]). Let σ be an endomorphism of a ring R such that R is a σ-rigid ring. Let

$$A = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix},\ a, b, c, d \in R \right\}$$

be a subring of $T_3(R)$, the ring of upper triangular matrices over R. Now σ can be extended to an endomorphism $\overline{\sigma}$ of A by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$. Then it can be seen that A is a weak $\overline{\sigma}$-rigid ring.

Theorem 2.3. *Let R be a Noetherian ring. Let σ be an automorphism of R. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R.*

Proof. See Theorem (3) of [3]. □

Proposition 2.4. *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then*

(1) *for any completely prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $O(P) = P[x;\sigma,\delta]$ is a completely prime ideal of $O(R)$,*
(2) *for any completely prime ideal U of $O(R)$, $U \cap R$ is a completely prime ideal of R.*

Proof. See Proposition (2.6) of [2]. □

Proposition 2.5. *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring. Then $U \in$ MinSpec(R) implies that $U(O(R)) = U[x;\sigma,\delta]$ is a completely prime ideal of $O(R) = R[x;\sigma,\delta]$.*

Proof. See Proposition (4) of [3]. □

Corollary 2.6. *Let R be a commutative Noetherian ring. Let σ be an automorphism of R. Then R is a 2-primal weak σ-rigid ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R.*

Proof. See Corollary (1) of [3]. □

Theorem 2.7 (Theorem (4) of [1])**.** *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a δ-ring and $\delta(P(R)) \subseteq P(R)$. Then R is 2-primal.*

Proof. Define a map $\rho\colon R/P(R) \to R/P(R)$ by $\rho(a + P(R)) = \delta(a) + P(R)$ for $a \in R$ and $\tau\colon R/P(R) \to R/P(R)$ a map by $\tau(a + P(R)) = \sigma(a) + P(R)$ for $a \in R$, then it can be seen that τ is an automorphism of $R/P(R)$ and ρ is a τ - derivation of $R/P(R)$. Now $a\delta(a) \in P(R)$ if and only if $(a + P(R))\rho(a + P(R)) = P(R)$ in $R/P(R)$. Thus as in Proposition (5) of Hong, Kim and Kwak [4], R is a reduced ring and, therefore, R is 2-primal. □

Corollary 2.8 (Corollary (2) of [1])**.** *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R such that R is moreover a δ-ring and $\delta(P(R)) \subseteq P(R)$. Let $P \in MinSpec(R)$ be such that $\sigma(P) = P$. Then $O(P)$ is a completely prime ideal of $O(R)$.*

Proof. R is 2-primal by Theorem (2.5), and so by Proposition (3) of [1] $\delta(P) \subseteq P$. Further more as mentioned in Proposition (2.3), P is a completely prime ideal of R. Now use Proposition (2.4), and the proof is complete. □

3. Main Results

Proposition 3.1. *Let R be a 2-primal ring. Then for any minimal prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $O(P)$ is a completely prime ideal of $O(R)$.*

Proof. Since R is a 2-primal ring, so every minimal prime ideal of R is completely prime by Proposition (1.11) of Shin [9]. This implies P is a completely prime ideal of R with $\delta(P) \subseteq P$. Therefore by Theorem (2.4) $O(P)$ is a completely prime ideal of $O(R)$. □

Proposition 3.2. *Let R be a Noetherian ring. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R such that $\delta(P) \subseteq P$ for all $P \in MinSpec(R)$. Then $O(P)$ is a completely prime ideal of $O(R)$.*

Proof. Since R is a $\sigma(*)$-ring, so by Theorem (2.3), we have $\sigma(P) = P$ and P is a completely prime ideal of R. So, P is a completely prime ideal of R with $\delta(P) \subseteq P$. Therefore, by Theorem (2.4), $O(P)$ is a completely prime ideal of $O(R)$. □

Proposition 3.3. *Let R be a commutative Noetherian ring. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ a σ-derivation of R such that R is a δ-ring with $\delta(P(R)) \subseteq P(R)$. Then $P \in MinSpec(R)$ with $\sigma(P) = P$ implies P is a completely prime ideal of R.*

Proof. Result is obvious by using Corollary (2.6). Clearly, R is 2-primal by Theorem (2.7). Now, let $P \in \mathrm{MinSpec}(R)$ be such that $\sigma(P) = P$. Then, by Corollary (2.6), P is a completely prime ideal of R. $\square$

Proposition 3.4. *Let R be a commutative Noetherian ring. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ a σ-derivation of R such that R is a δ-ring with $\delta(P(R)) \subseteq P(R)$. Then $\delta(P) \nsubseteq P$ implies that $\sigma(P) \neq P$.*

Proof. Suppose $\delta(P) \nsubseteq P$; i.e., let $a \in P$ be such that $\delta(a) \notin P$. Let us suppose $\sigma(P) = P$. Then by Theorem (3.3) P is a completely prime ideal of R. Therefore for any $a \in P$ there exists $b \notin P$ such that $ab \in P(R)$ by Corollary (1.10) of Shin [9]. Now $\delta(P(R)) \subseteq P(R)$, and therefore $\delta(ab) \in P(R)$ i.e., $\delta(a)\sigma(b)+a\delta(b) \in P(R) \subseteq P$. Now $a\delta(b) \in P$ implies $\delta(a)\sigma(b) \in P$ implies either $\delta(a) \in P$ or $\sigma(b) \in P$.

(1) If $\delta(a) \in P$, a contradiction.

(2) If $\sigma(b) \in P$, but $b \notin P$ implies $\sigma(b) \notin \sigma(P) = P$, a contradiction.

Therefore, $\sigma(P) \neq P$. $\square$

Theorem 3.5. *Let R be a commutative Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ a σ-derivation of R such that R is a δ-ring with $\delta(P(R)) \subseteq P(R)$. Then $O(R)$ is 2-primal if and only if $O(P(R)) = P(O(R))$.*

Proof. Let $O(R)$ be 2-primal. Now R is 2-primal by Theorem (2.7) and R is weak σ-rigid implies $\sigma(P) = P$ by Corollary (2.6). Then by Corollary (2.8) $P(O(R)) \subseteq O(P(R))$. Let $f(x) = \sum_{j=0}^{n} x^j a_j \in O(P(R))$. Now R is a 2-primal subring of $O(R)$ which implies that a_j is nilpotent and thus $a_j \in N(O(R)) = P(O(R))$, and so we have $x^j a_j \in P(O(R))$ for each j, $0 \leq j \leq n$, which implies that $f(x) \in P(O(R))$. Hence $O(P(R)) = P(O(R))$.

Conversely, suppose $O(P(R)) = P(O(R))$. We will show that $O(R)$ is 2-primal. Let $g(x) = \Sigma_{i=0}^{n} x^i b_i \in O(R), b_n \neq 0$, be such that $(g(x))^2 \in P(O(R)) = O(P(R))$. We will show that $g(x) \in P(O(R))$. Now leading coefficient $\sigma^{2n-1}(b_n)b_n \in P(R) \subseteq P$, for all $P \in \mathrm{MinSpec}(R)$. Now since R is 2-primal weak σ-rigid ring we have $\sigma(P) = P$ and P is completely prime. Therefore, we have $b_n \in P$, for all $P \in \mathrm{MinSpec}(R)$, i.e., $b_n \in P(R)$. Now since $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in \mathrm{MinSpec}(R)$, we get $(\Sigma_{i=0}^{n-1} x^i b_i)^2 \in P(O(R)) = O(P(R))$ and as above we get $b_{n-1} \in P(R)$. With the same process in a finite number of steps we get $b_i \in P(R)$ for all i, $0 \leq i \leq n$. Thus we have $g(x) \in O(P(R))$, i.e., $g(x) \in P(O(R))$. Therefore $P(O(R))$ is a completely semiprime ideal of $O(R)$. Hence $O(R)$ is 2-primal. $\square$

Theorem 3.6. *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ a σ-derivation of R such that R is a δ-ring with $\delta(P(R)) \subseteq P(R)$. Then $O(R)$ is 2-primal.*

Proof. Let $P_1 \in \mathrm{MinSpec}(R)$. Now R is 2-primal by Theorem (2.7) and Corollary (2.6) implies that $\sigma(P_1) = P_1$. Therefore Theorem (2.3) of [2] implies that $O(P_1) \in \mathrm{MinSpec}(O(R))$. Also for any $P \in \mathrm{MinSpec}(O(R))$ such that $\sigma(P \cap R) = P \cap R$, we have that $P \cap R \in \mathrm{MinSpec}(R)$ by Theorem (2.3) of [2]. Therefore, $O(P(R)) = P(O(R))$, and now the result is now obvious by using Theorem (3.5). □

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Received October 9, 2013.

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