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GRADIENT RICCI ALMOST SOLITONS IN SASAKIAN MANIFOLD

NIRABHRA BASU AND ARINDAM BHATTACHARYYA

ABSTRACT. In this paper we have shown that if a Sasakian manifold satisfies Ricci gradient almost soliton then the potential function f can not be constant.

1. INTRODUCTION

R. Hamilton introduced the concept of Ricci flow in [5]. Since then Ricci flow has enriched the ways of the study of Riemannian manifolds, especially for those manifolds with positive curvature. G. Perelman[8, 9] has made Ricci flow more interesting to scientists with the excellence of his work in this field. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2 \operatorname{Ric}\ g$$

Ricci soliton emerges as the limit of the solutions of Ricci flow [6]. A solution to the Ricci flow is called a Ricci soliton [4] if it moves only by a one-parameter group of diffeomorphism and scaling. A Riemannian manifold (M, g) is called Ricci soliton if there exists a smooth vector field X, such that the Ricci tensor satisfies the following equation

$$\operatorname{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1.1}$$

for some constant λ and $\mathcal{L}_X$ is the Lie-derivative.

Ricci soliton is called a gradient Ricci soliton if $X = \nabla f$, for some smooth function f on M. They are also natural generalizations of Einstein metrics. Note that a soliton is called shirking, steady and expanding according as $\lambda > 0, \lambda = 0$ and $\lambda < 0$. When $\lambda \leq 0$, all compact solitons are necessarily Einstein. Chenxu He and Meng Zhu [7] have shown that if a Sasakian manifold satisfies

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the gradient Ricci soliton, then f is a constant function and the manifold is an Einstein manifold.

Pigola and his co-researchers [10] have introduced a natural extension of the concept of gradient Ricci soliton; the Ricci almost soliton. R. Sharma [11], Barros [1, 2] did some significant research work on Ricci almost soliton. Pigola [10] defined Ricci almost soliton equation as a Riemannian manifold (M, g) satisfying the condition

$$2\,\mathrm{Ric} + \mathcal{L}_X g = 2\lambda g$$

where λ is a smooth function on M. For λ constant, the equation (1.2) will become the Ricci soliton equation (1.1). The Ricci almost soliton is said to be shrinking, steady or expanding according to the fact that λ is positive, zero or negative respectively; otherwise it is said to be indefinite. If the vector field X is gradient of a smooth function f, then we can replace the soliton vector field X by ∇f in the previous equation and (M, g, X, λ) is called a gradient Ricci almost soliton. The equation of gradient Ricci almost soliton is given below.

$$\mathrm{Ric} + \nabla(\nabla f) = \lambda g \tag{1.2}$$

where λ is a smooth function.

We have studied the nature of the function f in a Sasakian manifold which satisfies the gradient Ricci almost soliton equation.

2. Preliminaries

A Sasakian manifold $(M^{2n+1}, g, \xi, \eta; \phi)$ is a contact manifold on which there exists a unit Killing vector field ξ, η is the dual one-form of ξ and ϕ is a $(1, 1)$ tensor defined by $\phi(Y) = \nabla_Y \xi$ [3]. The metric (M, g) is called Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\phi - \eta(Y)X$$
$$R(X, Y)\xi = \eta(Y)X - \eta(x)Y$$
$$S(\phi X, \phi Y) = S(X, Y)$$
$$R(Y, \xi)Z = -g(Y, Z)\xi + g(Z, \xi)Y$$

for any vector fields $Y, Z \in TM$.

Let $D \subset TM$ be the distribution defined by $\eta(Y) = g(Y, \xi)$. Then D is nowhere integrable as η is a contact 1-form, Science ξ is a unit Killing vector field, we have $\phi(Y) = \nabla_Y \xi \in D$, for any $Y \in TM$.

Now $R(Y, \xi, Z, Y) = g(R(Y, \xi)Z, Y) = -g(Y, Z)g(\xi, Y) + g(Z, \xi)g(Y, Y)$. Using $\eta(Y) = g(\xi, Y) = 0$; we get

$$R(Y, \xi, Z, Y) = g(Z, \xi)|Y|^2.$$

If we take $Y = e_i$ and summing over i, we have

$$\mathrm{Ric}(\xi, Z) = 2mg(\xi, Z). \tag{2.1}$$

Now we shall prove the main result.

3. Result

Theorem 1. *If (M, g) is a Sasakian manifold satisfies gradient Ricci almost soliton equation, then the smooth function f of the gradient Ricci almost soliton is neither a constant nor ∇f is perpendicular to the vector field $Y \in D$, where $D \subset TM$ be the distribution and λ is the nonzero function of the gradient Ricci almost soliton.*

Proof. Let us start with $\mathcal{L}_\xi(\mathcal{L}_X g) = 0$ which implies

$$R(X, \xi, \xi, Y) + \nabla_Y g(\nabla_\xi X, \xi) + g(\nabla_\xi \nabla_\xi X, Y) = 0. \tag{3.1}$$

Now the Ricci almost soliton equation is

$$\mathcal{L}_X g(\xi, W) = 2\lambda g(\xi, W) - 2\,\mathrm{Ric}(\xi, W).$$

Using (2.1) we get

$$\mathcal{L}_X g(\xi, W) = 2(\lambda - 2m) g(\xi, W)$$

implies

$$Xg(\xi, W) - g(\mathcal{L}_X \xi, W) - g(\xi, \mathcal{L}_X W) = 2(\lambda - 2m) g(\xi, W)$$
$$\Rightarrow Xg(\xi, W) - g([X, \xi], W) - g(\xi, [X, W]) = 2(\lambda - 2m) g(\xi, W)$$
$$\Rightarrow Xg(\xi, W) - g(\nabla_X \xi - \nabla_\xi X, W) - g(\xi, \nabla_X W - \nabla_W X) = 2(\lambda - 2m) g(\xi, W)$$
$$\Rightarrow (\nabla_X g)(\xi, W) + g(\nabla_\xi X, W) + g(\xi, \nabla_W X) = 2(\lambda - 2m) g(\xi, W).$$

Science $(\nabla_X g)(\xi, W) = 0$; we have

$$g(\nabla_\xi X, W) + g(\xi, \nabla_W X) = 2(\lambda - 2m) g(\xi, W).$$

Now if we put $W = \xi$, we get

$$g(\nabla_\xi X, \xi) = (\lambda - 2m) g(\xi, \xi) \tag{3.2}$$

Now from (3.1), we get

$$R(X, \xi, \xi, Y) + \nabla_Y g(\nabla_\xi X, \xi) + g(\nabla_\xi \nabla_\xi, Y) = 0$$

i.e.

$$\Rightarrow g(X, Y) + \nabla_Y(\lambda - 2m) + g(\nabla_\xi \nabla_\xi X, Y) = 0. \tag{3.3}$$

For gradient almost soliton we put ∇f for X in (3.3), where f is a smooth function, we have

$$g(\nabla f, Y) + \nabla_Y(\lambda - 2m) + g(\nabla_\xi \nabla_\xi \nabla f, Y) = 0. \tag{3.4}$$

From (3.2), we get $\nabla_\xi X = (\lambda - 2m)\xi$; i.e $\nabla_\xi \nabla f = (\lambda - 2m)\xi$. Putting this in (3.4), we get

$$g(\nabla f, Y) + \nabla_Y \lambda + g(\nabla_\xi \lambda \xi, Y) = 0$$
$$g(\nabla f, Y) + Y\lambda + g(\nabla_\xi \lambda \xi, Y) = 0.$$

Putting $\nabla_\xi(\lambda\xi) = (\xi\lambda)\xi$ in the previous equation we get

$$g(\nabla f, Y) + Y\lambda + (\xi\lambda) g(\xi, Y) = 0$$
$$\Rightarrow g(\nabla f, Y) = -Y\lambda.$$

So, we can conclude that f is neither a constant nor ∇f is perpendicular to $Y \in D$ for the nonzero function λ. $\square$

References

[1] A. Barros, R. Batista, and E. Ribeiro, Jr. Compact almost Ricci solitons with constant scalar curvature are gradient. *Monatsh. Math.*, 174(1):29–39, 2014.
[2] A. Barros and E. Ribeiro, Jr. Some characterizations for compact almost Ricci solitons. *Proc. Amer. Math. Soc.*, 140(3):1033–1040, 2012.
[3] C. P. Boyer and K. Galicki. *Sasakian geometry.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008.
[4] H.-D. Cao. Recent progress on Ricci solitons. In *Recent advances in geometric analysis*, volume 11 of *Adv. Lect. Math. (ALM)*, pages 1–38. Int. Press, Somerville, MA, 2010.
[5] R. S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.
[6] R. S. Hamilton. The formation of singularities in the Ricci flow. In *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, pages 7–136. Int. Press, Cambridge, MA, 1995.
[7] C. He and M. Zhu. Ricci solitons on sasakian manifolds. arXiv:1109.4407v2, 2011.
[8] G. Perelman. The entropy formula for the ricci flow and its geometric applications. arXiv:math/0211159, 2002.
[9] G. Perelman. Ricci flow with surgery on three-manifolds. arXiv:math/0303109, 2003.
[10] S. Pigola, M. Rigoli, M. Rimoldi, and A. G. Setti. Ricci almost solitons. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 10(4):757–799, 2011.
[11] R. Sharma. Almost Ricci solitons and K-contact geometry. *Monatsh. Math.*, 175(4):621–628, 2014.

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Nirabhra Basu,
Department of Mathematics,
Bhowanipur Education Society College,
Kolkata-700020, West Bengal, India
E-mail address: nirabhra.basu@hotmail.com

Arindam Bhattacharyya,
Department of Mathematics,
Jadavpur University,
Kolkata-700032,India
E-mail address: bhattachar1968@yahoo.co.in