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EXISTENCE OF LOCAL SOLUTIONS FOR SOME INTEGRO-DIFFERENTIAL EQUATIONS OF ARBITRARY FRACTIONAL ORDER

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Abstract. In this paper we investigate the existence and uniqueness of a local solution for different types of fractional integro-differential equations of any order. The results are obtained by using fixed point theorems. An example is introduced to illustrate the theorem.

1. Introduction

Fractional differential equations have been gained much attention during the past decades. The extensive results on initial and boundary value problems of fractional order for the problems of existence and uniqueness of solutions are due to the appropriate applicability of such problems to many realistic phenomenon. More precisely, the fractional differential equations are appeared in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, and fitting of experimental data (see [7]-[9] and references therein). The fact that fractional differential equations are considered as alternative models to nonlinear differential equations which induced extensive researches in various fields including the theoretical part. The existence and uniqueness problems of fractional nonlinear differential equations as a basic theoretical part are investigated by many authors (see [1]-[15]) and references therein). In [3] and [14], the authors obtained sufficient conditions for the existence of solutions for a class of boundary value problem for fractional differential equations in the cases of $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$ respectively by using the Caputo fractional derivative and nonlocal conditions. Whereas, The authors in [1] considered the existence problem of solutions for a class of boundary value problems for fractional differential equations of order $2 < \alpha \leq 3$ involving the Caputo fractional derivative. The existence

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and uniqueness of initial value problems for some fractional differential equations are investigated by many authors (see [4], [6], and [8]). The fractional integro-differential equations in different orders are investigated by [10]-[13] using Banach, Schaefer and Krasnoleskii fixed point theorems. The existence of local solutions to initial value problem of Cauchy type for fractional differential equations involving Caputo definition are deeply investigated in the books ([7], [15], and references therein). In fact, the equivalent Volterra or Fredholm integral equations to Cauchy problem for nonlinear fractional differential equations introduced in the cited articles are essential to prove the existence of such systems. Motivated by these works, we study in this paper the existence and uniqueness of a local solution to initial-value and boundary-valued Cauchy problem for some fractional integro-differential equations at any inner point of a finite interval involving the Caputo derivative. The results are obtained by applying the fixed point theorems on the corresponding Volterra and Fredholm integral equations.

2. Equivalent integral forms

Let $X = C(J, \mathbb{R})$ be a Banach space of all continuous real valued functions defined on the interval $J = [t_0, T]$, $t_0 \geq 0$, $T < \infty$, $\alpha \in (n-1, n]$, and $n \in \mathbb{N}$.

A function f is said to be fractional integrable of order $\alpha > 0$ (see [7], and [15]) if for all $t > t_0$,

$$I^{\alpha} f(t) = \left(I^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) ds,$$

exists and if $\alpha = 0$, then $I^0 f(t) = f(t)$. The Caputo fractional derivative of x is defined as ${}^{C}D_{t_0}^{\alpha} x(t) = I^{n-\alpha}\left(\frac{d^n x}{dt^n}\right)(t)$, for $t > t_0$, provided that $D^n x$ is fractional integrable of order $n - \alpha$.

In what follows, we assume that all functions below are fractional integrable functions of any order less than or equal to n on their domains.

For a given fractional differential equation, we obtain in this section some equivalent integral forms in order to use in the proof of the existence problems. We begin these forms by the following basic linear form.

Theorem 1. *Let $f, g \in X$, and F be fractional integrable with order α. Then, the fractional integro-differential system*

$$ {}^{C}D_{t_0}^{\alpha} x(t) = g(t) + \int_{t_0}^{t} f(s) ds, t \in J - \{t_*\} \tag{2.1}$$

$$x^{(k)}(t_*) = b_k \in \mathbb{R},\ k = 0, 1, 2, \ldots, n-1, t_* \in J$$

is equivalent to the Volterra integral equation

$$x(t) = \sum_{k=0}^{n-1} \frac{(t-t_*)^k}{k!} \left(b_k - I^{\alpha-k} F(t_*)\right) + I^{\alpha} F(t), t \in J, \tag{2.2}$$

where $F(t) = g(t) + \int\limits_{t_0}^{t} f(s)ds$.

Proof. See [13, Theorem 3.1]. □

In accordance with the proof of Theorem 1, it is not hard to deduce some equivalent forms of different nonlinear integro-differential systems. In what follows, assume that $G(t,x(t)) = g(t,x(t)) + \int\limits_{t_0}^{t} f(s,x(t))ds$.

Corollary 1. *The nonlinear fractional integro-differential system*

$$^{C}D_{t_0}^{\alpha} x(t) = g(t,x(t)) + \int\limits_{t_0}^{t} f(s,x(t))ds, , t \in J - \{t_*\} \tag{2.3}$$

$$x^{(k)}(t_*) = b_k \in \mathbb{R},\ k = 0,1,2,\ldots,n-1, t_* \in J$$

is equivalent to the integral equation

$$x(t) = \sum_{k=0}^{n-1} \frac{(t-t_*)^k}{k!} b_k + \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-s)^{\alpha-1} G(s,x(s))ds$$
$$- \sum_{k=0}^{n-1} \frac{(t-t_*)^k}{k!\Gamma(\alpha-k)} \int\limits_{t_0}^{t_*} (t_*-s)^{\alpha-k-1} G(s,x(s))ds,$$

for $t \in J$.

Corollary 2. *The nonlinear fractional integro-differential system*

$$^{C}D_{t_0}^{\alpha} x(t) = g(t,x(t)) + \int\limits_{t_0}^{t} f(s,x(t))ds, t \in [t_0,T)$$

$$x^{(k)}(T) = b_k \in \mathbb{R},\ k = 0,1,2,\ldots,n-1,$$

is equivalent to the integral equation

$$x(t) = \sum_{k=0}^{n-1} (-1)^k \frac{(T-t)^k}{k!} b_k + \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-s)^{\alpha-1} G(s,x(s))ds \tag{2.4}$$
$$+ \sum_{k=0}^{n-1} (-1)^{k+1} \frac{(T-t)^k}{k!\Gamma(\alpha-k)} \int\limits_{t_0}^{T} (T-s)^{\alpha-k-1} G(s,x(s))ds,$$

for $t \in J$.

Corollary 3. *The nonlinear fractional integro-differential system*

$$
{}^{C}D_{t_0}^{\alpha}x(t) = g(t, x(t)) + \int_{t_0}^{t} f(s, x(t))ds, t \in (t_0, T) \tag{2.5}
$$

$$
x(t_0) = b_0 \in \mathbb{R}, x^{(k)}(T) = b_k \in \mathbb{R}, \ k = 1, 2, \ldots, n-1
$$

is equivalent to the integral equation

$$
\begin{aligned}
x(t) = {} & b_0 - \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \left((T - t_0)^k - (T - t)^k\right) b_k \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} G(s, x(s))ds \\
& + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!\Gamma(\alpha - k)} \left((T - t_0)^k - (T - t)^k\right) \int_{t_0}^{T} (T - s)^{\alpha - k - 1} G(s, x(s))ds,
\end{aligned} \tag{2.6}
$$

for $t \in J$.

Now, consider the fractional integro-differential equation

$$
\begin{aligned}
& {}^{C}D_{t_0}^{\alpha}x(t) = H(t, \varkappa(t), \varrho(t)), t \in J - \{t_*\} \\
& x^{(k)}(t_*) = b_k \in \mathbb{R}, \ k = 0, 1, 2, \ldots, n-1, t_* \in J
\end{aligned} \tag{2.7}
$$

where $\varkappa(t) = \int_{t_0}^{t} f(t, s, x(s))ds$, and $\varrho(t) = \int_{t_0}^{T} g(t, s, x(s))ds$ are respectively the Volterra and Fredholm integral operators with values in X. The function $H\colon J \times X \times X \to \mathbb{R}$ is fractional integrable with order α. In accordance of Theorem 1, one can deduce the equivalent mixed Volterra-Fredholm integral form of (2.7) as

$$
\begin{aligned}
x(t) = {} & \sum_{k=0}^{n-1} \frac{(t - t_*)^k}{k!} b_k + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} H(s, \varkappa(s), \varrho(s))ds \\
& - \sum_{k=0}^{n-1} \frac{(t - t_*)^k}{k!\Gamma(\alpha - k)} \int_{t_0}^{t_*} (t_* - s)^{\alpha - k - 1} H(s, \varkappa(s), \varrho(s))ds.
\end{aligned} \tag{2.8}
$$

3. Existence and uniqueness problems

We investigate in this section the existence of solution for the fractional integro-differential systems (2.3)-(2.7) by using the well-known Banach and Schaefer's fixed point Theorems. Let $J_1 = [t_* - h, t_* + h] \subset (t_0, T)$, where

$0 < h < \min\{t_* - t_0, T - t_*\}$, and $X_1 = C(J_1, \mathbb{R})$ be the space of all real valued continuous functions.

H1. Let $f, g \colon J \times X \to \mathbb{R}$ be jointly continuous Lipschitzian functions that is, there exist positive constants A and B such that

$$\|f(t,x) - f(t,y)\| \le A\|x-y\|,$$
$$\|g(t,x) - g(t,y)\| \le B\|x-y\|,$$

for any $t \in J$, and $x, y \in X$. Moreover, let $C = \sup_{t\in J} \|f(t,0)\|$, $D = \sup_{t\in J} \|g(t,0)\|$, and $L = \max\{A, B, C, D\}$.

Therefore, in accordance with Corollary 1, the fractional nonlinear system

$$(3.1) \qquad {}^{C}D^{\alpha}_{t_*-h} x(t) = g(t, x(t)) + \int_{t_*-h}^{t} f(s, x(t))ds, t \in J_1 - \{t_*\}$$
$$x^{(k)}(t_*) = b_k \in \mathbb{R},\ k = 0, 1, 2, \ldots, n-1,$$

is equivalent to the integral equation

$$(3.2) \qquad x(t) = \sum_{k=0}^{n-1} \frac{(t-t_*)^k}{k!} b_k + \frac{1}{\Gamma(\alpha)} \int_{t_*-h}^{t} (t-s)^{\alpha-1} G(s, x(s))ds$$
$$- \sum_{k=0}^{n-1} \frac{(t-t_*)^k}{k!\Gamma(\alpha-k)} \int_{t_*-h}^{t_*} (t_*-s)^{\alpha-k-1} G(s, x(s))ds.$$

where G is fractional integrable of order α. Accordingly, we define the operator Ψ on X_1 as follows:

$$(3.3) \qquad \Psi x(t) = \sum_{k=0}^{n-1} \frac{(t-t_*)^k}{k!} b_k + \frac{1}{\Gamma(\alpha)} \int_{t_*-h}^{t} (t-s)^{\alpha-1} G(s, x(s))ds$$
$$- \sum_{k=0}^{n-1} \frac{(t-t_*)^k}{k!\Gamma(\alpha-k)} \int_{t_*-h}^{t_*} (t_*-s)^{\alpha-k-1} G(s, x(s))ds.$$

The next hypothesis is essential to state and prove the first main result in this section.

H2. Let θ_1, and r_1 be positive real numbers such that

$$\theta_1 = L(1+2h)h^{\alpha} \left(\frac{2^{\alpha}}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1} \frac{1}{k!\Gamma(\alpha-k+1)} \right) < 1,$$
$$r_1 \ge \frac{\theta_1 + \sum_{k=0}^{n-1} \frac{h^k}{k!} |b_k|}{1-\theta_1}.$$

Moreover, let $\Omega_1 = \{x \in X_1 : \|x\| \le r_1\}$.

Theorem 2. *Let H1 and H2 be satisfied, then, there exists a unique solution for the fractional integro-differential system* (3.1) *in* X_1.

Proof. The Banach fixed point theorem is used to show that Ψ defined by (3.3) has a fixed point on the closed subspace Ω_1 of the Banach space X_1. This fixed point satisfies the integral equation (3.2), hence is a solution of (3.1). For any $t \in J_1$, the continuity of $f(t, x(t))$ and $g(t, x(t))$ implies the continuity of $G(t, x(t))$ and hence the continuity of $\Psi x(t)$. By using H1, we have

$$\begin{aligned}
|\Psi x(t)| \leq & \sum_{k=0}^{n-1} \frac{|t-t_*|^k}{k!} |b_k| \\
& + \sum_{k=0}^{n-1} \frac{|t-t_*|^k}{k!} \frac{(B\|x\| + D) + (A\|x\| + C)(t - t_* + h)}{\Gamma(\alpha - k + 1)} h^{\alpha - k} \\
& + \frac{(B\|x\| + D) + (A\|x\| + C)(t - t_* + h)}{\Gamma(\alpha + 1)} (t - t_* + h)^\alpha \\
\leq & \sum_{k=0}^{n-1} \frac{h^k}{k!} |b_k| + \sum_{k=0}^{n-1} \frac{h^\alpha}{k!} \frac{L(\|x\| + 1)(1 + 2h)}{\Gamma(\alpha - k + 1)} \\
& + \frac{L(\|x\| + 1)(1 + 2h)}{\Gamma(\alpha + 1)} (2h)^\alpha \\
\leq & \sum_{k=0}^{n-1} \frac{h^k}{k!} |b_k| \\
& + L(1 + 2h) h^\alpha \left(\frac{2^\alpha}{\Gamma(\alpha + 1)} + \sum_{k=0}^{n-1} \frac{1}{k! \Gamma(\alpha - k + 1)} \right) \\
& + L(1 + 2h) h^\alpha \left(\frac{2^\alpha}{\Gamma(\alpha + 1)} + \sum_{k=0}^{n-1} \frac{1}{k! \Gamma(\alpha - k + 1)} \right) \|x\|.
\end{aligned}$$

Hence, if $x \in \Omega_1$, it is obvious that $\Psi x \in \Omega_1$. Next, let $x, y \in \Omega_1$, then

$$\begin{aligned}
|\Psi x(t) - \Psi y(t)| \leq & \sum_{k=0}^{n-1} \frac{|t-t_*|^k}{k!} \frac{(B + A(t - t_* + h))\|x - y\|}{\Gamma(\alpha - k + 1)} h^{\alpha - k} \\
& + \frac{(B + A(t - t_* + h))\|x - y\|}{\Gamma(\alpha + 1)} (t - t_* + h)^\alpha \\
\leq & L(1 + 2h) h^\alpha \left(\frac{2^\alpha}{\Gamma(\alpha + 1)} + \sum_{k=0}^{n-1} \frac{1}{k! \Gamma(\alpha - k + 1)} \right) \|x - y\| \\
\leq & \theta_1 \|x - y\|,
\end{aligned}$$

since $\theta_1 < 1$, then Ψ is a contraction mapping on Ω_1. Hence, Ψ has a fixed point which is the unique solution to (3.1). $\square$

Next result is getting the existence of a local solution for the Cauchy problem (2.4). Let $J_2 = [T-h, T] \subset (t_0, T]$, where $0 < h < T - t_0$, and $X_2 = C(J_2, \mathbb{R})$ be the space of all real valued continuous functions on J_2. The system

$$
{}^{C}D^{\alpha}_{T-h}x(t) = g(t, x(t)) + \int\limits_{T-h}^{t} f(s, x(t))ds, t \in [T-h, T), \tag{3.4}
$$

$$
x^{(k)}(T) = b_k \in \mathbb{R},\ k = 0, 1, 2, \ldots, n-1
$$

is equivalent to the Volterra-Fredholm integral equation

$$
\begin{aligned}
x(t) = &\sum_{k=0}^{n-1}(-1)^k \frac{(T-t)^k}{k!} b_k + \frac{1}{\Gamma(\alpha)} \int\limits_{T-h}^{t} (t-s)^{\alpha-1} G(s, x(s))ds \\
&- \sum_{k=0}^{n-1}(-1)^k \frac{(T-t)^k}{k!\Gamma(\alpha-k)} \int\limits_{T-h}^{T} (T-s)^{\alpha-k-1} G(s, x(s))ds
\end{aligned}
$$

for $x \in X_2$, and $t \in J_2$.

The modified version of H2 can be given by the following:
H3. Let θ_2, and r_2 be positive real numbers such that

$$
\theta_2 = L(1+h)h^{\alpha} \left(\frac{1}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1} \frac{1}{k!\Gamma(\alpha-k+1)} \right) < 1,
$$

$$
r_2 \geq \frac{\theta_2 + \sum_{k=0}^{n-1} \frac{h^k}{k!} |b_k|}{1-\theta_2}.
$$

Moreover, let $\Omega_2 = \{x \in X_2 : \|x\| \leq r_2\}$.

The proof of the next result is similar to that one of Theorem 2, hence it is omitted.

Corollary 4. *Let H1 and H3 be satisfied, then, there exists a unique solution for the fractional integro-differential system* (3.4) *in* X_2.

Now, consider the initial value problem of the fractional integro-differential system

$$
{}^{C}D^{\alpha}_{t_0}x(t) = g(t, x(t)) + \int\limits_{t_*}^{t} f(s, x(t))ds, , t \in (t_0, T), \tag{3.5}
$$

$$
x^{(k)}(t_0) = b_k \in \mathbb{R},\ k = 0, 1, 2, \ldots, n-1,
$$

that has an equivalent Volterra integral equation given by

$$
x(t) = \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} b_k + \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-s)^{\alpha-1} G(s, x(s))ds
$$

for $t \in J_0 = [t_0, t_0 + h]$, $x \in X_0 = C(J_0, \mathbb{R})$.

To establish the existence and uniqueness results to the system (3.5), we replace the next hypothesis instead of H2.
H4. Let θ_0, and r_0 be positive real numbers such that

$$\theta_0 = \frac{L(h+1)}{\Gamma(\alpha+1)} h^\alpha < 1, \text{and } r_0 \geq \frac{\theta_0 + \sum_{k=0}^{n-1} \frac{h^k}{k!} |b_k|}{1 - \theta_0}.$$

Moreover, let $\Omega_0 = \{x \in X_0 : \|x\| \leq r_0\}$.

Corollary 5. *Let H1, and H4 be satisfied, then, there exists a unique solution for the fractional integro-differential system* (3.5) *in* X_0.

Next, we consider the system (2.5)-(2.6).
H5. Let θ_3, and r_3 be positive real numbers such that

$$\theta_3 = L(1 + T - t_0)(T - t_0)^\alpha \left(\frac{1}{\Gamma(\alpha+1)} + \sum_{k=1}^{n-1} \frac{1}{k! \Gamma(\alpha - k + 1)} \right) < 1,$$

$$r_3 \geq \frac{\theta_3 + \sum_{k=0}^{n-1} \frac{(T-t_0)^k}{k!} |b_k|}{1 - \theta_3}.$$

Moreover, let $\Omega_3 = \{x \in X_3 : \|x\| \leq r_3\}$.

Corollary 6. *Let H1, and H5 be satisfied, then, there exists a unique solution for the fractional integro-differential system* (2.5) *in* X.

Remark 1. By modifications on the hypotheses H1–H5, we can obtain all the previous results of the existence and uniqueness of solution for the fractional integro-differential systems (2.7) and (2.8).

We close this article by obtaining a sufficient condition of existence problem for the system (2.7) by using Schaefer's fixed point theorem.

Theorem 3 ([5])**.** *Let X be a Banach space. Assume that $\Psi : X \to X$ is completely continuous operator and the set $V = \{x \in X : x = \mu \Psi x, 0 < \mu < 1\}$ is bounded. Then Ψ has a fixed point in X.*

Define the operator Ψ on X by (see 2.8)

$$\Psi x(t) = \sum_{k=0}^{n-1} \frac{(t - t_*)^k}{k!} b_k + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} H(s, \varkappa(s), \varrho(s)) ds$$
$$- \sum_{k=0}^{n-1} \frac{(t - t_*)^k}{k! \Gamma(\alpha - k)} \int_{t_0}^{t_*} (t_* - s)^{\alpha - k - 1} H(s, \varkappa(s), \varrho(s)) ds.$$

H6. Let $f, g\colon D\times X \to X$ be continuous functions, where $D = \{(t,s) : t_0 \le s \le t \le T\}$, and assume that H satisfying a linear growth condition

$$\|H(\cdot, x, y)\| \le K(1 + \|x\| + \|y\|),$$

for any $x, y \in X$.

Theorem 4. *Assume that H6 is satisfied. Then the fractional integro-differential equation* (2.7) *has at least one solution.*

Proof. The continuity of f and g on $D \times X$ implies the continuity of H and hence the continuity of the operator Ψ on X. Define the nonempty closed convex subset $\Omega = \{x \in X : \|x\| \le r\}$ of the Banach space X. If

$$f_{\max} = \max\{\|f(t,s,x)\| : (t,s,x) \in D \times \Omega\}$$
$$g_{\max} = \max\{\|g(t,s,x)\| : (t,s,x) \in D \times \Omega\}$$

then for any $x \in \Omega$, $(t,s) \in D$, we have

$$|\Psi x(t)| \le \frac{K\left(1 + f_{\max}\left(T - t_0\right) + g_{\max}\left(T - t_0\right)\right)\left(t - t_0\right)^{\alpha}}{\Gamma\left(\alpha + 1\right)} + \sum_{k=0}^{n-1} \frac{(T - t_*)^k}{k!}\left(|b_k| + \frac{K\left(1 + f_{\max}\left(T - t_0\right) + g_{\max}\left(T - t_0\right)\right)\left(t_* - t_0\right)^{\alpha - k}}{\Gamma\left(\alpha - k + 1\right)}\right). \tag{3.6}$$

Hence Ψx has an upper bound $M < \infty$ on X. Furthermore, if $t_0 \le t_1 \le t_2 \le T$, then

$$\begin{aligned}
\left|\left(\Psi x\right)\left(t_2\right) - \left(\Psi x\right)\left(t_1\right)\right| &\le \sum_{k=0}^{n-1} \frac{\left|(t_2 - t_*)^k - (t_1 - t_*)^k\right|}{k!} \\
&\times \left(|b_k| + \frac{K\left(1 + f_{\max}\left(T - t_0\right) + g_{\max}\left(T - t_0\right)\right)\left(t_* - t_0\right)^{\alpha - k}}{\Gamma\left(\alpha - k + 1\right)}\right) \\
&+ \frac{K\left(1 + f_{\max}\left(T - t_0\right) + g_{\max}\left(T - t_0\right)\right)}{\Gamma\left(\alpha\right)} \int_{t_0}^{t_1} \left|\left(t_2 - s\right)^{\alpha - 1} - \left(t_1 - s\right)^{\alpha - 1}\right| ds \\
&+ \frac{K\left(1 + f_{\max}\left(T - t_0\right) + g_{\max}\left(T - t_0\right)\right)}{\Gamma\left(\alpha\right)} \int_{t_1}^{t_2} \left|t_2 - s\right|^{\alpha - 1} ds \\
&\le \sum_{k=0}^{n-1} \frac{\left|(t_2 - t_*)^k - (t_1 - t_*)^k\right|}{k!} \\
&\times \left(|b_k| + \frac{K\left(1 + f_{\max}\left(T - t_0\right) + g_{\max}\left(T - t_0\right)\right)\left(t_* - t_0\right)^{\alpha - k}}{\Gamma\left(\alpha - k + 1\right)}\right) \\
&+ \frac{K\left(1 + f_{\max}\left(T - t_0\right) + g_{\max}\left(T - t_0\right)\right)}{\Gamma\left(\alpha + 1\right)}
\end{aligned}$$

$$\times \left(2\left|t_2 - t_1\right|^\alpha + \left|(t_2 - t_0)^\alpha - (t_1 - t_0)^\alpha\right|\right)$$

which is independent of x, and tends to 0 as $t_1 \to t_2$. This implies that Ψ is equicontinuous on J. In consequence, it follows by the Arzela-Ascoli theorem that the operator Ψ is completely continuous. Next, let $x \in V = \{y \in \Omega : y = \mu\Psi y, 0 < \mu < 1\}$, then $x = \mu\Psi x$, for some $\mu \in (0,1)$. Using (3.6)), we have $|x(t)| = \mu\left|\Psi x(t)\right| \le M$, for any $t \in J$. Hence $\|x\| \le M$, which implies the boundedness of V. As a consequence of Theorem 3, the operator Ψ has at least one fixed point $x \in \Omega$, which is the solution of (2.7). This finishes the proof. □

Example 1. Consider the following fractional system

$$ {}^C D_0^{\frac{5}{2}} x(t) = \frac{t|x(t)|}{3+3|x(t)|} + \int_0^t s \sin\frac{x(s)}{3} ds, t \in (0,\frac{1}{2}) \cup (\frac{1}{2}, 1] \tag{3.7}$$

$$x\left(\frac{1}{2}\right) = x^{'}\left(\frac{1}{2}\right) = x^{''}\left(\frac{1}{2}\right) = 1.$$

The functions $f(t,x(t)) = t\sin\frac{x(t)}{3}$, and $g(t,x(t)) = \frac{t|x(t)|}{3+3|x(t)|}$are jointly continuous functions on $[0,1] \times [0,\infty)$. Moreover, the hypotheses (H1) and (H2) are satisfied such that $L = \frac{1}{3}, h < \frac{1}{2}$, and $\theta_1 < 1$. Hence for large r, there exist a unique solution for the fractional differential system (3.7) on $C[0,1]$.

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