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# ON A FINSLER SPACE WITH A SPECIAL METRICAL CONNECTION

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Abstract. In this paper, we consider a Finsler space with a special metrical connection and find necessary and sufficient condition when the (v)hv-torsion tensor ${}^*P^i_{jk}$ with respect to the special metrical connection coincides with the (v)hv-torsion $P^i_{jk}$ with respect to general Finsler connection. The relation in hv-curvature tensor, h-curvature tensor and v(h)-torsion tensor with respect to these two connection are also obtained.

## 1. Introduction

Let $F^n = (M^n, L)$ be an $n$-dimensional Finsler space equipped with the fundamental function $L(x, y)$. The metric tensor, the angular metric tensor and Cartan tensor are defined by

$$g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \quad h_{ij} = L\dot{\partial}_i\dot{\partial}_j L \quad \text{and} \quad C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$$

respectively, where $\dot{\partial}_i = \partial/\partial y^i$.

A Finsler connection is a triad $F\Gamma = (\Gamma^i_{jk}, N^i_j, C^i_{jk})$, where $\Gamma^i_{jk}$ are connection coefficients of h-connection, $N^i_j$ are connection coefficients of non-linear connection and $C^i_{jk}$ are connection coefficients of v-connection. For a given connection, the h- and v-covariant derivatives of any vector $X^i$ are given by

$$X^i_{|k} = \delta_k X^i + X^r \Gamma^i_{rk},$$

and

$$X^i|_k = \dot{\partial}_k X^i + X^r C^i_{rk},$$

where $\delta_k = \partial_k - N^r_k \dot{\partial}_r$ and $\partial_k = \frac{\partial}{\partial x^k}$.

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In 2006, H. S. Park et.al. [4] defined a new non-linear connection $\overline{N}^i_j$ with the help of given non-linear connection $N^i_j$ for an $(\alpha, \beta)$-metric as

$$\overline{N}^i_j = N^i_j + \nabla_j L \frac{y^i}{L}, \tag{1.1}$$

where '$\nabla_j$' denotes the covariant derivative with respect to the associated Riemannian connection.

In 2008, H. G. Nagaraja [3] defined a new non-linear connection ${}^*N^i_j$ with the help of given non-linear connection for Randers space as

$${}^*N^i_j = N^i_j + \frac{L_{|j} y^i}{L}, \tag{1.2}$$

where '${}_{|j}$' denote the covariant derivative with respect to Finsler connection $F\Gamma$, and find a new Finsler connection ${}^*F\Gamma = (\Gamma^i_{jk}, {}^* N^i_j, C^i_{jk})$.

In this paper, we consider a Finsler space $F^n$ admitting the Finsler connection ${}^*F\Gamma$ and we find a relation between $v(hv)$-torsion tensors with respect to these two Finsler connection connections $F\Gamma$ and ${}^*F\Gamma$. We obtain necessary and sufficient condition that two $(v)hv$-torsions coincides. We also find relation in hv-curvature tensor, h-curvature tensor and v(h) torsion tensor with respect to these two Finsler connections.

The Terminology and notion are referred to [2, 5].

## 2. A special metrical connection

Let $F^n = (M^n, L)$ be an $n$-dimensional Finsler space and $F\Gamma = (\Gamma^i_{jk}, N^i_j, C^i_{jk})$ be a Finsler connection. Let the Finsler space $F^n = (M^n, L)$ admits a new Finsler connection ${}^*F\Gamma = ({}^*\Gamma^i_{jk}, {}^* N^i_j, C^i_{jk})$, which is $h(h)$-torsion free and non-linear coefficients ${}^*N^i_j$ are given by (1.4). Then we have

$${}^*\delta_k = \partial_k - {}^* N^r_k \dot{\partial}_r = \partial_k - N^r_k \dot{\partial}_r - \frac{L_{|k} y^r}{L} \dot{\partial}_r = \delta_k - \frac{L_{|k} y^r}{L} \dot{\partial}_r. \tag{2.1}$$

The h-covariant derivative of $L$ with respect to ${}^*F\Gamma$ is given by

$$L_{{}^*|k} = {}^*\delta_k L = \delta_k L - \frac{L_{|k} y^r}{L} \dot{\partial}_r L = \delta_k L - L_{|k} = 0. \tag{2.2}$$

Therefore the Finsler connection ${}^*F\Gamma$ is h-metrical. Since ${}^*F\Gamma$ is h-metrical and $h(h)$- torsion ${}^*T^i_{jk}$ is zero, the linear connection coefficients ${}^*\Gamma^i_{jk}$ of ${}^*F\Gamma$ are given in [1] by

$$\Gamma^i_{jk} = g^{ir} \Gamma_{jrk} = \frac{1}{2} g^{ir} [\delta_j g_{rk} + \delta_k g_{rj} - \delta_r g_{jk}]. \tag{2.3}$$

Using (2.1) and $(\dot{\partial}_j g_{ik}) y^j = 0$ in (2.3), we have

$${}^*\Gamma^i_{jk} = \Gamma^i_{jk}. \tag{2.4}$$

Thus, the new connection ${}^*F\Gamma = ({}^*\Gamma^i_{jk}, {}^*N^i_j, C^i_{jk})$ reduces to ${}^*F\Gamma = (\Gamma^i_{jk}, {}^*N^i_j, C^i_{jk})$.

**The (v)hv-torsion tensor.** The (v)hv-torsion tensor $P^i_{jk}$ of a Finsler space with respect to connection $F\Gamma$ is defined by

$$P^i_{jk} = \dot{\partial}_k N^i_j - \Gamma^i_{jk}.$$

Therefore the (v)hv-torsion tensor ${}^*P^i_{jk}$ of a Finsler space with respect to the connection ${}^*F\Gamma$ is given by

$${}^*P^i_{jk} = \dot{\partial}^*_k N^i_j - {}^*\Gamma^i_{jk}.$$

Using (1.2) and (2.4), we get

$$(2.5) \qquad {}^*P^i_{jk} = P^i_{jk} + l_{k|j} l^i + L_{|j} \frac{h^i_k}{L}.$$

Let us suppose ${}^*P^i_{jk} = P^i_{jk}$. Then equation (2.5) implies that $l_{k|j} l^i + L_{|j} \frac{h^i_k}{L} = 0$. Transvecting by $y_i$ and using $h^i_k y_i = 0$, we get $L l_{k|j} = 0$, which gives $l_{k|j} = 0$.

Conversely, let $l_{k|j} = 0$. Also let us assume that deflection tensor $D^i_j$ for the Finsler connection $F\Gamma$ is zero, i.e. $D^i_j = 0$. Then we get $y^i_{|j} = 0$. Again $l_{k|j} = 0$ and $y^i_{|j} = 0$, implies that $L_{|j} = 0$, and hence equation (2.5) yields ${}^*P^i_{jk} = P^i_{jk}$. Thus, we have:

**Theorem 2.1.** *Let the Finsler space $F^n$ admits a Finsler connection $F\Gamma$ with zero deflection tensor and a Finsler connection ${}^*F\Gamma$. Then ${}^*P^i_{jk} = P^i_{jk}$ if and only if $l_{k|j} = 0$.*

Rephrasing, the theorem concludes that (v)hv-torsion tensors with respect to Finsler connections $F\Gamma$ and ${}^*F\Gamma$ are equal if and only if covariant differentiation of directional derivative of Fundamental metric function $L$ vanishes.

**The hv-curvature tensor.** The hv-curvature tensor $P^i_{hjk}$ of a Finsler space with respect to connection $F\Gamma$ is defined by

$$P^i_{hjk} = \dot{\partial}_k \Gamma^i_{hj} - C^i_{hk|j} + C^i_{hm} P^m_{jk}.$$

Therefore the hv-torsion tensor ${}^*P^i_{jk}$ of a Finsler space with respect to connection ${}^*F\Gamma$ is given by

$${}^*P^i_{hjk} = \dot{\partial}^*_k \Gamma^i_{hj} - C^i_{hk|j} + C^i_{hm} {}^*P^m_{jk}.$$

Using (2.4) and (2.5) we get

$$(2.6) \qquad {}^*P^i_{hjk} = P^i_{hjk} + C^i_{hm} h^m_k \frac{L_{|j}}{L}.$$

Thus, we have:

**Theorem 2.2.** *Let the Finsler space $F^n$ admits the Finsler connections $F\Gamma$ and ${}^*F\Gamma$. Then expression for ${}^*P^i_{hjk}$ is given by (2.6).*

Let us suppose that ${}^*P^i_{hjk} = P^i_{hjk}$. Then equation (2.6) implies that

$$C^i_{hm} h^m_k \frac{L_{|j}}{L} = 0,$$

which gives $C^i_{hm} = 0$, i.e. the space is Riemannian. Thus, we have:

**Theorem 2.3.** *If hv-curvature tensor with respect to Finsler connections $F\Gamma$ and ${}^*F\Gamma$ coincides, then the space will be Riemannian.*

**The v(h)-torsion tensor.** The v(h)-torsion tensor $R^i_{jk}$ of a Finsler space with respect to connection $F\Gamma$ is defined by

$$R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k.$$

Therefore the v(h)-torsion tensor ${}^*R^i_{jk}$ of a Finsler space with respect to connection ${}^*F\Gamma$ is given by

$${}^*R^i_{jk} = {}^*\delta_k {}^*N^i_j - {}^*\delta_j {}^*N^i_k.$$

Using (1.2) and (2.1) and simplification gives us

$$ {}^*R^i_{jk} = R^i_{jk} + [L_{|j|k} l^i + L_{|j} l^i_{|k} - L_{|k} l^r l_{r|j} l^i - j/k], \tag{2.7}$$

where $-j/k$ denotes interchange of j and k and subtract the terms within the bracket.

**Theorem 2.4.** *Let the Finsler space $F^n$ admits the Finsler connections $F\Gamma$ and ${}^*F\Gamma$. Then expression for ${}^*R^i_{jk}$ is given by* (2.7)*.*

Equation (2.7) can be re-written as

$$ {}^*R^i_{jk} = R^i_{jk} + [(L_{|j|k} - L_{|k|j}) l^i + L_{|j} \frac{y^i_{|k}}{L} - L_{|k} \frac{y^i_{|j}}{L} + L_{|k} \frac{y^r_{|j}}{L} l^r l^i - L_{|j} \frac{y^r_{|k}}{L} l^r l^i]. \tag{2.8}$$

Let us assume that deflection tensor $D^i_j$ for the Finsler connection $F\Gamma$ is zero i.e. $D^i_j = 0$. Then we get $y^i_{|j} = 0$ and then equation (2.8) becomes

$$ {}^*R^i_{jk} = R^i_{jk} + (L_{|j|k} - L_{|k|j}) l^i. \tag{2.9}$$

Thus, we have:

**Corollary 1.** *Let the Finsler space $F^n$ admits the Finsler connection $F\Gamma$ with zero deflection tensor and a Finsler connection ${}^*F\Gamma$. Then v(h)-torsion tensor ${}^*R^i_{jk}$ for the connection ${}^*F\Gamma$ is given by* (2.9)*.*

**The h-curvature tensor.** The h-curvature tensor $R^i_{hjk}$ of a Finsler space with respect to connection $F\Gamma$ is defined by

$$R^i_{hjk} = C^i_{hm} R^m_{jk} + \delta_k \Gamma^i_{hj} + \Gamma^m_{hj} \Gamma^i_{mk} - \delta_j \Gamma^i_{hk} - \Gamma^m_{hk} \Gamma^i_{mj}.$$

Therefore the h-curvature tensor ${}^*R^i_{jk}$ of a Finsler space with respect to connection ${}^*F\Gamma$ is given by

$${}^*R^i_{hjk} = C^i_{hm} {}^*R^m_{jk} + {}^*\delta_k \Gamma^i_{hj} + \Gamma^m_{hj} \Gamma^i_{mk} - {}^*\delta_j \Gamma^i_{hk} - \Gamma^m_{hk} \Gamma^i_{mj}.$$

Using (2.1) and (2.7) and simplifying, we obtain

$$ {}^*R^i_{hjk} = R^i_{hjk} - (L_{|k} l^r \dot{\partial}_r \Gamma^i_{hj} - L_{|j} l^r \dot{\partial}_r \Gamma^i_{hk}) + C^i_{hm}[L_{|j} l^m_{|k} - L_{|k} l^m_{|j}]. \tag{2.10} $$

Thus, we have:

**Theorem 2.5.** *Let the Finsler space $F^n$ admits Finsler connections $F\Gamma$ and ${}^*F\Gamma$. Then expression for ${}^*R^i_{hjk}$ is given by* (2.10).

Equation (2.10) can be re-written as

$$ {}^*R^i_{hjk} = R^i_{hjk} - L_{|k} l^r \dot{\partial}_r \Gamma^i_{hj} + L_{|j} l^r \dot{\partial}_r \Gamma^i_{hk} + C^i_{hm}[L_{|j} \frac{y^m_{|k}}{L} - L_{|k} \frac{y^m_{|j}}{L}]. \tag{2.11} $$

Also let us assume that deflection tensor $D^i_j$ for the Finsler connection $F\Gamma$ is zero, i.e. $D^i_j = 0$. Then we get $y^i_{|j} = 0$ and the equation (2.11) yields

$$ {}^*R^i_{hjk} = R^i_{hjk} - (L_{|k} l^r \dot{\partial}_r \Gamma^i_{hj}) + (L_{|j} l^r \dot{\partial}_r \Gamma^i_{hk}). \tag{2.12} $$

Thus, we have:

**Corollary 2.** *Let the Finsler space $F^n$ admits the Finsler connections $F\Gamma$ with deflection zero and ${}^*F\Gamma$. Then expression for ${}^*R^i_{hjk}$ is given by* (2.12).

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