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ON CHARACTER AND APPROXIMATE CHARACTER AMENABILITY OF VARIOUS SEGAL ALGEBRAS

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Abstract. We investigate character and approximate character amenability of various Segal algebras in both the group algebra $L^1(G)$, and the Fourier algebra $A(G)$, of a locally compact group G.

1. Introduction

The notion of amenability in Banach algebra was initiated by Johnson in [15]. Since then, amenability has become a major issue in Banach algebra theory and in harmonic analysis. For details on amenability in Banach algebras see [20].

In [11] Ghahramani and Loy introduced generalized notions of amenability with the hope that it will yield Banach algebra without bounded approximate identity which nonetheless had a form of amenability. All known approximate amenable Banach algebras have bounded approximate identities until recently when Ghahramani and Read in [12] give examples of Banach algebras which are boundedly approximately amenable but which do not have bounded approximate identities. This answers a question open since the year 2004 when Ghahramani and Loy founded the notion of approximate amenability.

Let A be a Banach algebra over $\mathbb{C}$ and $\varphi\colon A \to \mathbb{C}$ be a character on A, that is, an algebra homomorphism from A into $\mathbb{C}$, and let Φ_A denote the character space of A. In [22], Monfared introduced the notion of character amenability in Banach algebras. His definition of this notion requires continuous derivations from A into dual Banach A-bimodules to be inner, but only those modules are concerned where either of the left or right module action is defined by characters on A. As such character amenability is weaker than the classical amenability introduced by Johnson in [15], so all amenable Banach algebras are character amenable.

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In [21], Mewomo and Okoli applied the concept of approximate amenability to that of character amenability and introduced the notions of approximate left character amenability, approximately right character amenability and approximately character amenability. They developed general theory on these notions and studied them for Banach algebras defined over locally compact groups and second duals of Banach algebras.

In [1], the authors considered the character amenability and contractibility of abstract Segal algebras. In this paper, we shall extend the work in [1] and [21] by studying the character and approximate character amenability of various Segal algebras in both the group algebra $L^1(G)$ and the Fourier algebra $A(G)$ of a locally compact group G.

2. Preliminaries

First, we recall some standard notions; for further details, see [4], [6] and [20].

Let A be an algebra. The character space of A is denoted by Φ_A. Let X be an A-bimodule. A *derivation* from A to X is a linear map $D\colon A \to X$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in A)\,.$$

For example, for $x \in X$, the map $\delta_x\colon A \to X$ defined by

$$\delta_x(a) = a \cdot x - x \cdot a \quad (a \in A)$$

is a derivation; derivations of this form are called the *inner derivations.*

Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule if X is a Banach space and if there is a constant $k > 0$ such that

$$\|a \cdot x\| \leq k\,\|a\|\,\|x\|\,, \quad \|x \cdot a\| \leq k\,\|a\|\,\|x\| \quad (a \in A,\, x \in X)\,.$$

By renorming X, we can suppose that $k = 1$. For example, A itself is Banach A-bimodule, and X', the dual space of a Banach A-bimodule X, is a Banach A-bimodule with respect to the module operations specified for by

$$\langle x,\, a \cdot \lambda\rangle = \langle x \cdot a,\, \lambda\rangle, \quad \langle x,\, \lambda \cdot a\rangle = \langle a \cdot x,\, \lambda\rangle \quad (x \in X)$$

for $a \in A$ and $\lambda \in X'$; we say that X' is the *dual module* of X.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then $\mathcal{Z}^1(A, X)$ is the space of all continuous derivations from A into X, $\mathcal{N}^1(A, X)$ is the space of all inner derivations from A into X, and the first cohomology group of A with coefficients in X is the quotient space

$$\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X)\,.$$

The Banach algebra A is *amenable* if $\mathcal{H}^1(A, X') = \{0\}$ for each Banach A-bimodule X.

A derivation $D\colon A \to X$ is *approximately inner* if there is a net (x_v) in X such that

$$D(a) = \lim_v (a \cdot x_v - x_v \cdot a) \quad (a \in A),$$

the limit being taken in $(X, \|.\|)$. That is, $D(a) = \lim_v \delta_{x_v}(a)$, where (δ_{x_v}) is a net of inner derivations. The Banach algebra A is *approximately amenable* if, for each Banach A-bimodule X, every continuous derivation $D\colon A \to X'$ is approximately inner.

We let $\mathcal{M}^A_{\varphi_r}$ denote the class of Banach A- bimodule X for which the right module action of A on X is given by $x \cdot a = \varphi(a)x \quad (a \in A, x \in X, \varphi \in \Phi_A)$, and $\mathcal{M}^A_{\varphi_l}$ denote the class of Banach A- bimodule X for which the left module action of A on X is given by $a \cdot x = \varphi(a)x \quad (a \in A, x \in X, \varphi \in \Phi_A)$. If the right module action of A on X is given by $x \cdot a = \varphi(a)x$, then it is easy to see that the left module action of A on the dual module X' is given by $a \cdot f = \varphi(a)f \quad (a \in A, f \in X', \varphi \in \Phi_A)$. Thus, we note that $X \in \mathcal{M}^A_{\varphi_r}$ (resp. $X \in \mathcal{M}^A_{\varphi_l}$) if and only if $X' \in \mathcal{M}^A_{\varphi_l}$ (resp. $X' \in \mathcal{M}^A_{\varphi_r}$).

Let A be a Banach algebra and let $\varphi \in \Phi_A$, we recall from [14], see also [22]:

(i) A is left φ-amenable if every continuous derivation $D\colon A \to X'$ is inner for every $X \in \mathcal{M}^A_{\varphi_r}$
(ii) A is right φ-amenable if every continuous derivation $D\colon A \to X'$ is inner for every $X \in \mathcal{M}^A_{\varphi_l}$
(iii) A is left character amenable if it is left φ-amenable for every $\varphi \in \Phi_A$
(iv) A is right character amenable if it is right φ-amenable for every $\varphi \in \Phi_A$
(v) A is character amenable if it is both left and right character amenable.

Also, we recall the following definitions from [21].

Definition 2.1. Let A be a Banach algebra and let $\varphi \in \Phi_A$. Then we say that

(i) A is approximately left φ-amenable if every continuous derivation $D\colon A \to X'$ is approximately inner for every $X \in \mathcal{M}^A_{\varphi_r}$;
(ii) A is approximately right φ-amenable if every continuous derivation $D\colon A \to X'$ is approximately inner for every $X \in \mathcal{M}^A_{\varphi_l}$;
(iii) A is approximately left character amenable if it is approximately left φ-amenable for every $\varphi \in \Phi_A$;
(iv) A is approximately right character amenable if it is approximately right φ-amenable for every $\varphi \in \Phi_A$;
(v) A is approximately character amenable if it is both approximately left and approximately right character amenable.

Clearly, approximate character amenability is weaker than character amenability, approximate amenability, and amenability, and so, every character amenable, approximately amenable and amenable Banach algebra is approximately character amenable.

Let G be a locally compact group. We denote by $L^1(G)$ the group algebra of G. This is the Banach space

$$\{f\colon G\to\mathbb{C}, f \text{ measurable } : \|f\|_1 := \int_G |f(t)|d\mu(t) < \infty\},$$

where μ denotes left Haar measure on G and we equate functions that are equal almost everywhere with respect to μ. The product on $L^1(G)$ is defined by

$$(f*g)(t) = \int_G f(s)g(s^{-1}t)d\mu(s) \quad (t\in G, f,g\in L^1(G)).$$

$(L^1(G), *, \|f\|_1)$ is a Banach algebra. In the case where G is discrete, we write $l^1(G)$ for $L^1(G)$. For details, see [4]. We also denote by $A(G)$, the Fourier algebra on G. it is well known that that $A(G)\cong L^1(\hat{G})$ when G is abelian and $\hat{G}$ is its dual group.

3. Basic properties and results on algebras over locally compact group

The following result is useful in this work, and it is from [21].

Theorem 3.1. *Let A be an approximately left character amenable Banach algebra and let I be a weakly complemented left ideal of A. Then I has a right approximate identity. In particular, $\overline{I^2}=I$.*

Proposition 3.2. *Each Finite-dimensional, approximately left character amenable Banach algebra is semisimple, and hence amenable.*

Proof. Let A be a finite-dimensional Banach algebra with radical R. Since A is finite-dimensional, R is nilpotent, and R and R^2 are closed ideals in A. Since A is assume to be approximately left character amenable, then $\overline{R^2}=R$ by Theorem 3.1 and so $R^2=R$. Thus $R^n=R$ for each $n\in\mathbb{N}$, and so $R=\{0\}$. Thus A is semisimple and hence amenable. □

The next results follow from [22, Theorem 2.6 (i) and Theorem 2.3] and the fact any statement about left character amenability turns into an analogous statement about right character amenability.

Proposition 3.3. *(i) Let A and B be Banach algebras. Suppose A is character amenable and $\tau\colon A\to B$ is a continuous homomorphism with $\overline{\tau(A)}=B$, then B is character amenable.*
(ii) Each character amenable Banach algebra has a bounded approximate identity, and hence factors.

Proposition 3.4. *Let S be any infinite set, then $l^1(S)$ is not approximately character amenable.*

Proof. Suppose $l^1(S)$ is approximately character amenable. Since S is infinite, there is a continuous epimorphism $\tau\colon l^1(S) \to l^1(\mathbb{N})$, and so $l^1(\mathbb{N})$ is approximately character amenable by [21, Corollary 3.7]. This is a contradiction because $l^1(\mathbb{N})$ does not have a left approximate identity, so by [21, Proposition 3.4 (iii)], $l^1(S)$ is not approximately character amenable. □

For a locally compact group G. The amenability, approximate amenability and character amenability version of the result below on the group algebra $L^1(G)$ has been proved in [15, Theorem 2.5], [11, Theorem 3.2] and [22, Corollary 2.4] respectively. With these results, it follows that the amenability, approximate amenability and character amenability of $L^1(G)$ as a Banach algebra is equivalent to the amenability of G as a group. The proofs of the amenability, approximate amenability and character amenability cases carries over. Thus, the proof is similar to the argument used in [11, Theorem 3.2] and the famous result of Johnson [15, Theorem 2.5] to the effect that G is amenable if and only if $L^1(G)$ is amenable.

Theorem 3.5. *Let G be a locally compact group. Then $L^1(G)$ is approximately character amenable if and only if G is amenable.*

The next result describes an hereditary property of approximate character amenability of Fourier algebra $A(G)$. There is no known example of a locally compact group G, for which $A(G)$ does not have an approximate identity. In fact, Leptin's theorem asserts that $A(G)$ has a bounded approximate identity whenever G is amenable [18].

Proposition 3.6. *Let G be a locally compact group such that $A(G)$ has an approximate identity. Suppose H is a closed subgroup of G and $A(G)$ is approximately character amenable, then $A(H)$ is approximately character amenable.*

Proof. By [9, Lemma 3.8], $A(H)$ is a quotient of $A(G)$, thus, the result follows from [21, Corollary 3.8]. □

4. Character amenability of Segal algebras

In this section, we shall consider the character amenability properties of some Segal algebras. Segal algebras were first defined by Reiter for group algebras, see [24] and [25]. For the definition of abstract Segal algebras given below see [17].

Let A be a Banach algebra with norm $\|.\|_A$ and let B be a dense left ideal in A such that

(i) B is a Banach algebra with respect to some norm $\|.\|_B$,
(ii) there is a constant $K > 0$ such that
$$\|b\|_A \leq K\|b\|_B \quad \forall b \in B,$$
(iii) there is a constant $C > 0$ such that
$$\|ab\|_B \leq C\|a\|_A\|b\|_B \quad \forall a, b \in B.$$

Then we recall from [17] that B is called an abstract Segal algebra in A.

In the case $A = L^1(G)$, we write $S^1(G)$ instead of B and further insist that $S^1(G)$ is closed under left translation; $L_x f \in S^1(G)$ for all $x \in G$ and $f \in S^1(G)$, where $L_x f(y) = f(x^{-1}y)$ for $y \in G$. By well-known techniques, conditions (i)-(iii) above on $B = S^1(G)$, is equivalent to the map

$$(x, f) \to L_x f : G \times S^1(G) \to S^1(G)$$

is continuous with $\|L_x f\|_{S^1(G)} = \|f\|_{S^1(G)}$ for $f \in S^1(G), x \in G$.

Every Segal algebra $S^1(G)$ is an abstract Segal algebra in $L^1(G)$ but not conversely. It is shown in [25] that every Segal algebra has a left approximate identity which is bounded in L^1- norm and that it can never have a bounded approximate identity unless it coincides with $L^1(G)$.

For a locally compact group G, we recall from [10] that a Lebesgue-Fourier algebra $\mathcal{L}A(G)$ of a locally compact group G is

$$\mathcal{L}A(G) = L^1(G) \cap A(G),$$

where $|||f||| = \|f\|_1 + \|f\|_{A(G)} \quad (f \in \mathcal{L}A(G))$, and where the product is the convolution product. As shown in [10], pointwise product also provides a Banach algebra structure on $\mathcal{L}A(G)$.

It was also shown in [10], that $(\mathcal{L}A(G), |||.|||)$ with convolution product is a Segal algebra in $L^1(G)$ and that $(\mathcal{L}A(G), |||.|||)$ with pointwise multiplication is a commutative abstract Segal algebra in $A(G)$.

The following useful results are from [10].

Proposition 4.1. *Let G be a locally compact group. The following are equivalent:*

(i) G is discrete

(ii) $\mathcal{L}A(G) = L^1(G)$

(iii) $\mathcal{L}A(G)$ with convolution product has a bounded approximate identity.

Proposition 4.2. *Let G be a locally compact group. The following are equivalent:*

(i) G is compact

(ii) $\mathcal{L}A(G) = A(G)$

(iii) $\mathcal{L}A(G)$ with pointwise product has a bounded approximate identity.

With the above results, we have our results for this section.

Proposition 4.3. *Let G be a locally compact group. The following statements are equivalent:*

(i) $\mathcal{L}A(G)$ with convolution product is character amenable

(ii) G is discrete and amenable.

Proof. (i) $\implies$ (ii). Since $\mathcal{L}A(G)$ is character amenable, then it is both left and right character amenable, and so it has a bounded left and bounded right approximate identities by [22, Theorem 2.3], thus it has a bounded approximate identity by [2, Proposition 11.6], see also [8, Proposition 2.6]. Hence G

is discrete by Proposition 4.1 and $\mathcal{L}A(G) = l^1(G)$. Now the identity map is a topological isomorphism of $\mathcal{L}A(G)$ onto the group algebra $l^1(G)$. Hence $l^1(G)$ is character amenable also, and so the discrete group G is amenable by [22, Corollary 2.4].

(ii) $\Longrightarrow$ (i). If G is discrete and amenable, then $l^1(G)$ is character amenable by [22, Corollary 2.4] and $l^1(G) = \mathcal{L}A(G)$ by Proposition 4.1, and so $\mathcal{L}A(G)$ is character amenable. $\square$

Proposition 4.4. *Let G be a locally compact group. The following statements are equivalent:*

(i) $\mathcal{L}A(G)$ with pointwise product is character amenable
(ii) G is compact, amenable and the Fourier algebra $A(G)$ is character amenable.

Proof. (i) $\Longrightarrow$ (ii). Using argument similar to the proof of Proposition 4.3, $\mathcal{L}A(G)$ has a bounded approximate identity since it is character amenable. Hence G is compact by Proposition 4.2 and $\mathcal{L}A(G) = A(G)$. Thus the norms $|||.|||$ and $\|.\|_{A(G)}$ are equivalent by the open mapping theorem and so $A(G)$ is character amenable. Thus the compact group G is amenable [22, Corollary 2.4].

(ii) $\Longrightarrow$ (i). If G is compact, amenable and $A(G)$ is character amenable, then $\mathcal{L}A(G) = A(G)$ by Proposition 4.2 and $A(G)$ is amenable by [22, Corollary 2.4]. $\square$

Another examples of Segal algebras are the algebras $S_p(G)$ which were studied in [7]. We recall from [7] that for a locally compact group G, and $1 \leq p < \infty$,

$$S_p(G) = \{f \in L^1(G) : \hat{f} \in L^p(\Gamma)\}$$

and

$$\|f\|_p = \|f\|_1 + \|\hat{f}\|_p \quad (f \in S_p(G)),$$

where Γ denotes the dual group, and for $f \in L^1(G)$ the Fourier transform of f on Γ is noted by $\mathcal{F}(f) = \hat{f}$. The algebras $(S_p(G), \|.\|_p)$ are Segal algebras on G. For further details see [25] and [23].

Proposition 4.5. *Let G and H be locally compact abelian groups, and let $1 \leq p < \infty$. Suppose $S_p(G)$ and $S_p(H)$ are character amenable. Then $S_p(G \times H)$ is character amenable.*

Proof. Since $S_p(G)$ and $S_p(H)$ are character amenable, then $S_p(G)\hat{\otimes}S_p(H)$ is character amenable by [21, Corollary 3.11]. It was shown in [7] that $S_p(G)\hat{\otimes}S_p(H) \cong S_p(G \times H)$ and that the continuous linear map

$$T\colon S_p(G)\hat{\otimes}S_p(H) \to S_p(G \times H)$$

defined by $T(f \otimes g) = f \otimes g$ for $f \in S_p(G)$ and $g \in S_p(H)$ is a homomorphism with $\overline{S_p(G)\hat{\otimes}S_p(H)} = S_p(G \times H)$ (i.e. the image of T is dense in $S_p(G \times H)$). Thus, the result follows by Theorem 3.3(i). $\square$

5. Approximately character amenability of Segal algebras

Theorem 5.1. *Let A be a Banach algebra and B an abstract Segal algebra in A. Suppose B does not consist of left and right topological divisors of zero. Then the following statements are equivalent:*

(i) B is approximately character amenable;
(ii) $A = B$, and A is approximately character amenable;
(iii) A is Banach algebra isomorphic to B and A is approximately character amenable.

Proof. (i) $\implies$ (ii). Since B is approximately character amenable, then B has left and right approximate identities [21, Proposition 3.4], and since B does not consist of left and right topological divisors of zero, then [19, Proposition 3] shows that B has a bounded approximate identity (e_v). B is an abstract Segal algebra in A implies, there exists $C > 0$, such that

$$\|ab\|_B \leq C\|a\|_A\|b\|_B \quad (a, b \in B),$$

and there exists $M > 0$, such that

$$\|a\|_A \leq M\|a\|_B \quad (a \in B) \qquad (5.1).$$

Thus, for each $a \in B$, we have

$$\begin{aligned}\|a\|_B = \lim_v \|ae_v\|_B &\leq C\|a\|_A \liminf_v \|e_v\|_B \\ &\leq C\left(\sup_v \|e_v\|_B\right)\|a\|_A \qquad (5.2).\end{aligned}$$

And so, from (5.1) and (5.2), we have that $\|.\|_A$ and $\|.\|_B$ are equivalent on B. Since B is dense in A, It follows that $A = B$ and A is approximately character amenable.

(ii) $\implies$ (iii). Since B is an abstract Segal algebra in A, there exists $M > 0$, such that $\|a\|_A \leq M\|a\|_B \quad (a \in A)$. Thus B is a Banach algebra isomorphic to A by the open mapping theorem.

(iii) $\implies$ (i). This is trivial. □

Let A be a Banach algebra and B an abstract Segal algebra in A. it was shown in [1, Lemma 2.2] that $\Phi_B = \{\varphi \mid B : \varphi \in \Phi_A\}$. Using exactly the same argument as in the proof of [1, Proposition 2.3] in conjunction with [21, Theorem 3.3], we have the following result.

Proposition 5.2. *Let A be a Banach algebra, B an abstract Segal algebra in A and $\varphi \in \Phi_A$. Then A is approximately left [right] φ-amenable if and only if B is approximately left [right] $\varphi \mid B$-amenable.*

Corollary 5.3. *Let A be a Banach algebra, B an abstract Segal algebra in A. Suppose A is approximately character amenable. Then B is approximately φ-amenable for all $\varphi \in \Phi_B$. Moreover, B is approximately character amenable if and only if $A = B$.*

Proof. Follows from Proposition 5.2. □

Proposition 5.4. *Let G be an amenable locally compact group. Then the Segal algebra $S^1(G)$ in $L^1(G)$ is approximately φ-amenable for all $\varphi \in \Phi_{S^1(G)}$. Moreover, $S^1(G)$ is approximately character amenable if and only if $S^1(G) = L^1(G)$.*

Proof. Since G is amenable, then $L^1(G)$ is approximately character amenable by Theorem 3.5. Thus the result follows from Corollary 5.3. □

Proposition 5.5. *Let G be a locally compact group. Suppose $S^1(G)$ is approximately character amenable, then G is an amenable group.*

Proof. Let $I_0 = \{f \in L^1(G) : \int_G f(x)dx = 0\}$ be the augmentation ideal in $L^1(G)$, and let $I = I_0 \cap S^1(G)$. Then I is a codimension - 1 two-sided ideal in $S^1(G)$. Since $S^1(G)$ is approximately character amenable, then I has a right approximate identity by Theorem 3.1. Hence [17, Proposition 2.7] shows that I_0 must also have a right approximate identity. This implies that G is amenable by [26, Theorem 5.2]. □

Finally, by using Proposition 3.4, we show that the Feichtinger Segal algebra on an infinite compact abelian group G is not approximately character amenable. For the definition and details on Feichtinger Segal algebra see [24].

Proposition 5.6. *The Feichtinger algebra on an infinite compact abelian group is not approximately character amenable.*

Proof. For a compact and abelian group G, the Feichtinger algebra on G is

$$S_0(G) = \{f = \sum_{\gamma \in \hat{G}} C_\gamma X_\gamma : \|f\| = \sum_{\gamma \in \hat{G}} |C_\gamma| < \infty\},$$

where X_γ is the character of G associated with $\gamma \in \hat{G}$.
Hence $S_0(G) \cong l^1(\hat{G})$, where $l^1(\hat{G})$ is equipped with the pointwise product. But by Proposition 3.4, $l^1(S)$ is not approximately character amenable if S is an infinite set. So $S_0(G)$ is not approximately character amenable. □

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