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ESTIMATING THE ERROR OF THE NUMERICAL SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS VIA WALSH POLYNOMIALS

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Abstract. Our aim is to deal with the numerical solution of differential equations by Walsh polynomials approach. In this paper we simplify the procedure of C. F. Chen and C. H. Hsiao for one linear differential equation with constant coefficients in order to do a thorough analysis of errors. In this regard we introduce a faster method to obtain the values of the numerical solution.

The basic idea of using Fourier series for solving differential equations is to assume that the unknown solution can be approximated by the linear combination of basis functions and then this series is substituted into the equation. The point is to choose the coefficients of the series such that the residuum is minimized. However, the substitution is only possible if the basis functions are derivable and it is not true in case of locally constant orthonormal systems like systems formed by Walsh functions.

One approach to avoid the differentiation of basis functions is to consider the integral equation which is equivalent to the original differential equation. Thus, the substitution of the series in the integral equation reduces the problem to solve a linear system to obtain the coefficients. In 1975 C. F. Chen and C. H. Hsiao [1] used this idea to establish a procedure for the solution of state equations by Walsh polynomials approach. With this procedure we can obtain a numerical solution of the Cauchy problem of ordinary linear systems of differential equations with constant coefficients.

However, Chen and Hsiao did not deal with the analysis of the proposed numerical solution. We refer to determine if the linear system from which we obtain the coefficients of the Walsh polynomials is solvable or not and also to the estimation of errors. The aim of this paper is to deal with the questions above, but in case of only one equation, namely we study the numerical method based on

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Walsh polynomials approach of the Cauchy problem

$$\begin{aligned} y' + ay &= q(x) \\ y(0) &= \eta \end{aligned} \tag{1}$$

where $a \in \mathbf{R}$ and $q\colon [0,1[\to \mathbf{R}$ is a continuous function which is integrable on the interval $[0,1]$.

The Cauchy problem (1) is equivalent to the following integral equation

$$y(x) = \eta + \int_0^x q(t) - ay(t)\,dt \qquad (0 \le x < 1).$$

We propose to approach the solution of integral equation above by the Walsh polynomials

$$\overline{y}_n(x) = \sum_{k=0}^{2^n-1} c_k \omega_k(x),$$

satisfying the discretized integral equation

$$\overline{y}_n(x) = \eta + S_{2^n}\left(\int_0^{\cdot} S_{2^n}q(t) - a\overline{y}_n(t)\,dt\right)(x) \tag{2}$$

where $0 \le x < 1$, ω_k is the kth Walsh-Paley function and operators $S_{2^n}f$ are the 2^n-th partial sums of Walsh-Fourier series of an integrable function f on the interval $[0,1[$. We would point out that the expression

$$S_{2^n}\left(\int_0^{\cdot} S_{2^n}q(t) - a\overline{y}_n(t)\,dt\right)(x)$$

denotes the 2^n-th partial sums of Walsh-Fourier series of the integral function

$$\int_0^x S_{2^n}q(t) - a\overline{y}_n(t)\,dt$$

at the point x.

After the analysis of the above numerical method we obtain the following main result.

Theorem 1. *Let n be a non-negative integer such that $2^{n+1} \neq -a$. Then there exists an unique Walsh polynomial $\overline{y}_n$ which satisfies the discretized integral equation* (2)*. Moreover, $\overline{y}_n(x)$ converges uniformly to the solution of the Cauchy problem* (1) *on the interval* $[0,1[$.

In the next three sections we introduce the basic notations and statements that we use throughout this paper. Section 4 describes the method of Chen and Hsiao and proves the first part of Theorem 1. In Section 5 we propose a faster method to obtain directly the values of the numerical solution without needing to generate Walsh functions or solve the linear system (10).

Section 6 deals with the estimation of errors estimating the absolute difference between the solution the Cauchy problem (1) and the numerical solution. In this section we complete the proof of Theorem 1. In the last section we illustrate the established method through some examples.

1. The Walsh-Paley system

Every $n \in \mathbf{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_k 2^k,$$

where $n_k = 0$ or $n_k = 1$ for all $k \in \mathbf{N}$. This allows us to say that the sequence $(n_0, n_1, \dots)$ is the dyadic expansion of n. Similarly, the dyadic expansion $(x_0, x_1, \dots)$ of a real number $x \in [0, 1[$ is given by the sum

$$x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}},$$

where $x_k = 0$ or $x_k = 1$ for all $k \in \mathbf{N}$. This expansion is not unique if x is a dyadic rational, i.e. x is a number of the form $\frac{i}{2^k}$, where $i, k \in \mathbf{N}$ and $0 \le i < 2^k$. When this situation occurs we choose the expansion terminates in zeros.

The Walsh-Paley function ω_n is obtained by the finite product of Rademacher functions

$$r_k(x) := (-1)^{x_k} \qquad (x \in [0, 1[,\ k \in \mathbf{N}),$$

namely

$$\omega_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (x \in [0, 1[, n \in \mathbf{N}).$$

The Walsh-Paley system is an orthonormal and complete system on $L^1([0, 1[)$.

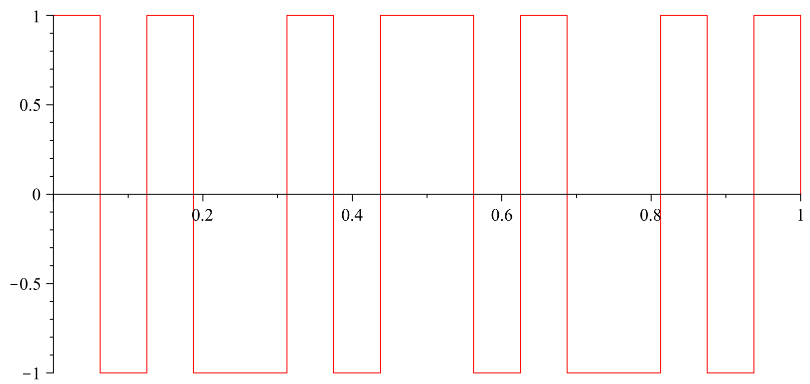

Figure 1. The Walsh-Paley function ω_{10}

For an integrable function f on the interval $[0, 1[$ we define the Fourier coefficients and partial sums of Fourier series by

$$\widehat{f}_k := \int_0^1 f(x)\omega_k(x)\,dx \quad (k \in \mathbf{N}),$$

$$S_n f(x) := \sum_{k=0}^{n-1} \widehat{f}_k \omega_k(x) \quad (n \in \mathbf{N}, x \in [0, 1[).$$

It is important to note that the partial sums $S_{2^n}f$ can be written as

$$S_{2^n}f(x) = 2^n \int_{I_n(x)} f(y)\, dy$$

where the sets

$$I_k(i) := \left[\frac{i-1}{2^k}, \frac{i}{2^k}\right[\qquad (i = 1, \dots, 2^k)$$

are called dyadic intervals and $I_n(x)$ is the dyadic interval which contains the value of x. For this reason, the operator S_{2^n} is the conditional expectation with respect to the σ-algebra generated by the sets $I_n(x)$, $x \in [0,1[$. Thus, by the martingale convergence theorem we obtain that $S_{2^n}f$ converge to f in L^p-norm and a.e. for all functions $f \in L^p([0,1[)$, $p \geq 1$ (see [4] p. 29).

2. The dyadic modulus of continuity

The topology generated by the collection of dyadic intervals on $[0,1[$ is called dyadic topology. Define the dyadic sum of two numbers $x, y \in [0,1[$ with expansion $(x_0, x_1, \dots)$ and $(y_0, y_1, \dots)$ respectively by

$$x \dot{+} y := \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}.$$

Then $\rho(x,y) := x \dot{+} y$ is a metric on $[0,1[$ which generates the dyadic topology.

Let C_W be the set of real-valued functions on the interval $[0,1[$ which are continuous at every dyadic irrational, continuous from the right on $[0,1[$ and have a finite limit from the left on $]0,1]$, all this on the usual topology. It is not hard to see that every continuous function $f\colon [0,1[\to \mathbf{R}$ on the usual topology belongs to C_W if f has a finite limit from the left of 1. Moreover, every function in C_W is continuous on the dyadic topology. On the other hand, every continuous function $f\colon [0,1[\to \mathbf{R}$ on the usual topology is also continuous on the dyadic topology (see [4] p. 11), but it is necessary to have a finite limit from the left of 1 in order to be in C_W.

A function belongs to the Walsh-Paley system is called Walsh function. The finite linear combinations of Walsh functions

$$f(x) = \sum_{k=0}^{n} a_k \omega_k(x)$$

are called Walsh polynomials. Every Walsh polynomial is a dyadic step function and vice versa. Every continuous function on the interval $[0,1[$ can be approximated by Walsh polynomials at every point, but this can be done uniformly only for functions belongs to C_W.

Define the dyadic modulus of continuity of an $f \in C_W$ by

$$\omega_n f := \sup\{|f(x \dot{+} h) - f(x)| \colon x \in [0,1[, 0 \leq h < 2^{-n}\}$$

and the local modulus of continuity on the dyadic interval $I_n(i)$ of a continuous function f on the dyadic topology by

$$\omega_{n,i} f := \sup\{|f(x \dotplus h) - f(x)| \colon x \in I_n(i), 0 \le h < 2^{-n}\}.$$

for all $i = 1, 2, \ldots, 2^n$. Not every $f \colon [0,1[\to \mathbf{R}$ continuous function on the usual topology has a finite modulus of continuity, it is also necessary the existence of the limit of f from the left of 1.

The dyadic modulus of continuity can be written as follows

$$\omega_n f = \sup\{|f(x_1) - f(x_2)| \colon x_1, x_2 \in I_n(i), i = 1, 2, \ldots, 2^n\}. \tag{3}$$

and similarly,

$$\omega_{n,i} f = \sup\{|f(x_1) - f(x_2)| \colon x_1, x_2 \in I_n(i)\}. \tag{4}$$

for all $i = 1, 2, \ldots, 2^n$. Obviously,

$$\omega_n f = \max\{\omega_{n,i} f \colon i = 1, 2, \ldots, 2^n\}.$$

Since every function $f \in C_W$ is uniformly continuous on the interval $[0,1[$ with respect to the dyadic topology, we have $\omega_n f \searrow 0$. If we have the sequence of dyadic intervals $I_k(i_k) \supseteq I_{k+1}(i_{k+1}) \supseteq \ldots$ and f is a continuous function on $[0,1[$ such that $\omega_{k,i_k} f$ is finite then $\omega_{n,i_n} f \searrow 0$.

For every $f \in C_W$ and $x \in [0,1[$ we have (see [4])

$$\frac{1}{2}\omega_n f \le |f(x) - S_{2^n} f(x)| \le \omega_n f. \tag{5}$$

The inequalities above imply the fact that for all function f belongs to the space C_W, especially if f is continuous on the usual topology, the partial sums $S_{2^n} f$ converges uniformly to the function f. Moreover, the dyadic modulus of continuity indicates the rate of convergence. Similarly, if $x \in I_n(i)$ we have

$$\frac{1}{2}\omega_{n,i} f \le |f(x) - S_{2^n} f(x)| \le \omega_{n,i} f. \tag{6}$$

It is also important to note that if the function f satisfies the Lipschitz condition, i.e. there is a constant L such that

$$|f(x_1) - f(x_2)| \le L|x_1 - x_2|$$

for all $x_1, x_2 \in [0,1[$, then $\omega_n f \le \frac{L}{2^n}$. Similar statement is also true for the local moduli of continuity.

3. The triangular functions

Triangular functions are the set of integral functions of the Walsh-Paley system. We denote them by

$$J_k(x) := \int_0^x \omega_k(t)\, dt \qquad (k \in \mathbf{N}, 0 \le x < 1).$$

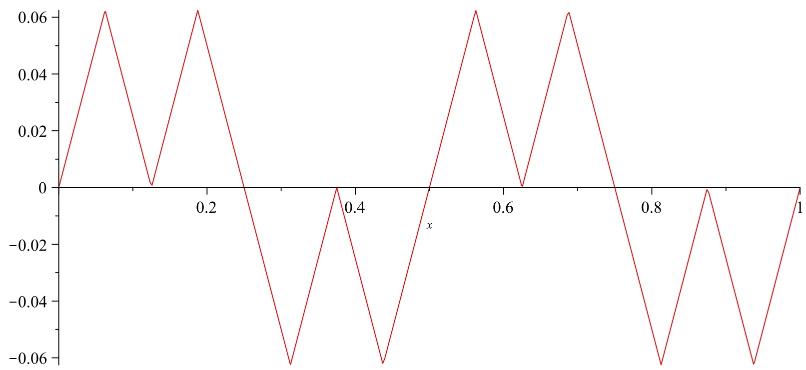

FIGURE 2. The triangular function J_{10}

Let us consider the Walsh-Fourier series of the triangular functions J_k, denoting the Fourier coefficients by $\widehat{J}_{k,j}$, and hence

$$J_k(x) = \sum_{j=0}^{\infty} \widehat{J}_{k,j}\omega_j(x).$$

Coefficients $\widehat{J}_{k,j}$ often take the value 0, in particular they have the following properties.

Lemma 1. *For every $n \in \mathbf{N}$ we have*

$$\begin{aligned}
&\widehat{J}_{0,0} = \frac{1}{2}, \\
&\widehat{J}_{k,j} = \frac{1}{2^{n+1}} && \textit{if} \quad 0 \le j < 2^{n-1} \le k < 2^n, \ k = 2^{n-1} + j, \\
&\widehat{J}_{k,j} = 0 && \textit{if} \quad 0 \le j < 2^{n-1} \le k < 2^n, \ k \ne 2^{n-1} + j, \\
&\widehat{J}_{k,j} = -\frac{1}{2^{n+1}} && \textit{if} \quad 0 \le k < 2^{n-1} \le j < 2^n, \ k = j - 2^{n-1}, \\
&\widehat{J}_{k,j} = 0 && \textit{if} \quad 0 \le k < 2^{n-1} \le j < 2^n, \ k \ne j - 2^{n-1}, \\
&\widehat{J}_{k,j} = 0 && \textit{if} \quad 2^{n-1} \le k, j < 2^n.
\end{aligned}$$

Proof. The values of coefficients $\widehat{J}_{k,j}$ can be obtained directly by the Fine's formulae (see [3])

$$J_0(x) = \frac{1}{2} - \frac{1}{4}\sum_{i=0}^{\infty} \frac{1}{2^i}\omega_{2^i}(x)$$

and

$$J_k(x) = \frac{1}{2^{n+1}}\left(\omega_l(x) - \sum_{i=1}^{\infty} \frac{1}{2^i}\omega_{2^{n-1+i}+k}(x)\right) \tag{7}$$

where $k = 2^{n-1} + l$, $0 \le l < 2^{n-1}$.

Indeed, if $2^{n-1} \leq k < 2^n$ then $2^{n-1+i} + k \geq 2^n$ for all $i \geq 1$, so in (7) only the coefficient with index $l = 2^{n-1} - k$ is not zero if we only consider indexes less than 2^n. Hence, $\widehat{J}_{k,j} \neq 0$ only if $j = l$, that is $k = 2^{n-1} + j$ and $\widehat{J}_{k,j} = \frac{1}{2^{n+1}}$.

On the other hand, if $k < 2^{n-1}$ and $k = 2^{n'} + l$, where $0 \leq l < 2^{n'}$, then $2^{n-1} \leq 2^{n'+i} + k < 2^n$ only for $i = n - 1 - n'$. For this reason, the Walsh series

$$J_k(x) = \frac{1}{2^{n'+1}} \left(\omega_l(x) - \sum_{i=1}^{\infty} \frac{1}{2^i} \omega_{2^{n'+i}+k}(x) \right).$$

only have one non-zero coefficient with index j such that $2^{n-1} \leq j < 2^n$. In this case, $j = 2^{n'+(n-1-n')} + k = 2^{n-1} + k$ and $\widehat{J}_{k,j} = \frac{1}{2^{n'+1}} \cdot \frac{-1}{2^{n-1-n'}} = -\frac{1}{2^{n+1}}$.

This completes the proof of the lemma. $\square$

Denote by $\widehat{J}^{(n)}$ the matrix of size 2^n with entries $\widehat{J}_{k,j}$, where $k, j = 0, 1, \ldots, 2^n - 1$. The properties of $\widehat{J}_{k,j}$ allow us to construct the matrices $\widehat{J}^{(n)}$ in an easier way as follows:

$$\left(\begin{array}{c|c} \begin{array}{c|c} \begin{array}{c|c} \frac{1}{2} & \ddots \\ \hline \ddots & \ddots \end{array} & -\frac{1}{2^n}\mathcal{I}_{2^{n-2}} \\ \hline \frac{1}{2^n}\mathcal{I}_{2^{n-2}} & 0_{2^{n-2}} \end{array} & -\frac{1}{2^{n+1}}\mathcal{I}_{2^{n-1}} \\ \hline \frac{1}{2^{n+1}}\mathcal{I}_{2^{n-1}} & 0_{2^{n-1}} \end{array} \right)$$

where $\mathcal{I}_j$ and 0_j are the identity and null matrix of size j.

For example

$$\widehat{J}^{(3)} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & 0 & -\frac{1}{16} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & -\frac{1}{8} & 0 & -\frac{1}{16} & 0 & 0 \\ \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 \\ 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} \\ \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the matrices $\widehat{J}^{(n)}$ can be also constructed by the iteration

$$\widehat{J}^{(0)} = \left(\frac{1}{2}\right), \qquad \widehat{J}^{(n)} = \left(\begin{array}{c|c} \widehat{J}^{(n-1)} & -\frac{1}{2^{n+1}}\mathcal{I}_{2^{n-1}} \\ \hline \frac{1}{2^{n+1}}\mathcal{I}_{2^{n-1}} & 0_{2^{n-1}} \end{array}\right).$$

The following lemma is the key to determine the solvability of the discretized integral equation (2).

Lemma 2. *Let $a, b \in \mathbf{R}$. Then*

$$\det(b\mathcal{I}_{2^n} + a\widehat{J}^{(n)\top}) = \left(b + \frac{a}{2^{n+1}}\right)^{2^n}.$$

Proof. We prove the statement of the lemma by induction.

$$\widehat{J}^{(1)\top} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & 0 \end{pmatrix}, \qquad b\mathcal{I}_2 + a\widehat{J}^{(1)\top} = \begin{pmatrix} b + \frac{a}{2} & \frac{a}{4} \\ -\frac{a}{4} & b \end{pmatrix},$$

and

$$\det(b\mathcal{I}_2 + a\widehat{J}^{(1)\top}) = b^2 + \frac{1}{2}ab + \frac{a^2}{16} = \left(b + \frac{a}{2^2}\right)^2,$$

so the statement of the lemma holds for $n = 1$. Moreover,

$$\widehat{J}^{(n+1)\top} = \left(\begin{array}{c|c} \widehat{J}^{(n)\top} & \frac{1}{2^{n+2}}\mathcal{I}_{2^n} \\ \hline -\frac{1}{2^{n+2}}\mathcal{I}_{2^n} & 0_{2^n} \end{array}\right),$$

and hence

$$b\mathcal{I}_{2^{n+1}} + a\widehat{J}^{(n+1)\top} = \left(\begin{array}{c|c} b\mathcal{I}_{2^n} + a\widehat{J}^{(n)\top} & \frac{a}{2^{n+2}}\mathcal{I}_{2^n} \\ \hline -\frac{a}{2^{n+2}}\mathcal{I}_{2^n} & b\mathcal{I}_{2^n} \end{array}\right).$$

The determinant of matrices partitioned in four blocks with the same size satisfies Schur's formula

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$$

whenever the matrices A, B, C and D commute pairwise. It is clear that the matrices in $b\mathcal{I}_{2^{n+1}} + a\widehat{J}^{(n+1)\top}$ satisfy the condition above, so from Schur's formula we have

$$\begin{aligned}\det(b\mathcal{I}_{2^{n+1}} + a\widehat{J}^{(n+1)\top}) &= \det((b\mathcal{I}_{2^n} + a\widehat{J}^{(n)\top})b\mathcal{I}_{2^n} - (-\frac{a}{2^{n+2}}\mathcal{I}_{2^n})(\frac{a}{2^{n+2}}\mathcal{I}_{2^n})) \\ &= \det\left(\left(b^2 + \left(\frac{a}{2^{n+2}}\right)^2\right)\mathcal{I}_{2^n} + ab\widehat{J}^{(n)\top}\right).\end{aligned}$$

Under the assumption that the statement of the lemma holds for n, we obtain

$$\det(b\mathcal{I}_{2^{n+1}} + a\widehat{J}^{(n+1)\top}) = \left(b^2 + \left(\frac{a}{2^{n+2}}\right)^2 + \frac{ab}{2^{n+1}}\right)^{2^n}$$

$$= \left(\left(b + \frac{a}{2^{n+2}}\right)^2\right)^{2^n}$$
$$= \left(b + \frac{a}{2^{n+2}}\right)^{2^{n+1}},$$

which implies that the statement of the lemma also holds for $n+1$. This completes the proof of the lemma. $\square$

4. The numerical solution of the integral equation

Our aims is to approach the solution of the integral equation

$$y(x) = \eta + \int_0^x q(t) - ay(t)\,dt \qquad (0 \le x < 1) \tag{8}$$

by Walsh polynomials, where $0 \le x < 1$, $a \in \mathbf{R}$, q is a continuous function on the interval $[0,1[$ such that

$$\int_0^1 q(t)\,dt < \infty.$$

In this regard, consider the Walsh polynomials

$$\overline{y}_n(x) = \sum_{k=0}^{2^n-1} c_k \omega_k(x) \tag{9}$$

satisfying the discretized integral equation (2), that is

$$\overline{y}_n(x) = \eta + S_{2^n}\left(\int_0^{\cdot} S_{2^n} q(t) - a\overline{y}_n(t)\,dt\right)(x)$$

where ω_k is the kth Walsh-Paley function and operators $S_{2^n} f$ are the 2^n-th partial sums of Walsh-Fourier series of an integrable function f on the interval $[0,1[$.

In order to find the coefficients of the Walsh polynomial $\overline{y}_n$ for a fixed natural number n we introduce the following column vectors:

$$\begin{aligned}
\mathbf{c}^\top &:= (c_0, c_1, \dots, c_{2^n-1}),\\
\widehat{\mathbf{q}}^\top &:= (\widehat{q}_0, \widehat{q}_1, \dots, \widehat{q}_{2^n-1}),\\
\boldsymbol{\omega}(x)^\top &:= (\omega_0(x), \omega_1(x), \dots, \omega_{2^n-1}(x)),\\
\mathbf{e_0}^\top &:= (1, 0, \dots, 0) \text{ with size } 2^n
\end{aligned}$$

and the matrix

$$\widehat{J} := (\widehat{J}_{k,j})_{k,j=0}^{2^n-1}.$$

With these matrix notations the discretized integral equation (2) can be written as follows

$$\begin{aligned}
\boldsymbol{\omega}(x)^\top \mathbf{c} &= \eta + S_{2^n}\left(\int_0^{\cdot} \boldsymbol{\omega}(t)^\top \widehat{\mathbf{q}} - a\boldsymbol{\omega}(t)^\top \mathbf{c}\,dt\right)(x)\\
&= \boldsymbol{\omega}(x)^\top \eta \mathbf{e_0} + S_{2^n}\left(\int_0^{\cdot} \boldsymbol{\omega}(t)^\top\,dt\right)(x) \cdot (\widehat{\mathbf{q}} - a\mathbf{c})
\end{aligned}$$

$$= \boldsymbol{\omega}(x)^\top \eta \mathbf{e_0} + \boldsymbol{\omega}(x)^\top \widehat{J}^\top(\widehat{\mathbf{q}} - a\mathbf{c})$$
$$= \boldsymbol{\omega}(x)^\top (\eta \mathbf{e_0} + \widehat{J}^\top(\widehat{\mathbf{q}} - a\mathbf{c})).$$

at every point of $[0, 1[$. Thus, the equation above also holds for the coefficients of the Walsh polynomials from which we obtain the linear system

$$\mathbf{c} = \eta \mathbf{e_0} + \widehat{J}^\top(\widehat{\mathbf{q}} - a\mathbf{c})$$

involving the variables $c_0, c_1, \dots, c_{2^n-1}$ and it can be written as follows

$$(\mathcal{I} + a\widehat{J}^\top)\mathbf{c} = \eta \mathbf{e_0} + \widehat{J}^\top \widehat{\mathbf{q}}, \tag{10}$$

where $\mathcal{I}$ is the identity matrix of size 2^n.

The solvability of the linear system (10) only depend on whether the value of $\det(\mathcal{I} + a\widehat{J}^\top)$ is zero or not. By Lemma 2 for $b = 1$ we obtain

$$\det(\mathcal{I} + a\widehat{J}^\top) = \Big(1 + \frac{a}{2^{n+1}}\Big)^{2^n}$$

from which we obtain that if $2^{n+1} \neq -a$, then the linear system (10) has an unique solution given by the formula

$$\mathbf{c} = (\mathcal{I} + a\widehat{J}^\top)^{-1}(\eta \mathbf{e_0} + \widehat{J}^\top \widehat{\mathbf{q}}). \tag{11}$$

This completes the proof of the first part of Theorem 1.

5. A multistep method to obtain the numerical solution

Summarizing the last section, the method of C. F. Chen and C. H. Hsiao is based on discretizing the integral equation (8) to obtain the linear system (10). After solving it, we have to generate all of Walsh functions with indices up to $2^n - 1$ and take their linear combination with the solution of the linear system, according with

$$\overline{y}_n(x) = \sum_{k=0}^{2^n-1} c_k \omega_k(x).$$

We propose a faster method to obtain directly the values of $\overline{y}_n$ without needing to generate Walsh functions or solve the linear system (10). The method is based on the fact that $\overline{y}_n$ is constant on the dyadic intervals $I_n(i)$, so the sequence $\overline{y}_n(\frac{i-1}{2^n})$ $(i = 1, 2, \dots, 2^n)$ contains all values of $\overline{y}_n$. The point is to calculate the value of $\overline{y}_n(\frac{i-1}{2^n})$ directly from the values $\overline{y}_n(\frac{k}{2^n})$ $(k = 0, 1, \dots, i-2)$ starting from the value of $\overline{y}_n(0)$.

Let $\chi_{I_n(k)}$ be the characteristic function of the interval $I_n(k)$, where $k = 1, 2, \dots, 2^n$. It is not hard to see that

$$S_{2^n}\Big(\int_0^\cdot \chi_{I_n(k)}(t)\,dt\Big)(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{k-1}{2^n}, \\ \frac{1}{2^{n+1}} & \text{if } \frac{k-1}{2^n} \le x < \frac{k}{2^n}, \\ \frac{1}{2^n} & \text{if } \frac{k}{2^n} \le x < 1. \end{cases} \tag{12}$$

In order to simplify the notations denote by $\widetilde{q}_n := S_{2^n} q$ the 2^n-th partial sums of the Walsh series of the function q. Thus,

$$\widetilde{q}_n(x) = 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} q(t)\,dt$$

for all $x \in I_n(i)$, specially if $x = \frac{i-1}{2^n}$.

The function $\widetilde{q}_n(x) - a\overline{y}_n(x)$ is also constant on the dyadic intervals $I_n(i)$. For this reason

$$\widetilde{q}_n(x) - a\overline{y}_n(x) = \sum_{k=1}^{2^n} \left(\widetilde{q}_n(\frac{k-1}{2^n}) - a\overline{y}_n(\frac{k-1}{2^n}) \right) \chi_{I_n(k)}(x).$$

Suppose $x \in I_n(i)$ for some $i = 1, 2, \ldots, 2^n$. Then by (2) we have

$$\begin{aligned}
\overline{y}_n(x) &= \eta + S_{2^n} \left(\int_0^{\cdot} \sum_{k=1}^{2^n} \left(\widetilde{q}_n(\frac{k-1}{2^n}) - a\overline{y}_n(\frac{k-1}{2^n}) \right) \chi_{I_n(k)}(t)\,dt \right)(x) \\
&= \eta + \sum_{k=1}^{2^n} \left(\widetilde{q}_n(\frac{k-1}{2^n}) - a\overline{y}_n(\frac{k-1}{2^n}) \right) S_{2^n} \left(\int_0^{\cdot} \chi_{I_n(k)}(t)\,dt \right)(x) \\
&= \eta + \frac{1}{2^n} \sum_{k=1}^{i-1} \left(\widetilde{q}_n(\frac{k-1}{2^n}) - a\overline{y}_n(\frac{k-1}{2^n}) \right) + \frac{1}{2^{n+1}} \left(\widetilde{q}_n(\frac{i-1}{2^n}) - a\overline{y}_n(\frac{i-1}{2^n}) \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
(13) \quad & \overline{y}_n(\frac{i-1}{2^n}) \\
&= \frac{1}{1 + \frac{a}{2^{n+1}}} \left(\eta + \frac{1}{2^n} \sum_{k=1}^{i-1} \left(\widetilde{q}_n(\frac{k-1}{2^n}) - a\overline{y}_n(\frac{k-1}{2^n}) \right) + \frac{1}{2^{n+1}} \widetilde{q}_n(\frac{i-1}{2^n}) \right),
\end{aligned}$$

which is easy to obtain by a recursive algorithm starting from the value

$$\overline{y}_n(0) = \frac{1}{1 + \frac{a}{2^{n+1}}} \left(\eta + \frac{1}{2^{n+1}} \widetilde{q}_n(0) \right).$$

6. Estimation of errors

The aim of this section is to find an upper estimation of the difference

$$|y(x) - \overline{y}_n(x)|$$

at every point $x \in [0, 1[$, where $\overline{y}_n$ is the Walsh polynomial which satisfies the discretized integral equation (2) and y is at the same time the unique solution of the Cauchy problem (1), i.e.

$$y' + ay = q(x)$$
$$y(0) = \eta$$

and the equivalent integral equation (8), where q is a continuous function on the interval $[0, 1[$ and integrable on the interval $[0, 1]$.

The estimate will be established by using the values of a and η, as well as the 2^n-th partial sums of the Walsh series of the function $|q|$, denoted by $\widetilde{|q|}_n$. Since $|q|$ is integrable on the interval $[0,1]$, the function $\widetilde{|q|}_n$ is constant on the dyadic intervals $I_n(i)$ and every values

$$\widetilde{|q|}_n(\frac{i-1}{2^n}) = 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} |q(t)|\,dt \qquad (i = 1,2,\dots,2^n)$$

are finite.

The conditions assumed for the function q do not ensure that the dyadic modulus of continuity of q is finite. For this to happen, the limit of the function from the left of 1 needs to be finite. Nevertheless, the solution y always has a finite dyadic modulus of continuity. Indeed, it is a very well known fact that the solution of the Cauchy problem (1) is given by the formula

$$y(x) = e^{-ax}\left(\eta + \int_0^x q(t)e^{at}\,dt\right) \qquad (0 \le x < 1). \tag{14}$$

Thus, the limit

$$\lim_{x\to 1^-} y(x) = e^{-a}\left(\eta + \int_0^1 q(t)e^{at}\,dt\right)$$

is finite since q is integrable on the interval $[0,1]$. The fact above allows us to establish the estimation in two steps as follows

$$|y(x) - \overline{y}_n(x)| \le |y(x) - S_{2^n}y(x)| + |S_{2^n}y(x) - \overline{y}_n(x)| \tag{15}$$

for all $x \in [0,1[$.

We start first with the estimation of the first part. Note that this part does not depend on the numerical solution $\overline{y}_n$. Suppose that x is in the dyadic interval $I_n(i)$ for some $i = 1,2,\dots,2^n$. Then by (6) we obtain

$$|y(x) - S_{2^n}y(x)| \le \omega_{n,i}y.$$

So we will estimate the value of $\omega_{n,i}y$ and in this regard we study the supremum of $|y(x_2) - y(x_1)|$, where $x_1, x_2 \in I_n(i)$ and $x_1 < x_2$, according with (4). By (14) we have

$$y(x_2) - y(x_1) = e^{-ax_2}\left(\eta + \int_0^{x_2} q(t)e^{at}\,dt\right) - e^{-ax_1}\left(\eta + \int_0^{x_1} q(t)e^{at}\,dt\right).$$

The expression above can be written in two different forms as follows

$$\begin{aligned} y(x_2) - y(x_1) =& (e^{-ax_2} - e^{-ax_1})\left(\eta + \int_0^{x_1} q(t)e^{at}\,dt\right) + e^{-ax_2}\int_{x_1}^{x_2} q(t)e^{at}\,dt \\ =& (e^{-a(x_2-x_1)} - 1)\left(e^{-ax_1}\eta + e^{-ax_1}\int_0^{x_1} q(t)e^{at}\,dt\right) \\ &+ e^{-ax_2}\int_{x_1}^{x_2} q(t)e^{at}\,dt \end{aligned} \tag{16}$$

or

$$\begin{aligned} y(x_2)-y(x_1) =&(e^{-ax_2}-e^{-ax_1})\left(\eta+\int_0^{x_2} q(t)e^{at}\,dt\right)+e^{-ax_1}\int_{x_1}^{x_2} q(t)e^{at}\,dt \\ =&(1-e^{a(x_2-x_1)})\left(e^{-ax_2}\eta+e^{-ax_2}\int_0^{x_2} q(t)e^{at}\,dt\right) \\ &+e^{-ax_1}\int_{x_1}^{x_2} q(t)e^{at}\,dt. \end{aligned} \tag{17}$$

Let us suppose that $a \geq 0$. In this case e^{ax} is a positive and monotone increasing function. By the fact that

$$1-e^{-x}<x \qquad (x>0),$$

from (16) we obtain the following estimation.

$$\begin{aligned} |y(x_2)-y(x_1)| \leq&(1-e^{-a(x_2-x_1)})\left(e^{-ax_1}|\eta|+e^{-ax_1}\int_0^{x_1} |q(t)|e^{at}\,dt\right) \\ &+e^{-ax_2}\int_{x_1}^{x_2} |q(t)|e^{at}\,dt \\ \leq&a(x_2-x_1)\left(e^{-ax_1}|\eta|+e^{-ax_1}\int_0^{x_1} |q(t)|e^{ax_1}\,dt\right) \\ &+e^{-ax_2}\int_{x_1}^{x_2} |q(t)|e^{ax_2}\,dt \\ =&a(x_2-x_1)\left(e^{-ax_1}|\eta|+\int_0^{x_1} |q(t)|\,dt\right)+\int_{x_1}^{x_2} |q(t)|\,dt \\ \leq&\frac{a}{2^n}\left(e^{-a\frac{i-1}{2^n}}|\eta|+\int_0^{\frac{i}{2^n}} |q(t)|\,dt\right)+\int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} |q(t)|\,dt \\ =&\frac{a}{2^n}\left(e^{-a\frac{i-1}{2^n}}|\eta|+\int_0^{\frac{i}{2^n}} |q(t)|\,dt\right)+\frac{1}{2^n}\widetilde{|q|}_n(\frac{i-1}{2^n}). \end{aligned}$$

Similarly, if $a<0$ then e^{ax} is a positive and monotone decreasing function, hence from (17) we obtain the following estimation.

$$\begin{aligned} |y(x_2)-y(x_1)| \leq&(1-e^{a(x_2-x_1)})\left(e^{-ax_2}|\eta|+e^{-ax_2}\int_0^{x_2} |q(t)|e^{at}\,dt\right) \\ &+e^{-ax_1}\int_{x_1}^{x_2} |q(t)|e^{at}\,dt \\ \leq&-a(x_2-x_1)\left(e^{-ax_2}|\eta|+e^{-ax_2}\int_0^{x_2} |q(t)|\,dt\right) \\ &+e^{-ax_1}\int_{x_1}^{x_2} |q(t)|e^{ax_1}\,dt \end{aligned}$$

$$\begin{aligned}
&= - a(x_2 - x_1)e^{-ax_2}\left(|\eta| + \int_0^{x_2} |q(t)|\,dt\right) + \int_{x_1}^{x_2} |q(t)|\,dt \\
&\leq - \frac{a}{2^n}e^{-a\frac{i}{2^n}}\left(|\eta| + \int_0^{\frac{i}{2^n}} |q(t)|\,dt\right) + \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} |q(t)|\,dt \\
&= - \frac{a}{2^n}e^{-a\frac{i}{2^n}}\left(|\eta| + \int_0^{\frac{i}{2^n}} |q(t)|\,dt\right) + \frac{1}{2^n}\widetilde{|q|}_n(\frac{i-1}{2^n}).
\end{aligned}$$

In summary, by the notation

$$E_1^{(n)}(x) := \begin{cases} \frac{a}{2^n}\left(e^{-a\frac{i-1}{2^n}}|\eta| + \int_0^{\frac{i}{2^n}} |q(t)|\,dt\right) + \frac{1}{2^n}\widetilde{|q|}_n(\frac{i-1}{2^n}) & \text{if } a \geq 0, \\ \frac{-ae^{-a\frac{i}{2^n}}}{2^n}\left(|\eta| + \int_0^{\frac{i}{2^n}} |q(t)|\,dt\right) + \frac{1}{2^n}\widetilde{|q|}_n(\frac{i-1}{2^n}) & \text{if } a < 0. \end{cases} \tag{18}$$

if $x \in I_n(i)$ we obtain

$$|y(x) - S_{2^n}y(x)| \leq E_1^{(n)}(x). \tag{19}$$

We shall now continue with the estimation of the second part of (15). For this purpose, we introduce the function

$$z_n(x) := \overline{y}_n(x) - S_{2^n}y(x) \qquad (x \in [0,1[).$$

Thus, by (2) and (8) we obtain

$$\begin{aligned}
z_n(x) &= \eta + S_{2^n}\left(\int_0^{\cdot} S_{2^n}q(t) - a\overline{y}_n(t)\,dt\right)(x) - S_{2^n}\left(\eta + \int_0^{\cdot} q(t) - ay(t)\,dt\right)(x) \\
&= S_{2^n}\left(\int_0^{\cdot} S_{2^n}q(t) - q(t)\,dt\right)(x) + aS_{2^n}\left(\int_0^{\cdot} y(t) - \overline{y}_n(t)\,dt\right)(x) \\
&= S_{2^n}\left(\int_0^{\cdot} S_{2^n}q(t) - q(t)\,dt\right)(x) + aS_{2^n}\left(\int_0^{\cdot} y(t) - S_{2^n}y(t)\,dt\right)(x) \\
&\quad - aS_{2^n}\left(\int_0^{\cdot} z_n(t)\,dt\right)(x).
\end{aligned}$$

Since z_n is constant on the dyadic intervals, it can be written as

$$z_n(x) = \sum_{k=1}^{2^n} z_n(\frac{k-1}{2^n})\chi_{I_n(k)}(x) \qquad (x \in [0,1[).$$

Suppose that x is in the dyadic interval $I_n(i)$ for some $i = 1, 2, \ldots, 2^n$. Then, by (12) we obtain (see the method applied in Section 5)

$$S_{2^n}\left(\int_0^{\cdot} z_n(t)\,dt\right)(x) = \frac{1}{2^n}\sum_{k=1}^{i-1} z_n(\frac{k-1}{2^n}) + \frac{1}{2^{n+1}}z_n(\frac{i-1}{2^n}).$$

Thus,

$$z_n(x) = z_n(\frac{i-1}{2^n}) = S_{2^n}\left(\int_0^{\cdot} S_{2^n}q(t) - q(t)\,dt\right)(\frac{i-1}{2^n})$$

$$+ aS_{2^n}\left(\int_0^{\cdot} y(t) - S_{2^n}y(t)\,dt\right)(\frac{i-1}{2^n})$$
$$- \frac{a}{2^n}\sum_{k=1}^{i-1} z_n(\frac{k-1}{2^n}) - \frac{a}{2^{n+1}} z_n(\frac{i-1}{2^n}),$$

from which we obtain

$$\begin{aligned}\left(1+\frac{a}{2^{n+1}}\right) z_n(\frac{i-1}{2^n}) =& S_{2^n}\left(\int_0^{\cdot} S_{2^n}q(t) - q(t)\,dt\right)(\frac{i-1}{2^n})\\ &+ aS_{2^n}\left(\int_0^{\cdot} y(t) - S_{2^n}y(t)\,dt\right)(\frac{i-1}{2^n})\\ &- \frac{a}{2^n}\sum_{k=1}^{i-1} z_n(\frac{k-1}{2^n}).\end{aligned}\tag{20}$$

If $a < 0$ and $2^{n+1} = -a$ then the value of the expression $1 + \frac{a}{2^{n+1}}$ can be 0. In this case, we can not calculate the value of $z_n(\frac{i-1}{2^n})$ from (20). From this point forward, we suppose that n is large enough to satisfy the condition $2^{n+1} \geq -3a$.

Now we introduce the following notations:

$$\begin{aligned}M_{n,i} := \frac{1}{1+\frac{a}{2^{n+1}}}\Bigg| & S_{2^n}\left(\int_0^{\cdot} S_{2^n}q(t) - q(t)\,dt\right)(\frac{i-1}{2^n})\\ &+aS_{2^n}\left(\int_0^{\cdot} y(t) - S_{2^n}y(t)\,dt\right)(\frac{i-1}{2^n})\Bigg|,\end{aligned}$$

$$z_n^* := |z_n| \qquad \text{and} \qquad b_n := \frac{\frac{|a|}{2^n}}{1+\frac{a}{2^{n+1}}} = \frac{|a|}{2^n + \frac{a}{2}}.$$

With the new notations from (20) we obtain

$$z_n^*(\frac{i-1}{2^n}) \leq M_{n,i} + b_n \sum_{k=1}^{i-1} z_n^*(\frac{k-1}{2^n})$$

for all $i = 1, 2, \ldots, 2^n$, from which by induction we can prove that

$$z_n^*(\frac{i-1}{2^n}) \leq \max_{j\leq i}\{M_{n,j}\}(1+b_n)^{i-1}.$$

Since $x \in I_n(i)$, for which $i - 1 \leq x2^n$ holds, we obtain

$$z_n^*(\frac{i-1}{2^n}) \leq \max_{j\leq i}\{M_{n,j}\}(1+b_n)^{x2^n}.\tag{21}$$

First we estimate the expression $(1+b_n)^{x2^n}$. If $a \geq 0$ then

$$(1+b_n)^{x2^n} = \left(1+\frac{a}{2^n+\frac{a}{2}}\right)^{x2^n} \leq \left(1+\frac{a}{2^n}\right)^{x2^n} < e^{ax}.$$

If $a < 0$ then

$$(1+b_n)^{x2^n} = \left(1+\frac{|a|}{2^n - \frac{|a|}{2}}\right)^{x2^n} = \left(1+\frac{|a|}{2^n - \frac{|a|}{2}}\right)^{x2^n - \frac{|a|x}{2}} \left(1+\frac{|a|}{2^n - \frac{|a|}{2}}\right)^{\frac{|a|x}{2}}$$

$$\leq \left(1+\frac{|a|}{2^n-\frac{|a|}{2}}\right)^{x2^n-\frac{|a|x}{2}}\left(1+\frac{|a|}{|a|}\right)^{\frac{|a|x}{2}} < e^{|a|x}2^{\frac{|a|x}{2}} = (\sqrt{2}e)^{|a|x},$$

since $2^n - \frac{|a|}{2} \geq |a|$ according with the assumption $2^{n+1} \geq -3a$.

With respect to the estimation of $M_{n,i}$ note that for all integrable function f we have

$$\int_0^{\frac{j-1}{2^n}} S_{2^n}f(t) - f(t)\,dt = 0 \qquad (j = 1, 2, \ldots, 2^n),$$

hence

$$\begin{aligned} S_{2^n}\left(\int_0^{\cdot} S_{2^n}f(t) - f(t)\,dt\right)\left(\frac{i-1}{2^n}\right) &= 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \int_0^x S_{2^n}f(t) - f(t)\,dt\,dx \\ &= 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \int_{\frac{i-1}{2^n}}^x S_{2^n}f(t) - f(t)\,dt\,dx. \end{aligned} \tag{22}$$

We use the formula above in case of $f = y$ and $f = q$ to estimate the value of $M_{n,i}$. First, note that the solution y has a finite dyadic modulus of continuity. Thus, by (22) we have

$$\begin{aligned} \left|S_{2^n}\left(\int_0^{\cdot} S_{2^n}y(t) - y(t)\,dt\right)\left(\frac{i-1}{2^n}\right)\right| &\leq 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \int_{\frac{i-1}{2^n}}^x |S_{2^n}y(t) - y(t)|\,dt\,dx \\ &\leq 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \omega_{n,i}y\,dt\,dx \\ &\leq \frac{1}{2^n}\omega_{n,i}y \leq \frac{1}{2^n}E_1^{(n)}(x). \end{aligned} \tag{23}$$

However, the dyadic modulus of continuity of the function q may not be finite, so we estimate this part in a different way. By (22) we have

$$\begin{aligned} S_{2^n}\left(\int_0^{\cdot} S_{2^n}q(t) - q(t)\,dt\right)\left(\frac{i-1}{2^n}\right) &= 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \int_{\frac{i-1}{2^n}}^x S_{2^n}q(t) - q(t)\,dt\,dx \\ &= 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \int_{\frac{i-1}{2^n}}^x S_{2^n}q(t)\,dt\,dx - 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \int_{\frac{i-1}{2^n}}^x q(t)\,dt\,dx. \end{aligned}$$

$S_{2^n}q$ is constant on the interval $I_n(i)$, hence for the first integral we have

$$\begin{aligned} 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \int_{\frac{i-1}{2^n}}^x S_{2^n}q(t)\,dt\,dx &= 2^n S_{2^n}q\left(\frac{i-1}{2^n}\right) \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \int_{\frac{i-1}{2^n}}^x 1\,dt\,dx \\ &= 2^n S_{2^n}q\left(\frac{i-1}{2^n}\right) \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} x - \frac{i-1}{2^n}\,dx \\ &= 2^n S_{2^n}q\left(\frac{i-1}{2^n}\right) \int_0^{\frac{1}{2^n}} x\,dx \\ &= \frac{1}{2^{n+1}} S_{2^n}q\left(\frac{i-1}{2^n}\right) \end{aligned}$$

$$= 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \frac{1}{2^{n+1}} q(t)\, dt.$$

For the second integral we obtain

$$2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \int_{\frac{i-1}{2^n}}^{x} q(t)\, dt\, dx = 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \int_{t}^{\frac{i}{2^n}} q(t)\, dx\, dt = 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} q(t)(\frac{i}{2^n} - t)\, dt.$$

By subtracting the two integrals we obtain

$$\begin{aligned} S_{2^n}\left(\int_0^{\cdot} S_{2^n} q(t) - q(t)\, dt\right)(\frac{i-1}{2^n}) &= 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \frac{1}{2^{n+1}} q(t)\, dt - 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} q(t)(\frac{i}{2^n} - t)\, dt \\ &= 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \left(\frac{1}{2^{n+1}} - \frac{i}{2^n} + t\right) q(t)\, dt. \end{aligned}$$

Thus,

$$\begin{aligned} \left|S_{2^n}\left(\int_0^{\cdot} S_{2^n} q(t) - q(t)\, dt\right)(\frac{i-1}{2^n})\right| &\le 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \left|\frac{1}{2^{n+1}} - \frac{i}{2^n} + t\right| |q(t)|\, dt \\ &\le 2^n \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \frac{1}{2^{n+1}} |q(t)|\, dt \\ &= \frac{1}{2^{n+1}} \widetilde{|q|}_n(\frac{i-1}{2^n}), \end{aligned} \tag{24}$$

since

$$-\frac{1}{2^{n+1}} \le \frac{1}{2^{n+1}} - \frac{i}{2^n} + t \le \frac{1}{2^{n+1}}$$

for all $t \in I_n(i)$.

By the results of (23) and (24) we obtain

$$\begin{aligned} M_{n,i} &\le \frac{1}{1 + \frac{a}{2^{n+1}}} \left(\frac{1}{2^{n+1}} \widetilde{|q|}_n(\frac{i-1}{2^n}) + \frac{|a|}{2^n} E_1^{(n)}(x)\right) \\ &= \frac{1}{2^{n+1} + a} \left(\widetilde{|q|}_n(\frac{i-1}{2^n}) + 2|a| E_1^{(n)}(x)\right). \end{aligned}$$

In summary, by the notation

(25)

$$E_2^{(n)}(x) := \begin{cases} \frac{1}{2^{n+1}+a}\left(\max_{j\le i}\{\widetilde{|q|}_n(\frac{j-1}{2^n})\} + 2|a| \sup_{t\le x}\{E_1^{(n)}(t)\}\right) e^{ax} & \text{if } a \ge 0, \\ \frac{1}{2^{n+1}+a}\left(\max_{j\le i}\{\widetilde{|q|}_n(\frac{j-1}{2^n})\} + 2|a| \sup_{t\le x}\{E_1^{(n)}(t)\}\right) (\sqrt{2}e)^{-ax} & \text{if } a < 0, \end{cases}$$

if $x \in I_n(i)$ we obtain from (21) the fact that

$$|S_{2^n} y(x) - \overline{y}_n(x)| \le E_2^{(n)}(x) \tag{26}$$

if $2^{n+1} \ge -3a$.

From (18) and (25) we can estimate of errors uniformly on $[0, 1[$. Indeed, by the notation

$$E_1^{(n)} := \begin{cases} \frac{a}{2^n}\left(e^{-a}|\eta| + \int_0^1 |q(t)|\, dt\right) + \frac{1}{2^n} \max_i \left\{\widetilde{|q|}_n(\frac{i-1}{2^n})\right\} & \text{if } a \ge 0, \\ \frac{-ae^{-a}}{2^n}\left(|\eta| + \int_0^1 |q(t)|\, dt\right) + \frac{1}{2^n} \max_i \left\{\widetilde{|q|}_n(\frac{i-1}{2^n})\right\} & \text{if } a < 0. \end{cases} \tag{27}$$

and

$$E_2^{(n)} := \begin{cases} \frac{1}{2^{n+1}+a}\left(\max_i\{\widetilde{|q|}_n(\frac{i-1}{2^n})\} + 2aE_1^{(n)}\right)e^a & \text{if } a \geq 0, \\ \frac{1}{2^{n+1}+a}\left(\max_i\{\widetilde{|q|}_n(\frac{i-1}{2^n})\} - 2aE_1^{(n)}\right)(\sqrt{2}e)^{-a} & \text{if } a < 0, \end{cases} \tag{28}$$

we obtain that

$$|y(x) - S_{2^n}y(x)| \leq E_1^{(n)}, \qquad |S_{2^n}y(x) - \overline{y}_n(x)| \leq E_2^{(n)} \quad \text{if } 2^{n+1} \geq -3a$$

for all $x \in [0,1[$. The fact that $E_1^{(n)}$ and $E_2^{(n)}$ tend to 0 completes the second part of Theorem 1.

7. Examples

In this section we test the effectiveness of the developed method. First we consider the following Cauchy problem.

$$\begin{aligned} y' + y &= (x+1)^2, \\ y(0) &= 1. \end{aligned} \tag{29}$$

The solution of the Cauchy problem is $y(x) = x^2 + 1$. The function $q(x) = (x+1)^2$ is continuous on the close interval $[0,1]$ and the value of the coefficient a is 1.

Figure 3 shows the approximation of $\overline{y}_n$ to the solution y in case of $n = 5$.

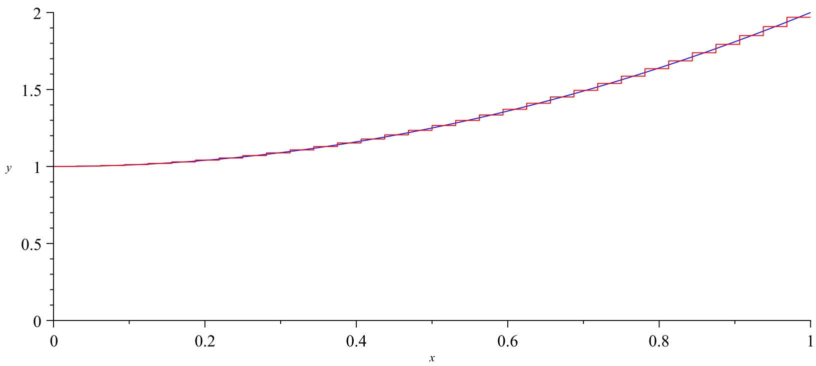

Figure 3. Approximation to the solution of the Cauchy problem (29) for $n = 5$.

Table 1 contains the supremum of the error of the approximation on the interval $[0,1[$, furthermore showing the two steps of the approximation and their respective estimates E_1 and E_2 for n from 3 to 10.

We observe that the error in first step halves and the error in second step is reduced to quarter with the increment of n.

The second example is the Cauchy problem

$$\begin{aligned} y' - 10y &= -\frac{10x+11}{(x+1)^2}, \\ y(0) &= 1. \end{aligned} \tag{30}$$

Table 1. Errors related to the Cauchy problem (29).

n	$\|y-\overline{y}_n\|$	$\|y-S_{2^n}y\|$	$E_1^{(n)}$	$\|\overline{y}_n-S_{2^n}y\|$	$E_2^{(n)}$
3	0.11877111	0.11979166	0.80705263	0.00245098	0.85854750
4	0.06095076	0.06119791	0.41109467	0.00063131	0.38702496
5	0.03086365	0.03092447	0.20746994	0.00016025	0.18203143
6	0.01552853	0.01554361	0.10421943	0.00004037	0.08802323
7	0.00778839	0.00779215	0.05223131	0.00001013	0.04324773
8	0.00390022	0.00390116	0.02614611	0.00000253	0.02143089
9	0.00195161	0.00195185	0.01308067	0.00000063	0.01066695
10	0.00097618	0.00097624	0.00654224	0.00000015	0.00532132

The solution of the Cauchy problem is $y(x)=\dfrac{1}{x+1}$. The function $q(x)=-\dfrac{10x+11}{(x+1)^2}$ is continuous on the close interval $[0,1]$ and $a=-10$.

Figure 4 shows the approximation of $\overline{y}_n$ to the solution y in case of $n=6$.

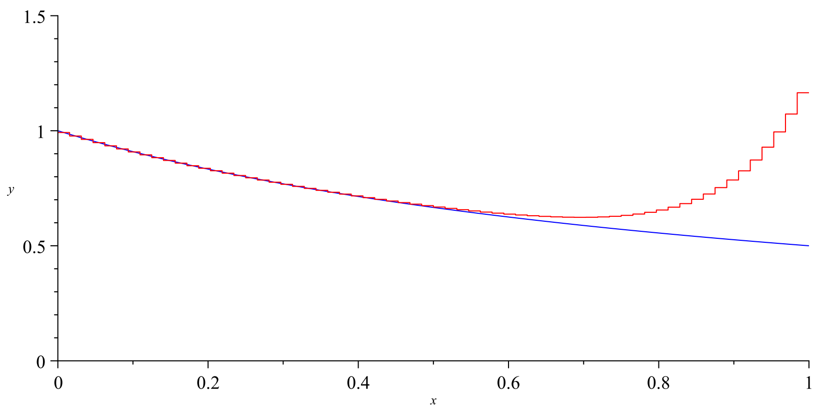

Figure 4. Approximation to the solution of the Cauchy problem (30) for $n=6$.

We can see that the approximation does not seem to be as effective for values close to 1. This may be a consequence of the fact that the value of a is negative and in this case the expression $2^{n+1}+a$ is not sufficiently large. Despite this problem, Table 2 shows that the approximation is uniform on the interval $[0,1[$.

TABLE 2. Errors related to the Cauchy problem (30).

n	$\|y-\overline{y}_n\|$	$\|y-S_{2^n}y\|$	$E_1^{(n)}$	$\|\overline{y}_n-S_{2^n}y\|$	$E_2^{(n)}$
4	12.04297028	0.03000606	$1.160728 \cdot 10^5$	12.03496554	$7.437634 \cdot 10^{10}$
5	2.63427763	0.01530692	$5.803643 \cdot 10^4$	2.63037763	$1.515080 \cdot 10^{10}$
6	0.66505331	0.00773206	$2.901822 \cdot 10^4$	0.66308995	$3.466743 \cdot 10^9$
7	0.17039122	0.00388602	$1.450911 \cdot 10^4$	0.16941211	$8.314705 \cdot 10^8$
8	0.04349030	0.00194805	$7.254555 \cdot 10^3$	0.04300138	$2.037345 \cdot 10^8$
9	0.01108835	0.00097529	$3.627277 \cdot 10^3$	0.01084405	$5.043516 \cdot 10^7$
10	0.00284565	0.00048796	$1.813638 \cdot 10^3$	0.00272354	$1.254882 \cdot 10^7$

The estimates $E_1^{(n)}$ and $E_2^{(n)}$ are very large with respect to the real errors in both steps. This is mainly due to the factor e^{-a} which appears in the formulae (27) and (28).

The last example is the Cauchy problem

$$\begin{aligned} y' + 10y &= \frac{1}{\sqrt{1-x}}, \\ y(0) &= 2. \end{aligned} \tag{31}$$

The solution of the Cauchy problem can be determined only numerically. The function $q(x) = \dfrac{1}{\sqrt{1-x}}$ has not finite limit from the left of 1, but it is integrable on the close interval $[0,1]$. Figure 5 and Table 3 shows that the developed method also works properly in this case, but the errors do not tends as fast to zero as the another examples in this section.

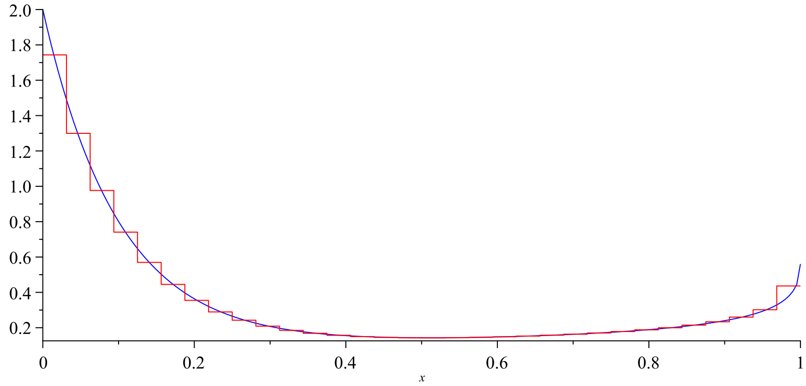

FIGURE 5. Approximation to the solution of the Cauchy problem (31) for $n = 5$.

TABLE 3. Errors related to the Cauchy problem (31).

n	$\lvert y-\overline{y}_n\rvert$	$\lvert y-S_{2^n}y\rvert$	$E_1^{(n)}$	$\lvert\overline{y}_n-S_{2^n}y\rvert$	$E_2^{(n)}$
3	0.72948575	0.81445644	3.20722028	0.08497069	$5.913365 \cdot 10^4$
4	0.45199682	0.48690734	1.75005675	0.05448919	$2.255150 \cdot 10^4$
5	0.25664949	0.26815629	0.97858176	0.04630729	$9.193175 \cdot 10^3$
6	0.13765262	0.14097948	0.56251418	0.03648475	$4.349474 \cdot 10^3$
7	0.07171151	0.09913818	0.33303378	0.02742667	$2.425237 \cdot 10^3$
8	0.05349971	0.07355717	0.20312854	0.02005746	$1.521706 \cdot 10^3$
9	0.03943617	0.05387821	0.12745262	0.01444204	$1.018327 \cdot 10^3$
10	0.02877332	0.03908371	0.08203213	0.01031039	$7.025419 \cdot 10^2$

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