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RICCI TENSOR OF A FINSLER SPACE WITH SPECIAL (α, β)-METRICS

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Abstract. In this paper, we investigate the Ricci tensor of a Finsler space of a special (α, β)-metric $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ be a Riemannian metric and β be a 1-form. We also prove that if α is a positive (negative) sectional curvature and F is of α-parallel Ricci curvature with constant Killing 1-form β, then (M, F) is a Riemannian Einstein space.

1. Introduction

One of the most important problems in Finsler geometry is to understand the geometric meanings of various quantities and their impacts on the global geometric structures. The flag curvature K, which is obtained by the Riemannian curvature, tells us how curved the Finsler manifold is at a specific point. Moreover, there are several important non- Riemannian quantities in Finsler geometry: the Cartan torsion C, the Berwald curvature B, the Landsberg curvature L, and the well-known S curvature, etc. They all vanish for Riemannian metrics, hence they are said to be non-Riemannian. These quantities interact with the flag curvature in a fragile way.

Finsler space with (α, β) -metrics were introduced in 1972 by M. Matsumoto [9]. The study of Finsler spaces with (α, β) -metrics is a very important aspect of Finsler geometry and its applications (see [2, 5]). An (α, β) -metric is a scalar function on TM defined by

$$F = \alpha\phi\left(\frac{\beta}{\alpha}\right),\ s = \frac{\beta}{\alpha},$$

where $\phi = \phi(s)$ is a C^∞ on (b_0, b_0) with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form in the manifold M. Therefore, (M, F) is called the associated Riemannian manifold. A Finsler space is a manifold M equipped with a family of smoothly varying Minkowski norms;

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one on each tangent space, Riemannian metrics are examples of Finsler norms that are induced from an inner-product.

Some interesting examples of (α, β)-metrics are the Randers metric, Matsumoto metric and Berwald metric, etc., Randers metric and its Ricci tensor are related via their applications in physics. The well-known Ricci tensor was introduced in 1904 by G. Ricci. Nine years later Ricci's work was used to formulate Einstein's theory of gravitation [1]. Einstein metrics are defined in the next section but, roughly, we will say a Finsler metric F is Einstein if the average of its flag curvatures at a flag pole y is a function of position x alone, rather than the a priori position x and flag pole y. C. Robles [13] investigated Randers Einstein metrics in her thesis in 2003. She obtained the necessary and sufficient conditions for Randers metric to be Einstein and by using Einstein Zermelo navigation description, she proved the pair (h, W) of a Riemannian metric and an appropriate vector field W has been founded in [6].

Put $H_{ij} = H^k_{ikj}$; denote the canonical section of the vector bundle π^*TM and the vertical derivation with respect to y^i by v and $\dot{\partial}_i$ respectively. For an (α, β) -metric $F = \alpha\phi(\beta/\alpha)$, by using the geodesic coefficient of α, we can introduce a new geometric quantity. Let us denote the Levi-Civita connection of α by $\widetilde{\nabla}$. We define the Ricci tensor $\bar{H}$ and $\widetilde{H}$ on π^*TM as follows:

$$\begin{aligned}
&\bar{H}_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j H(v, v),\\
&\widetilde{H}(X, Y) = \widetilde{\nabla}_{\hat{v}}\bar{H}(X, Y),\ X = \pi^*(\hat{X}), Y = \pi^*(\hat{Y}),
\end{aligned}$$

where, $\hat{X}$, $\hat{Y} \in TM_0$ and $\hat{v}$ is the geodesic spray associated with α. The curvature $\widetilde{H}$ is closely related to the Ricci curvature and its related to (α, β) -metrics, especially to the associated Riemannian manifold (M, α). In this paper we investigate an (α, β) - metric of α -parallel Ricci curvature, and we prove the following main theorem:

Theorem. *Let $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ be a Finsler metric on a connected manifold M of dimension n. Suppose that α is a positive (negative) sectional curvature and Ricci tensor $\widetilde{H} = 0$, $(H(v, v) = 0)$ and β is a constant Killing 1-form. Then (M, F) is a Riemannian Einstein space.*

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM be the tangent space at $x \in M$ and $TM = \bigcup_{x \in M} T_xM$ be the tangent bundle of M. Each element of TM has the form (x, y) where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM\backslash\{0\}$. The natural projection $\pi\colon TM \to M$ is given by, $\pi(x, y) = x$. The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber π^*_vTM at $v \in TM_0$ is just T_xM, where $\pi(v) = x$. Then

$$\pi^*TM = \{(x, y, v) | y \in T_xM_0, v \in T_xM\}.$$

A Finsler metric on a manifold M is a function ([9, 10]) $F\colon TM \to [0,\infty)$ which has the following properties:

(i) F is C^∞ on TM_0,

(ii) $F(x,\lambda y) = \lambda F(x,y)$, $\lambda > 0$,

(iii) for any tangent vector $y \in T_xM$, the vertical Hessian of $\frac{F^2}{2}$ given by

$$g_{ij} = \left(\frac{1}{2}F^2\right)_{y^iy^j},$$

is positive definite.

Every Finsler metric F induces a spray [7]:

$$G = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

is defined by

$$G^i(x,y) = \frac{1}{4}g^{il}(x,y)\left\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\right\}y^jy^k,$$

where the matrix (g^{ij}) means the inverse of matrix (g_{ij}) and the coefficients G^i_j, G^i_{jk} and hv-curvature G^i_{jkl} of the Berwald connection can be derived from the spray G^i as follows:

$$G^i_j = \frac{\partial G^i}{\partial y^j},\ G^i_{jk} = \frac{\partial G^i_j}{\partial y^k},\ G^i_{jkl} = \frac{\partial G^i_{jk}}{\partial y^l}.$$

When $F = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $K^i_k = R^i_{jkl}(x)y^jy^l$, where $R^i_{jkl}(x)$ denote the coefficients of the usual Riemannian curvature tensor. Thus, the Ricci scalar function of F is given by

$$\rho = \frac{1}{F^2}K^i_i,\ \ H(v,v) = K^i_i.$$

Therefore, the Ricci scalar function is positive homogeneous of degree 0 in y. This means $\rho(x,y)$ depends on the direction of the flag pole y, but not its length.

$$H_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_jH(v,v).$$

A Finsler manifold (M,F) is called an Einstein space if there exists a differentiable function c defined on M such that $H(v,v) = cF^2$. The Ricci identity for a tensor W_{jm} of π^*TM is given by the following formula [11]:

$$D_kD_lW_{jm} - D_lD_kW_{jm} = -W_{rm}H^r_{jkl} - W_{jr}H^r_{mkl} - \frac{\partial W_{jm}}{\partial y^r}H^r_{0kl},$$

where D_k denotes the horizontal covariant derivative with respect to $\{\frac{\delta}{\delta x^k}\}$ in the Berwald connection. Let (M,F) be an n -dimensional Finsler space. For every $x \in M$, assume $S_xM = \{y \in T_xM\backslash F(x,y) = 1\}$. S_xM is called the indicatrix of F at $x \in M$ and is a compact hyper surface of T_xM, for every $x \in M$. Let $v\colon S_xM \to T_xM$ be its canonical embedding; where $\|v\| = 1$ and

(t, U) be a coordinate system on S_xM. Then, S_xM is represented locally by $v^i = v^i(t^\alpha), (\alpha = 1, 2, \ldots, (n-1))$. One can easily show that:

$$\frac{\partial}{\partial v^i} = F\frac{\partial}{\partial y^i},$$

The $(n-1)$ vectors $\{(v^i_\alpha)\}$ from a basis for the tangent space of S_xM in each point, where $v^i_\alpha = \frac{\partial v^i}{\partial t^\alpha}$, put $\partial_\alpha = \frac{\partial}{\partial t^\alpha}$. It implies that:

$$\partial_\alpha = F v^i_\alpha \frac{\partial}{\partial y^i}.$$

$g = g_{ij}(x, y)y^iy^j$ is a Riemannian metric on T_xM. Inducing g in S_xM, one gets the Riemannian metric $\bar{g} = \bar{g}_{\alpha\beta}dt^\alpha dt^\beta$, where $\bar{g}_{\alpha\beta} = v^i_\alpha v^i_\beta g_{ij}$. The canonical unit vertical vector field $V(x, y) = y^i\frac{\partial}{\partial y^i}$ together the $(n-1)$ vectors, ∂_α from the local basis for T_xM, $B = \{u^1, u^2, \ldots, u^n\}$ where, $u^\alpha = (v^i_\alpha)$ and $u^n = V$. We conclude that $g(V, \partial_\alpha) = 0$, that is $y_i v^i_\alpha = 0$.

Let (M, F) be an n-dimensional Finsler space equipped with an (α, β) - metric F, where

$$\alpha(x, y) = \sqrt{a_{ij}(x)y^iy^j},\ \beta(x, y) = b_i(x)y^i.$$

M. Matsumoto ([2, 9]) proved that, the spray G^i of Finsler space with (α, β)-metrics are given by

$$2G^i = \gamma^i_{00} + 2B^i,$$

where

$$\begin{aligned}
B^i &= \frac{E}{\alpha}y^i + \left(\frac{\alpha F_\beta}{F_\alpha}\right)s^i_0 - \left(\frac{\alpha F_{\alpha\alpha}}{F_\alpha}\right)C\left\{\frac{y^i}{\alpha} - \frac{\alpha}{\beta}b^i\right\},\\
E &= \left(\beta\frac{F_\beta}{F}\right)C,\\
C &= \alpha\beta(r_{00}F_\alpha - 2\alpha s_0F_\beta)/2(\beta^2F_\alpha + \alpha\gamma^2F_{\alpha\alpha}),\\
b^i &= a^{ir}b_r,\ b^2 = b^rb_r,\ \gamma^2 = b^2\alpha^2 - \beta^2,\\
r_{ij} &= \frac{1}{2}(\widetilde{\nabla}_jb_i + \widetilde{\nabla}_ib_j),\ s_{ij} = \frac{1}{2}(\widetilde{\nabla}_jb_i - \widetilde{\nabla}_ib_j),\\
s^i_j &= a^{ih}s_{hj},\ s_j = b_is^i_j.
\end{aligned}$$

The matrix (a^{ij}) means the inverse of matrix (a_{ij}). The function γ^i_{jk} stands for the Christoffel symbols in the space (M, α), and the suffix 0 means transvecting length with respect to α, equivalently

$$r_{ij} = 0 \text{ and } s_i = 0.$$

In an n-dimensional coordinate neighborhood U, we consider a liner partial differential equation of second order,

$$L(\varphi) = g^{ik}\frac{\partial^2\varphi}{\partial x^ix^k} + h^i\frac{\partial\varphi}{\partial x^i}.$$

where $g^{ik}(x)$ and $h^i(x)$ are continuous function of point x in U, and quadratic form $g^{jk}Z_jZ_k$ is supposed to be positive definite everywhere in U. Then we call L an elliptic differential operator.

Strong maximum principle. *In coordinate neighborhood U, if a function $\varphi(p)$ of class C^2 satisfies*

$$L(\varphi) \geq 0,$$

where $\varphi\colon M \to R^n$, and if there exists a fixed point p_0 in U such that $\varphi(p) \leq \varphi(p_0)$ then we have $\varphi(p) = \varphi(p_0)$, $\forall p \in U$. If φ have absolute maximum in U, then φ is constant on U.

The following are the simple examples for existence of constant sectional curvature and constant flag curvature:

Example 1 ([8]). Consider the family of Riemannian metrics:

$$\alpha_\mu = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - <x, y>^2)}}{1 + \mu|x|^2},\ y \in T_xB^n(r_\mu) \cong R^n.$$

The spray coefficients $G^i = Py^i$, where

$$P = -\frac{\mu <x, y>}{1 + \mu|x|^2}.$$

Using the equation,

$$K = \frac{P^2 - P_{x^k}y^k}{F^2}.$$

we obtain that $K = \mu$. Thus α_μ has constant sectional curvature.

Example 2 ([4]). Let $F = \alpha + \beta$ be the family of Randers metric on S^3 constructed in [4]. β satisfies that $r_{ij} = 0$ and $s_i = 0$. Moreover, the authors have found a special family of these Randers metrics with constant flag curvature $K = 1$.

Now, we consider the (α, β) -metrics where α is of positive (negative) sectional curvature. Let $\{\frac{\delta}{\delta x^i}\}$ and $\left\{\frac{\hat{\delta}}{\hat{\delta} x^i}\right\}$ be the natural locally horizontal basis of TM_0 with respect to F and α, respectively. To prove the main theorem , we use the following proposition, proved by [11]:

Proposition 2.1. *Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be an (α, β) - metric on a connected manifold M. Suppose that α is of positive (negative) sectional curvature. Then, we have $H(v, v) = c\alpha^2$ $c \in R$, if and only if $\bar{H} = 0$.*

3. Einstein criterion for special (α, β)-metric

In this section, we consider the Finsler space with special (α, β)-metric is of the form $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$. In [12], we obtained the following relation between

$H(v,v)$ and $\widetilde{R}(v,v)$ for special (α,β)- metrics with constant Killing 1-form β:

$$(3.1) H(v,v)=\widetilde{R}(v,v)+\frac{2\alpha^2(3\alpha+2\beta)}{\alpha^2-\beta^2}\widetilde{\nabla}_i s^i_0-\frac{2\alpha^2(3\alpha+2\beta)^2}{(\alpha^2-\beta^2)^2}$$
$$\left\{-\alpha\left(\frac{9}{2}\alpha+2\beta\right)^2+\beta\left(\frac{9}{2}\alpha+2\beta\right)^2+\alpha(3\alpha+2\beta)^2\right\}s^i_0 s_{0i}$$
$$+\frac{\alpha^4(3\alpha+2\beta)^2}{(\alpha^2-\beta^2)^2}s^{ij}s_{ij}.$$

Let $H(v,v)=c\alpha^2$, where $c\in R$, we obtain

$$0=\widetilde{R}(v,v)+\frac{2\alpha^2(3\alpha+2\beta)}{\alpha^2-\beta^2}\widetilde{\nabla}_i s^i_0-\frac{2\alpha^2(3\alpha+2\beta)^2}{(\alpha^2-\beta^2)^2}$$
$$\left\{-\alpha\left(\frac{9}{2}\alpha+2\beta\right)^2+\beta\left(\frac{9}{2}\alpha+2\beta\right)^2+\alpha(3\alpha+2\beta)^2\right\}s^i_0 s_{0i}$$
$$+\frac{\alpha^4(3\alpha+2\beta)^2}{(\alpha^2-\beta^2)^2}s^{ij}s_{ij}-c\alpha^2,$$

$$(3.2)\ 0=\widetilde{R}(v,v)+\frac{2\alpha^2(3\alpha+2\beta)}{\alpha^2-\beta^2}\widetilde{\nabla}_i s^i_0-\frac{\alpha}{2(3\alpha+2\beta)^3}$$
$$(413\alpha^3-1152\alpha^3\beta^3-488\alpha\beta^4+51\alpha^4\beta-1260\alpha^2\beta^3-64\beta^5)s^i_0 s_{0i}$$
$$+\frac{\alpha^4(3\alpha+2\beta)^2}{(\alpha^2-\beta^2)^2}s^{ij}s_{ij}-c\alpha^2.$$

Multiplying (3.2) by $2(\alpha^2-\beta^2)^3$ removes y from the denominators and we can derive the following identity:

$$\mathrm{Rat}+\alpha\,\mathrm{Irrat}=0,$$

where Rat and Irrat are polynomials of degree 6 and 5 in y respectively and are given as follows:

$$\begin{aligned}\mathrm{Rat}=&(2\alpha^6-6\alpha^4\beta^2+6\alpha^2\beta^4-2\beta^6)\widetilde{R}(v,v)\\&+(4\alpha^8\beta+4\alpha^4\beta^5-8\alpha^6\beta^3)\widetilde{\nabla}_i s^i_0\\&-(413\alpha^6-1152\alpha^4\beta^2-488\alpha^2\beta^4)s^i_0 s_{0i}\\&+(24\alpha^6\beta-18\alpha^6\beta^2-8\alpha^4\beta^2)s^{ij}s_{ij}\\&-c\alpha^2(2\alpha^6-6\alpha^4\beta^2+6\alpha^2\beta^4-2\beta^6),\end{aligned}$$

$$\begin{aligned}\mathrm{Irrat}=&(6\alpha^4\beta^4-4\alpha^6\beta^2)\widetilde{\nabla}_i s^i_0\\&-(51\alpha^4\beta-1260\alpha^2\beta^3-64\beta^5)s^i_0 s_{0i}\\&+(8\alpha^4\beta^2-24\alpha^4\beta^3)s^{ij}s_{ij}.\end{aligned}$$

We have the following lemma:

Lemma 3.1. *Let $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ be a metric with constant Killing from β , and $H(v,v) = c\alpha^2$ for some constants $c \in R$. Then, (M, F) is a Riemannian Einstein space.*

Proof. We know that α can never be a polynomial in y. Otherwise the quadratic $\alpha^2 = a_{ij}(x)y^i y^j$ would have been factored into two linear terms. Its zero set would then consist of a hyper-plan, contradicting the positive definiteness of a_{ij}. Now suppose the polynomial Rat were not zero. The above equation would imply that it is the product of polynomial Irrat with a non-polynomial factor α. This is not possible. So Rat must vanish and, since α is positive at all $y \neq 0$, we see that Irrat must be zero as well. Notice that Rat $= 0$ shows that α^2 divides $\beta^6 \widetilde{R}(v,v)$. Since α^2 is an irreducible degree two polynomial in y, and β^6 factors into six linear terms, it must be the case that α^2 divides $\widetilde{R}(v,v)$, that is, (M,α) is an Einstein space.

Therefore, $\widetilde{R}(v,v) = k\alpha^2$, where the function k must be a constant by the Riemannian Schur's Lemma for the case $n > 2$. But, we can easily reform Rat $= 0$ as the following relation:

$$\begin{aligned}&-2k\beta^6 - 488\beta\beta^4 s^i_0 s_{0i} + 2c\beta^6\\ &= \alpha^2(-6k\beta^4 + 8\beta^2 s^{ij}s_{ij} - 1152\beta^2 s^i_0 s_{0i} + 4\beta^5\widetilde{\nabla}_i s^i_o + 2c\beta^4),\end{aligned}$$

which results in, α^2 divides β. From the irreducibility of α^2, it shows that, $\beta = 0$ and F is a Riemannian Einstein metric. □

4. Proof of The Main Theorem

Theorem 4.1. *Let $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$ be a Finsler metric on a connected manifold M of dimension n. Suppose that α is a positive (negative) sectional curvature and Ricci tensor $\widetilde{H} = 0$, $(H(v,v) = 0)$ and β is a constant Killing 1-form. Then (M,F) is a Riemannian Einstein space.*

Proof. By proposition 2.1 says that, α is of positive (negative) sectional curvature. Then, $H(v,v) = c\alpha^2$, where c is a non-zero constant and by lemma 3.1 it shows that if F is Einstein then, it is Riemannian Einstein space, that means: if F is Einstein if and only if Rat $= 0$ and Irrat $= 0$ are holds. Again from proof of lemma, we know that α^2 divides $\widetilde{R}(v,v)$, then their is a function k defined on M and $F = \alpha$, it implies that, $\widetilde{R}(v,v) = k\alpha^2$. Hence (M,α) is an Einstein space. Therefore, we conclude that main theorem, $(M, F = \frac{(\alpha+\beta)^2}{\alpha} + \beta)$ is a Riemannian Einstein space. □

5. Conclusion

The Einstein metrics comprise a major focus in differential geometry and mainly connect with gravitational in general relativity. In particular, Einstein metric are solutions to Einstein field equations in general relativity containing Ricci tensor [1]. Einstein Finsler metric which represent a non-Riemannian stage for the extensions of metric gravity provide an interesting source of geometric issues and the (α, β)-metric is an important class of Finsler metric appearing frequently in the study of applications in physics [4].

In this paper, we consider a special (α, β)-metric such as $F = \frac{(\alpha+\beta)^2}{\alpha} + \beta$. For this , we investigate an (α, β)-metric of α-parallel Ricci curvature. Finally, we prove that the above mentioned Einstein metric must be Riemannian.

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