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OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF HIGHER ORDER NEUTRAL DIFFERENCE EQUATIONS

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Abstract. In this paper, sufficient conditions are obtained so that every solution of neutral functional difference equation

$$\Delta^m(y_n - p_n y_{\tau(n)}) + v_n G(y_{\sigma(n)}) - u_n H(y_{\alpha(n)}) = f_n,$$

oscillates or tends to zero or $\pm\infty$ as $n \to \infty$, where different symbols have their usual meaning. In particular, we extend the results of [14] Rath et al. (2010) to the case when G has sublinear growth at ∞. Our results also apply to the neutral equation

$$\Delta^m(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n,$$

where q_n has sign changes. This paper expands some recent results.

1. Introduction

This article concerns the oscillation of solutions to the neutral functional difference equation

$$\Delta^m(y_n - p_n y_{\tau(n)}) + v_n G(y_{\sigma(n)}) - u_n H(y_{\alpha(n)}) = f_n, \tag{1.1}$$

where Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$, p_n, v_n, u_n and f_n are infinite sequences of real numbers with $v_n > 0, u_n \geq 0, G, H \in C(R, R)$. Further, we assume $\{\tau(n)\}, \{\sigma(n)\}$, and $\{\alpha(n)\}$ are monotonic increasing and unbounded sequences such that $\tau(n) \leq n, \sigma(n) \leq n$ and $\alpha(n) \leq n$ for every n. Different ranges of $\{p_n\}$ are considered. The positive integer $m \geq 2$, can take both odd and even values.

Let N_1 be a fixed non-negative integer. Let $N_0 = \min\{\tau(N_1), \sigma(N_1), \alpha(N_1)\}$. By a solution of (1.1) we mean a real sequence $\{y_n\}$ which is defined for all positive integer $n \geq N_0$ and satisfies (1.1) for $n \geq N_1$. Clearly, if the initial condition

$$y_n = a_n \quad \text{for} \quad N_0 \leq n \leq N_1 + m - 1, \tag{1.2}$$

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is given then the equation (1.1) has a unique solution satisfying the given initial condition (1.2). A solution $\{y_n\}$ of (1.1) is said to be oscillatory if for every positive integer $n_0 \geq N_1$, there exists $n \geq n_0$ such that $y_n y_{n+1} \leq 0$, otherwise $\{y_n\}$ is said to be non-oscillatory.

In this work we assume the existence of solutions and study only their qualitative behaviour. For existence and uniqueness of solutions, we refer the reader to [2, 5]. The function G is said to have linear growth (or to be linear) at infinity, if $\lim_{x\to\infty} |G(x)|/x$ is a positive constant. G is super-linear if $\lim_{x\to\infty} |G(x)|/x = \infty$, and G is sub-linear if $\lim_{x\to\infty} |G(x)|/x = 0$.

In the sequel, unless otherwise specified, when we write a functional inequality, it will be assumed to hold for all n sufficiently large. Our results will use the following hypothesis:

(H0) For any sequence $\{x_n\}$, if $\liminf_{n\to\infty} |x_n| > 0$ then $\liminf_{n\to\infty} |G(x_n)| > 0$.
(H1) $xG(x) > 0$ for $x \neq 0$.
(H2) H is bounded.
(H3) $\sum_{n=n_0}^{\infty} v_n = \infty$.
(H4) $\sum_{n=n_0}^{\infty} n^{m-1} u_n < \infty$.
(H5) There exists a bounded sequence $\{F_n\}$ such that $\Delta^m F_n = f_n$.
(H6) The sequence $\{F_n\}$ satisfies $\lim_{n\to\infty} F_n = 0$.

We observe that there are only few publications [12, 15, 13, 14, 16, 18] on the oscillatory behaviour of higher order ($m \geq 2$) neutral difference equations (1.1) with positive and negative coefficients. The article [16] study (1.1) for $m = 1$ with restrictions $G(u) = u$ and $f \equiv 0$. In a recent publication [14], Rath et al. obtained some results assuming the conditions

$$\liminf_{|u|\to\infty} G(u)/u > 0, \tag{1.3}$$

and

$$G \quad \text{is non-decreasing}, \tag{1.4}$$

in order to study the oscillatory behaviour of solutions of the neutral equation (1.1). The motivation for this work is derived from the following example.

Example 1.1.

$$\Delta^m\left(y_n - \frac{1}{8}y_{n-1}\right) + \frac{1}{2^{9\alpha}} y_{n-3}^{\alpha} = 2^{-3n\alpha}, \tag{1.5}$$

where m is any integer ≥ 1, α is the quotient of any two odd integers. If $\alpha < 1$ then here, in this example, we find $G(u) = u^\alpha$, does not satisfy (1.3) and clearly $y_n = 2^{-3n}$ is a solution of (1.5), which tends to zero as $n \to \infty$.

It is clear from the above example that, the study in [14] has left out a class of neutral equations because of the assumption (1.3).

Since the conditions (1.3) or (1.4) are incompatible to the condition that G is bounded, we relaxed these conditions and thus, could generalize, improve the work in [14] and apply it to study

$$\Delta^m(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n, \tag{1.6}$$

where q_n is allowed to change sign. There is almost no result for (1.6)($m > 2$) with oscillatory q_n. As majority of the existing publications concerned with (1.6) (see [8, 7, 9, 11, 10, 17, 18]) have results for positive q_n, this article generalizes these results.

2. Neutral equations with positive and negative coefficients

To begin with, we state some lemmas which would be useful for our work.

Lemma 2.1. [1] *Let $\{f_n\}, \{q_n\}$ and $\{p_n\}$ be sequences of real numbers defined for $n \geq N_0 > 0$ such that*

$$f_n = q_n - p_n q_{\tau(n)}, \quad n \geq N_1 \geq N_0,$$

where $\{\tau(n)\}$ is an increasing unbounded sequence such that $\tau(n) \leq n$. Suppose that p_n satisfies one of the following three conditions

$$-1 < -b_1 \leq p_n \leq 0, \quad -b_2 \leq p_n \leq -b_3 < -1, \quad \text{and} \quad 0 \leq p_n \leq b_4 < \infty,$$

$\forall n$, where b_1, b_2, b_3 and b_4 are constants. If $q_n > 0$ for $n \geq N_0, \liminf_{n\to\infty} q_n = 0$ and $\lim_{n\to\infty} f_n = L$ exists then $L = 0$.

Lemma 2.2. [6] *If $\sum u_n$ and $\sum v_n$ are two positive term series such that $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = l$, where l is a non-zero finite number, then the two series converge or diverge together. If $l = 0$ then $\sum v_n$ is convergent implies the convergence of $\sum u_n$. If $l = \infty$ then $\sum v_n$ is divergent implies the divergence of $\sum u_n$.*

Lemma 2.3. [1, 11] *Let z_n be a real valued function defined for $n \in N(n_0) = \{n_0, n_0+1, \ldots\}, n_0 \geq 0$ and $z_n > 0$ with $\Delta^m z_n$ of constant sign on $N(n_0)$ and not identically zero.Then there exists an integer p, $0 \leq p \leq m-1$, with $m+p$ odd for $\Delta^m z_n \leq 0$ and $(m+p)$ even for $\Delta^m z_n \geq 0$, such that*

$$\Delta^i z_n > 0 \quad \text{for} \quad n \geq n_0, 0 \leq i \leq p,$$

and

$$(-1)^{p+i}\Delta^i z_n > 0, \quad \text{for} \quad n \geq n_0,\, p+1 \leq i \leq m-1.$$

Before we state and prove our next result, we need the following definition and further discussion.

Definition 2.1. Define the factorial function(cf [5, page-20]) by

$$n^{(k)} := n\,(n-1)\cdots(n-k+1)\,,$$

where $k \leq n$ and $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Note that $n^{(k)} = 0$, if $k > n$.

Then we have

$$\Delta n^{(k)} = k n^{(k-1)}, \tag{2.1}$$

where $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and Δ is the forward difference operator. Next, we present our last lemma, where $\Phi(\infty)$ means $\lim_{n\to\infty} \Phi(n)$.

Lemma 2.4. [14] *Let $p \in \mathbb{N}$ and $x(n)$ be a non oscillatory sequence which is positive for large n. If there exists an integer $p_0 \in \{0, 1, \dots, p-1\}$ such that $\Delta^{p_0} w(\infty)$ exits (finite) and $\Delta^i w(\infty) = 0$ for all $i \in \{p_0 + 1, \dots, p-1\}$. Then*

$$\Delta^p w(n) = -x(n), \tag{2.2}$$

implies

$$\Delta^{p_0} w(n) = \Delta^{p_0} w(\infty) + \frac{(-1)^{p-p_0-1}}{(p-p_0-1)!} \sum_{i=n}^{\infty} (i + p - p_0 - 1 - n)^{(p-p_0-1)} x(i), \tag{2.3}$$

for all sufficiently large n.

Now, we state our first main result.

Theorem 2.5. *Suppose that* (H0)–(H5) *hold. Assume that there exists a positive constant b_1 such that the sequence $\{p_n\}$ satisfies the condition*

$$0 \le p_n \le b_1 < 1, \quad or \quad -1 < -b_1 \le p_n \le 0. \tag{2.4}$$

Then every non-oscillatory solution of (1.1) *is bounded.*

Proof. Let $y = \{y_n\}$ be any non-oscillatory solution of (1.1) for $n \ge N_1$, where N_1 is a fixed positive integer. Then $y_n > 0$ or $y_n < 0$. Suppose $y_n > 0$ eventually. There exits positive integer $n_0 \ge N_1 > 0$ such that $y_n > 0, y_{\tau(n)} > 0$, $y_{\sigma(n)} > 0$ and $y_{\alpha(n)} > 0$ for $n \ge n_0$. For simplicity of notation, define for $n \ge n_0$,

$$z_n = y_n - p_n y_{\tau(n)}. \tag{2.5}$$

Further, we define for $n \ge n_0$

$$T_n = \frac{(-1)^{m-1}}{(m-1)!} \sum_{i=n}^{\infty} (i - n + m - 1)^{(m-1)} u_i H(y_{\alpha(i)}). \tag{2.6}$$

Note that, due to the assumptions (H2) and (H4), $\{T_n\}$ is a well defined real sequence which is convergent. This implies

$$\lim_{n\to\infty} T_n = 0 \tag{2.7}$$

and

$$\Delta^m T_n = -u_n H(y_{\alpha(n)}). \tag{2.8}$$

Set,

$$w_n = y_n - p_n y_{\tau(n)} + T_n - F_n. \tag{2.9}$$

From (1.1), (2.8), and (2.9), it follows due to (H1) that

$$\Delta^m w_n = -v_n G(y_{\sigma(n)}) \leq 0. \tag{2.10}$$

Then there exists $n_1 \geq n_0$ such that $w_n, \Delta w_n, \Delta^2 w_n, \dots, \Delta^{m-1} w_n$ are monotonic and of constant sign for $n \geq n_1$. For the sake of a contradiction assume that y_n is not bounded. Then there exists a sub sequence $\{y_{n_k}\}$ such that

$$n_k \to \infty,\, y_{n_k} \to \infty \quad \text{as} \quad k \to \infty,$$

and

$$y(n_k) = \max\{y_n : n_1 \leq n \leq n_k\}. \tag{2.11}$$

Since $\tau(n) \to \infty$, $\sigma(n) \to \infty$, and $\alpha(n) \to \infty$ as $n \to \infty$, we may choose k large enough so that for $\tau(n_k) \geq n_1$, $\sigma(n_k) \geq n_1$ and $\alpha(n_k) \geq n_1$. For $0 < \epsilon$, because of (2.7) and (H5), we can find a positive integer n_2 and a constant γ such that, for $k \geq n_2 \geq n_1$ implies $|T_{n_k}| < \epsilon$ and $|F_{n_k}| < \gamma$. If the condition $0 \leq p_n \leq b_1 < 1$ holds, then using (2.9) and (2.11) we obtain

$$w_{n_k} \geq y_{n_k}(1 - b_1) - \epsilon - \gamma,$$

for $k \geq n_2$. Similarly, if $-1 < -b_1 \leq p_n \leq 0$ holds, then for $k \geq n_2$, we have

$$w_{n_k} \geq y_{n_k} - \epsilon - \gamma.$$

Taking $k \to \infty$, we find $\lim_{n\to\infty} w_n = \infty$. Since $w_n, \Delta w_n, \dots, \Delta^{m-1} w_n$ are monotonic and of constant sign, it follows that $w_n > 0$ and $\Delta w_n > 0$.

Consequently, by Lemma 2.3, $w_n > 0$, and $\Delta^m w_n \leq 0$ imply $\Delta^{m-1} w_n > 0$ for $n \geq n_2 \geq n_1$.

Next, we show that y_n is bounded below by a positive constant, which will be used for bounding the G term from below. Using that w_n is positive and increasing, and that $\tau(n) \leq n$, we have for sufficiently large n: for the case $0 \leq p_n \leq b_1 < 1$,

$$\begin{aligned} w_n &\leq w_n + p_n w_{\tau(n)} \\ &= y_n + T_n - F_n + p_n[-p_{\tau(n)} y_{\tau(\tau(n))} + T_{\tau(n)} - F_{\tau(n)}], \end{aligned}$$

and for the case $-1 < -b_1 \leq p_n \leq 0$,

$$\begin{aligned} (1 - b_1) w_n &\leq w_n - b_1 w_{\tau(n)} \\ &\leq w_n + p_n w_{\tau(n)} \\ &= y_n + T_n - F_n + p_n[-p_{\tau(n)} y_{\tau(\tau(n))} + T_{\tau(n)} - F_{\tau(n)}]. \end{aligned}$$

We may note that p_n and $p_{\tau(n)}$ have the same sign and $y_n > 0$ in each of the two inequalities above, and $1 - b_1 \leq 1$ which implies

$$(1 - b_1) w_n \leq y_n + \epsilon + \gamma + b_1 \epsilon + b_1 \gamma, \quad \text{for } n \geq n_1.$$

As $\lim_{n\to\infty} w_n = \infty$, it follows that $\lim_{n\to\infty} y_n = \infty$. Then there exists $n_2 \geq n_1$ such that for $n \geq n_2$: $y_n, y_{\tau(n)} y_{\sigma(n)}$ and $y_{\alpha(n)}$ are bounded below by a positive constant.

By (H0)–(H1), for $i \geq n_2$, $G(y_{\sigma(i)})$ is bounded below by a positive constant c. Summing (2.10) from $n = n_2$ to $n = k-1$, we obtain

$$\Delta^{m-1} w_k = \Delta^{m-1} w_{n_2} - \sum_{n=n_2}^{k-1} v_n G(y_{\sigma(n)}) \leq \Delta^{m-1} w_{n_2} - c \sum_{n=n_2}^{k-1} v_n.$$

Note that by (H3), the right-hand side approaches $-\infty$, while the left-hand side is positive. This contradiction implies that the non-oscillatory positive solution y_n of (1.1) is bounded.

If y_n is an eventually negative solution of (1.1) for large n then we set $x_n = -y_n$ to obtain $x_n > 0$ and then (1.1) reduces to

$$\Delta^m \big(x_n - p_n x_{\tau(n)}\big) + v_n \tilde{G}(x_{\sigma(n)}) - u_n \tilde{H}(x_{\alpha(n)}) = \tilde{f}_n,$$

where

$$\tilde{f}_n = -f_n, \quad \tilde{G}(v) = -G(-v) \quad \tilde{H}(v) = -H(-v).$$

Further,

$$\tilde{F}_n = -F_n \quad \text{implies} \quad \Delta^m(\tilde{F}_n) = \tilde{f}_n.$$

In view of the above facts, it can be easily verified that $\tilde{G}$, $\tilde{H}$ and $\tilde{F}$ satisfy the corresponding conditions satisfied by the functions G, H and F in the theorem. Proceeding as in the proof for the case $y_n > 0$, we may complete the proof of the theorem. $\square$

The following result follows immediately from the above theorem.

Corollary 2.6. *Suppose that* (H0)–(H5) *and* (2.4) *hold. Then every unbounded solution of* (1.1)*, (if exists) is oscillatory.*

Also note that by setting $p_n = 0$, Theorems 2.5 can be applied to the equation

$$\Delta^m(y_n) + v_n G(y_{\sigma(n)}) - u_n G(y_{\alpha(n)}) = f_n.$$

2.1. **Results for bounded solutions.** In this subsection, we study the behaviour of bounded solutions of (1.1) and we do not require the assumption (H2). However, we need a condition

$$\sum_{n=n_0}^{\infty} n^{m-1} v_n = \infty, \tag{2.12}$$

which is less restrictive than (H3).

Theorem 2.7. *Assume* (H0), (H1), (H4), (H6) *and* (2.12)*. Then every bounded solution of* (1.1) *is oscillatory or tends to zero as $n \to \infty$, for each one of the following cases:*

$$0 \leq p_n \leq b_1 < 1 \quad \forall n\,; \tag{2.13}$$

$$-1 < -b_1 \leq p_n \leq 0 \quad \forall n\,; \tag{2.14}$$

$$b_2 \leq p_n \leq b_3 < -1 \quad \forall n\,; \tag{2.15}$$

$$1 < b_4 \leq p_n \leq b_5 \quad \forall n\,; \tag{2.16}$$

where b_1, b_2, b_3, b_4, b_5 are constants.

Proof. Let $y = y_n$ be a bounded solution of (1.1) for $n \geq N_1$. If it oscillates then there is nothing to prove. If it does not oscillate then $y_n > 0$ or $y_n < 0$ eventually. Suppose $y_n > 0$ for large n. There exits positive integer $n_0 \geq N_1 > 0$ such that $y_n > 0, y_{\tau(n)} > 0$, and $y_{\sigma(n)} > 0$ for $n \geq n_0$. Set z_n, T_n and w_n as in (2.5), (2.6) and (2.9) respectively, to obtain (2.10). T_n is well defined due to the boundedness of y_n and note that it satisfies (2.7). Then $w_n, \Delta w_n, \Delta^2 w_n, \ldots, \Delta^{m-1} w_n$ are monotonic and single sign for $n \geq n_1 \geq n_0$. Boundedness of y_n implies that of z_n and w_n. Using (2.7), (H6) and monotonic nature of w_n, we obtain $\lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = \lambda$, which exists finitely. Then applying Lemma 2.4 to (2.10), we obtain for $n \geq n_1$,

$$w_n = \lambda + \frac{(-1)^{m-1}}{(m-1)!} \sum_{i=n}^{\infty} (i - n + m - 1)^{(m-1)} v_i G(y_{\sigma(i)}). \tag{2.17}$$

As $\lim_{n\to\infty} w_n$ exists, from (2.17) it follows that

$$\sum_{i=n}^{\infty} (i - n + m - 1)^{(m-1)} v_i G(y_{\sigma(i)}) < \infty, \quad n \geq n_1. \tag{2.18}$$

Using Lemma 2.2 in (2.18), we obtain

$$\sum_{i=n}^{\infty} i^{m-1} v_i G(y_{\sigma(i)}) < \infty, \quad n \geq n_1. \tag{2.19}$$

From (2.19), it follows due to (2.12) that $\liminf_{n\to\infty} G(y_{\sigma(n)}) = 0$. Further, as $\lim_{n\to\infty} \sigma(n) = \infty$, we have $\liminf_{n\to\infty} G(y_n) = 0$. This implies due to (H0) that $\liminf_{n\to\infty} y_n = 0$. Then using Lemma 2.1, we may obtain $\lim_{n\to\infty} z_n = 0$. If (2.13) holds then

$$\begin{aligned} 0 = \lim_{n\to\infty} z_n &= \limsup_{n\to\infty} (y_n - p_n y_{\tau(n)}) \\ &\geq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty} (-p_n y_{\tau(n)}) \\ &\geq (1 - b_1) \limsup_{n\to\infty} y_n. \end{aligned}$$

This implies $\limsup_{n\to\infty} y_n = 0$ and consequently $y_n \to 0$ as $n \to \infty$. If (2.14) or (2.15) holds then, since $y_n \leq z_n$, it follows that $y_n \to 0$ as $n \to \infty$. If p_n satisfies (2.16), then $z_n \leq y_n - b_4 y_{\tau(n)}$, and it follows that

$$\begin{aligned} 0 = \liminf_{n\to\infty} z_n &\leq \liminf_{n\to\infty} [y_n - b_4 y_{\tau(n)}] \\ &\leq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty} [-b_4 y_{\tau(n)}] \\ &= (1 - b_4) \limsup_{n\to\infty} y_n. \end{aligned}$$

Then $\limsup_{n\to\infty} y_n = 0$, which implies $\lim_{n\to\infty} y_n = 0$.

If y_n is eventually negative for large n, then we may proceed with $x_n = -y_n$. □

2.2. **Results for bounded or unbounded solutions.** Clearly, the condition (H3) implies (2.12), so we combine the results Corollary 2.6 and Theorem 2.7 to state the following result.

Theorem 2.8. *Suppose that either* (2.13) *holds or* (2.14) *holds. Further assume (H0)–(H6) to hold. Then every solution of* (1.1) *is oscillatory or tends to zero as* $n \to \infty$.

The following oscillation result does not assume that $\tau(n)$ is increasing, but restricts p_n further than the theorem above.

Theorem 2.9. *Assume* (H0)–(H6) *and that* $0 \le p_n \le b_2 < 1$. *Then every solution of* (1.1) *is oscillatory or tends to zero as* $n \to \infty$.

Proof. By contradiction assume $y = y_n$ is an eventually positive solution of (1.1), which does not tend to zero as $n \to \infty$. Then there exists a n_0 such that for $n \ge n_0$: $y_n, y_{\alpha(n)}, y_{\sigma(n)}, y_{\tau(n)}$ are positive and $\limsup_{n\to\infty} y_n > 0$. Define w_n by (2.9). Then, as above, $\Delta^m w_n \le 0$ and $w, \Delta w_n, \dots, \Delta^{m-1} w_n$ are monotonic and of constant sign for $n \ge n_1$. We do not know that $w_n > 0$ yet. Since $0 \le p_n \le b_2 < 1$ and $y_n > 0$,

$$w_n \ge y_n - b_2 y_{\tau(n)} + T_n - F_n.$$

Taking the limit superior, using that w_n is monotonic and that T_n and F_n converge to zero, we have

$$\lambda = \lim_{n\to\infty} w_n \ge (1 - b_2) \limsup_{n\to\infty} y_n > 0.$$

Then w_n is positive for n large enough. By Lemma 2.3, $w^{(n)} \le 0$ and $w_n > 0$ imply the existence of n_1 such that $\Delta^{m-1} w_n > 0$ for $n \ge n_1$. Next we show that $\liminf_{n\to\infty} y_n > 0$, which will be used for bounding $G(y_{\sigma(n)})$ from below by a positive constant. Using that $0 \le p_n$ and $y_n > 0$, we have

$$w_n \le y_n + T_n - F_n.$$

Taking the limit inferior, using that w_n is monotonic and that T_n and F_n approach zero, we have

$$0 < \lambda = \lim_{n\to\infty} w_n \le \liminf_{n\to\infty} y_n.$$

Then there exists a $n_2 \ge n_1$ such that for $n \ge n_2$: $y_n, y_{\alpha(n)}, y_{\sigma(n)}, y_{\tau(n)}$ are bounded below by a positive constant. By (H0)–(H1), for $n \ge n_2$, $G(y_{\sigma(n)}$ is bounded below by a positive constant c. Taking sum in (2.10), we obtain

$$\Delta^{m-1} w_n = \Delta^{m-1} w_{n_2} - \sum_{k=n_2}^{n} v_k G(y_{\sigma(k)}) \le \Delta^{m-1} w_{n_2} - c \sum_{k=n_2}^{n} v_k.$$

Note that by (H3), the right-hand side approaches $-\infty$, while the left-hand side is positive. This contradiction implies that the solution can not be eventually positive without approaching zero.

If $y = y_n$ is an eventually negative solution of (1.1) that does not tend to zero as $n \to \infty$, then we may proceed with $x_n = -y_n$. This completes the proof. $\square$

Next, we give some examples to signify the importance of the above results.

Example 2.1. Consider the neutral equation

$$\Delta^m \left(y_n - \frac{1}{2} y_{n-1} \right) + n^{-m} y_{n-2}^{\alpha} = n^{-m} 2^{\alpha(2-n)}, \tag{2.20}$$

where $m \geq 2, \alpha$ is a positive rational, being the quotient of two odd integers. Here, $p_n = \frac{1}{2}$, and $v_n = n^{-m}, f_n = n^{-m} 2^{\alpha(2-n)}$. It is clear that

$$\sum_{n=n_0}^{\infty} n^{m-1} f_n < \infty.$$

Hence,it follows that

$$F_n = \frac{(-1)^m}{(m-1)!} \sum_{j=n}^{\infty} (j-n+m-1)^{(m-1)} j^{-m} 2^{\alpha(2-j)}.$$

Obviously, $|F_n| < \infty$. Clearly, the equation (2.20) satisfies all the conditions of Theorem 2.7. Hence every bounded non-oscillatory solution tends to zero as $n \to \infty$. In particular $y_n = 2^{-n}$ is a bounded solution of (2.20), which tends to zero as $n \to \infty$. However, the results in [18, Theorem 2.2] cannot be applied to (2.20) if $\alpha > 1$, because G is sub-linear in [18]. If $\alpha < 1$ then $m \geq 2$ implies $m - \alpha m + \alpha > 1$. Therefore the following condition

$$\sum_{n=n_0}^{\infty} v_n (n-k)^{\alpha(m-1)} = \infty, \tag{2.21}$$

does not hold. Hence, the result [18, Theorem 2.2] cannot be applied to (2.20). Also,in the case when $\alpha < 1$, the results in [4, 14] cannot be applied to (2.20) as G is super-linear there.

Example 2.2. Consider the neutral equation

$$\Delta^m \left(y_n + \frac{1}{2} y_{n-1} \right) + n^{-1} y_{n-2}^{\alpha} = (-1)^m 2^{-n-m+1} + n^{-1} 2^{\alpha(2-n)}, \tag{2.22}$$

where $m \geq 2, \alpha$ is a positive rational, which is the quotient of two odd integers. Here, $p_n = -\frac{1}{2}$, $v_n = n^{-1}, G(x) = x^{\alpha}$ and $f_n = (-1)^m 2^{-n-m+1} + n^{-1} 2^{\alpha(2-n)}$. Easily, we can verify that, $\sum_{n=n_0}^{\infty} n^{m-1} f_n < \infty$ and the equation (2.22) satisfies all the conditions of Theorem 2.8. Hence $y_n = 2^{-n}$ is a positive solution of (2.22), which tends to zero as $n \to \infty$. However, if $\alpha < 1$, then the results of

[4, 14] cannot be applied to this equation, because (1.3) is not satisfied. Again if $\alpha \geq 1$ then results of [17, 18] fail as G is sublinear there.

In the next result we remove the barrier at -1 for p_n. However, we assume additional hypotheses.

Theorem 2.10. *Assume* (H0)–(H2), (H4)–(H6), $b_4 \leq p_n \leq 0$, *and the delay functions satisfy* $\sigma(\tau(n)) = \tau(\sigma(n))$. *Also assume that*

$$\sum_{n_0}^{\infty} \min\{v_n, v_{\tau(n)}\} = \infty; \tag{2.23}$$

that there exists a positive constant δ, *such that for* $x, y, z > 0$,

$$G(x+y) \leq \delta(G(x) + G(y)), \quad G(zx) \leq G(z)G(x); \tag{2.24}$$

and that for $x, y < 0$ *and* $z > 0$,

$$G(x+y) \geq \delta(G(x) + G(y)), \quad G(zx) \geq G(z)G(x). \tag{2.25}$$

Then every solution of (1.1) *is oscillatory or tends to zero as* $n \to \infty$.

Proof. By contradiction assume $y = y_n$ is an eventually positive solution of (1.1), which does not tend to zero as $n \to \infty$. Then there exists a n_0 such that for $n \geq n_0$: $y_n, y_{\alpha(n)}, y_{\sigma(n)}, y_{\tau(n)}$ are positive and $\limsup_{n\to\infty} y_n > 0$. Define w_n by (2.9). Then, as above, $\Delta^m w_n \leq 0$ and $w_n, \Delta w_n, \dots, \Delta^{m-1} w_n$ are monotonic and of constant sign for $n \geq n_1$. From $p_n \leq 0$ and $y_n > 0$, it follows that $w_n \geq y_n + T_n - F_n$. In the limit

$$\lambda = \lim_{n\to\infty} w_n \geq \limsup_{n\to\infty} y_n > 0.$$

Since T_n and F_n approach zero, $y_n - p_n y_{\tau(n)}$ is bounded below by a positive constant, for all n large. Using $y_n - b_4 y_{\tau(n)} \geq y_n - p_n y_{\tau(n)}$, $\lim_{n\to\infty} \sigma(n) = \infty$, and $\sigma(\tau(n)) = \tau(\sigma(n))$, it follows that $y_{\sigma(n)} - b_4 y_{\sigma(\tau(n))}$ is also bounded below by a positive constant, for some $n \geq n_2$. Then by (H0)–(H1), there exist a positive constant c such that $c \leq G\left(y_{\sigma(n)} - b_4 y_{\sigma(\tau(n))}\right)$. Using (2.24)

$$\begin{aligned} c &\leq G\left(y_{\sigma(n)} - b_4 y_{\sigma(\tau(n))}\right) \\ &\leq \delta[G(y_{\sigma(n)}) + G(-b_4 y_{\sigma(\tau(n))})] \\ &\leq \delta\left[G(y_{\sigma(n)}) + G(-b_4) G\left(y_{\sigma(\tau(n))}\right)\right] \end{aligned}$$

From (2.10),

$$\begin{aligned} \Delta^m w_n + G(-b_4)\Delta^m w_{\tau(n)} &\leq -\min\{v_n, v_{\tau(n)}\}\left[G(y_{\sigma(n)}) + G(-b_4) G\left(y_{\sigma(\tau(n))}\right)\right] \\ &\leq -\min\{v_n, v_{\tau(n)}\} c/\delta \end{aligned}$$

Taking the sum,

$$\begin{aligned} &\Delta^{m-1} w_n + G(-b_4)\Delta^{m-1} w_{\tau(n)} \\ &\leq \Delta^{m-1} w_{n_2} + G(-b_4)\Delta^{m-1} w_{\tau(n_2)} - (c/\delta) \sum_{n_2}^{n} \min\{v_k, v_{\tau(k)}\}. \end{aligned}$$

In the limit as $n \to \infty$, by (2.23), the right-hand side approaches $-\infty$ while the left-hand side is positive. This contradiction proves that eventually positive solutions must converge to zero. For eventually negative solutions, we proceed as above. Thus the proof is complete. □

As prototypes of functions G satisfying the conditions (H0), (H1), (2.24)–(2.25), we have $G(x) = |x|^\lambda \operatorname{sgn}(x)$ and $G(x) = (\beta + |x|^\mu)|x|^\lambda \operatorname{sgn}(x)$ with $\lambda > 0$, $\mu > 0$, $\lambda + \mu \geq 1$, $\beta \geq 1$. For verifying these conditions, we may use the well known inequality [3, p. 292]

$$x^p + y^p \geq \begin{cases} (x+y)^p, & \text{for } 0 \leq p < 1, \\ 2^{1-p}(x+y)^p, & \text{for } 1 \leq p. \end{cases}$$

Clearly, (2.23) implies (H3), but not the other way around. However, when v is monotonic, (2.23) is equivalent to (H3).

A result similar to Theorem 2.10 is shown in [14, Theorem 2.20]. There it is assumed that

$$\sum_{n=n_0}^{\infty} n^{m-2} \min\{v_n, v_{\tau(n)}\} = \infty$$

which is less restrictive than (2.23). This is a trade off for G being non-decreasing and of super-linear growth there.

Next, we would like to prove a result when p_n is in a different range, not yet, considered in this work. For that we need the following definition.

Definition 2.2. For any positive integer $n \geq n_0$, define

$$\tau_{-1}(n) = \{m : m \text{ is an integer } \geq n \text{ and } \tau(m) = n\}.$$

Theorem 2.11. *Suppose that m is odd. Assume that (H0)–(H4) hold. Let $1 \leq p_n \leq b_2$. Then every non-oscillatory solution of*

$$\Delta^m \left(y_n - p_n y_{\tau(n)}\right) + u_n G(y_{\sigma(n)}) - v_n H(y_{\alpha(n)}) = 0 \tag{2.26}$$

tends to $\pm\infty$.

Setting z_n and T_n by (2.5) and (2.6) respectively and Proceeding as in the proof of theorem 2.7 we obtain (2.7) and (2.8). Then using all these in (2.26) we obtain

$$\Delta^m z_n = -v_n G(y_{\sigma(n)}) \leq 0. \tag{2.27}$$

This implies $z_n, \Delta z_n, \Delta^2 z_n, \ldots, \Delta^{m-1} z_n$ are monotonic and single sign for large n. As $\Delta^{m-1} z_n$ is decreasing we have

$$\lim_{n\to\infty} \Delta^{m-1} z_n = \lambda, \quad -\infty \leq \lambda < \infty.$$

We claim $\lambda = -\infty$. Otherwise, $-\infty < \lambda < \infty$.

Next, we show that $\liminf_{n\to\infty} y_n = 0$ as in the proof of theorem 2.7. Then there exists a subsequence y_{n_k} such that $n_k \to \infty$ and $y_{n_k} \to 0$ as $k \to \infty$. As $z_{n_k} < y_{n_k}$, we have $\limsup z_{n_k} \le 0$. Again,

$$z_{\tau^{-1}(n_k)} = y_{\tau^{-1}(n_k)} + p_{\tau^{-1}(n_k)} y_{n_k} > -b_2 y_{n_k}. \tag{2.28}$$

This implies $\liminf_{n\to\infty} z_{\tau^{-1}(n_k)} \ge 0$. As z_n is monotonic and $\lim_{n\to\infty} z_n$ exists (finitely or infinitely), we have $\lim_{n\to\infty} z_n = 0$. Consequently, by applying Lemma 2.3 we obtain

$$(-1)^{m+k}\Delta^k z_n < 0, \quad k = 0, 1, \dots, m-1, \text{for large } n,$$

and

$$\lim_{n\to\infty} \Delta^k z_n = 0, \quad k = 0, 1, \dots, m-1.$$

Since m is odd, $\Delta z_n < 0$ for large n. Hence $z_n > 0$ for $n \ge n_2$. This implies $y_n > y_{\tau_n}$ which further implies $\liminf_{n\to\infty} y_n > 0$, a contradiction. Hence our claim holds. Thus, $\lambda = -\infty$. From (2.28), we have $y_{n_k} > -\frac{1}{b_2} z_{\tau^{-1}(n_k)}$. This implies that $\lim_{n\to\infty} y_n = \infty$. Thus the proof of the theorem is complete.

Corollary 2.12. *Under the assumptions of the above theorem every bounded solution of* (2.26) *oscillates.*

Remark 2.1. The results in this section hold when G is linear, sub-linear or super linear.

Remark 2.2. Note that, even in the particular cases of our results for $u \equiv 0$ in (1.1); i.e., for the equation

$$\Delta^m \big[y_n - p_n y_{\tau(n)}\big] + v_n G(y_{\sigma(n)}) = f(t),$$

Theorems 2.5, 2.7, 2.9 and 2.10 generalize the results in [4]. Due to this generalization, particularly, by relaxing the conditions that G is non-decreasing and super linear, it is now possible to apply these results to the oscillatory and asymptotic behaviour of higher-order neutral equation (1.6) with oscillating coefficient q_n in our next section, which was not hitherto possible.

3. Application to neutral equations with oscillating coefficients

In this section, we find sufficient conditions so that every solution of the higher order ($m \ge 2$) neutral differential equation

$$\Delta^m \left[y_n - p_n y_{\tau(n)}\right] + q_n G(y_{\sigma(n)}) = f_n \tag{3.1}$$

oscillates or tends to zero as $n \to \infty$, where q_n is allowed to change sign. Let $q_n^+ = \max\{q_n, 0\}$ and $q_n^- = \max\{-q_n, 0\}$. Then $q_n = q_n^+ - q_n^-$ and the above equation can be written as

$$\Delta^m \left[y_n - p_n y_{\tau(n)}\right] + q_n^+ G(y_{\sigma(n)}) - q_n^- G(y_{\sigma(n)}) = f_n. \tag{3.2}$$

Now, we proceed as in the previous section by setting $v_n = q_n^+$, $u_n = q_n^-$ and $H(x) = G(x)$. Assumptions (H3), (H4) become

(C3) $$\sum_{n_0}^{\infty} q_n^{+} = \infty,$$

(C4) $$\sum_{n_0}^{\infty} n^{m-1} q_n^{-} < \infty,$$

respectively, which are feasible conditions. Therefore, the study of (3.1) reduces to the study of (1.1) in Theorems 2.5, 2.7, 2.8, 2.10. Thus we can have the following results for (3.1), where q_n changes sign.

Theorem 3.1. *Suppose that* (H0),(H1)*, (H5),(C3) and (C4) hold. Suppose that G is bounded. Assume that the sequence $\{p_n\}$ satisfies one of the conditions* (2.13) *or* (2.14)*. Then every unbounded solution of* (3.1) *oscillates.*

Theorem 3.2. *Assume* (H0), (H1), (C4), (H6)*. Further assume that* (2.12) *holds for $v_n = q_n^{+}$. If p_n satisfies one of the conditions of* (2.13),(2.14)*,* (2.15)*, and* (2.16) *then every bounded solution of* (3.1) *oscillates or tends to zero as $n \to \infty$.*

Theorem 3.3. *Assume* (H0), (H1)(H5), (H6), (C3) and (C4) *to hold. Suppose that p_n lies in the range given by* (2.13)*or* (2.14)*. Let G be bounded. Then every solution of* (3.1) *oscillates or tends to zero as $n \to \infty$.*

Theorem 3.4. *Assume* (H0), (H1), (C4), (H5),(H6)*. Suppose $b_4 \leq p_n \leq 0$, and the delay functions satisfy $\sigma(\tau(n)) = \tau(\sigma(n))$. Further, assume that G is bounded. Suppose that* (2.23) *holds for $v_n = q_n^{+}$. Let* (2.24)*, and* (2.25) *hold. Then every solution of* (3.1) *oscillates or tends to zero as $n \to \infty$.*

However, technique in [14] can not be applied to get results for (3.1) because (H2) and (1.3) are incompatible conditions if $G \equiv H$.

For the results in this section, we need G to be bounded, continuous, and to satisfy (H0) and (H1). The prototype of such a function $G(y)$ is $y^{2n}\operatorname{sgn}(y)/(1+y^{2n})$. Next, we present some examples to illustrate our results and to prove their significance.

Remark 3.1. The results in this work hold for $f_n \equiv 0$.

All the results in the cited papers fail to apply to the neutral equations given in the following examples, which illustrate some of the results of this section.

3.1. **Examples.**

Example 3.1. Consider the linear delay equation

(3.3) $$\Delta^m(y_n) + q_n y_{n-k} = \frac{(-1)^m}{2^{n+m}},$$

where m is any positive integer,k is a small positive odd integer and q_n is as given below.

$$q_n = \begin{cases} (-1)^m 2^{-(k+m)}, & n \text{ is even,} \\ (-1)^{1+m} 2^{-(k+m)}, & n \text{ is odd.} \end{cases} \tag{3.4}$$

It is easily verified that eq. (3.3) satisfies all the conditions of Theorem 3.2. As such, every bounded solution of (3.3) oscillates or tends to zero as $n \to \infty$ and in fact, this equation admits a bounded non-oscillatory solution given by

$$y_n = \begin{cases} 2^{-n}, & n \,\text{is odd,} \\ 0, & n \,\text{is even,} \end{cases}$$

which tends to zero as $n \to \infty$. However, none of the results in the papers under our reference can be applied to (3.3).

Example 3.2. Consider the non-linear neutral equation

$$\Delta^m(y_n) + q_n G(y_{n-k}) = 0, \tag{3.5}$$

where m is any positive integer, k is a small positive integer odd or even and $G(u) = \frac{u^2}{(1+u^2)} \operatorname{sgn} u$, q_n is as given below.

$$q_n = \begin{cases} (-1)^m \frac{1+2^{2k-2n}}{2^{2k-n+m}}, & n \,\text{is odd,} \\ (-1)^{m+1} \frac{1+2^{2k-2n}}{2^{2k-n+m}}, & n \,\text{is even.} \end{cases} \tag{3.6}$$

It is easily verified that eq. (3.5) satisfies all the conditions of Theorem 3.3. As such, every solution of (3.5) oscillates or tends to zero as $n \to \infty$ and in fact, this equation admits a oscillatory solution given by

$$y_n = \begin{cases} -2^{-n}, & n \,\text{is odd,} \\ 2^{-n}, & n \,\text{is even,} \end{cases}$$

which tends to zero as $n \to \infty$. However, none of the results in the articles of our reference can be applied to (3.5).

Remark 3.2. The results in this section extend the results of [15] to higher order neutral difference equations.

Before we close, we would like to share a few things with the reader that may be helpful for further research.

Final Comments

It would be interesting to study the oscillation of solutions to (3.1) or to (3.2) for the case when q_n oscillates and $G(x) = x$. It is so because when we compare (3.2) with (1.1), we have $G(x) = H(x) = x$ and (H2) not being satisfied.

While studying (1.1) and (3.1), we assumed (H4). However, we do not know yet, what would happen, if this condition is not met. Hence it would be very interesting, to do some work in this direction.

We observe that in the majority of the results for forced equations, non-oscillatory solutions tend to zero at ∞. Can we change this asymptotic behaviour of the non-oscillatory solutions, by imposing additional conditions on the coefficient functions of (1.1) or (3.1).

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