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A NOTE ON MAXIMAL OPERATORS OF VILENKIN – NÖRLUND MEANS

I. BLAHOTA AND G. TEPHNADZE

Dedicated to Professor Ferenc Schipp on the occasion of his 75th birthday, to Professor William Wade on the occasion of his 70th birthday and to Professor Péter Simon on the occasion of his 65th birthday.

Abstract. In this paper we prove and discuss some new (H_p, L_p)-type inequalities of weighted maximal operators of Vilenkin – Nörlund means with non-increasing coefficients. These results are the best possible in a special sense. As applications, both some well-known and new results are pointed out in the theory of strong convergence of Vilenkin – Nörlund means with non-increasing coefficients.

1. Introduction

The definitions and notations used in this introduction can be found in our next section. In the one-dimensional case the weak $(1,1)$-type inequality for maximal operator of Fejér means $\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|$ can be found in Schipp [18] for Walsh series and in Pál, Simon [17] for bounded Vilenkin series. Fujji [6] and Simon [19] verified that σ^* is bounded from H_1 to L_1. Weisz [28] generalized this result and proved boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Simon [20] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. A counterexample for $p = 1/2$ was given by Goginava [9]. Weisz [31] proved that the maximal operator of the Fejér means σ^* is bounded from the Hardy space $H_{1/2}$ to the space $weak - L_{1/2}$.

In [8] Goginava investigated the behaviour of Cesàro means in detail. In the two-dimensional case approximation properties of Nörlund and Cesàro means

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was considered by Nagy [13]. Weisz [30] proved that the maximal operator of Cesàro means $\sigma^{\alpha,*}f := \sup_{n\in\mathbb{N}} |\sigma_n^{\alpha} f|$ is bounded from the martingale space H_p to the space L_p for $p > 1/(1+\alpha)$. Goginava [10] gave a counterexample, which shows that boundedness does not hold for $0 < p \leq 1/(1+\alpha)$. Simon and Weisz [22] showed that the maximal operator $\sigma^{\alpha,*}$ $(0 < \alpha < 1)$ of the (C,α) means is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $weak-L_{1/(1+\alpha)}$. In [4] and [25] it was also proved that the maximal operator

$$\widetilde{\sigma}_p^{\alpha,*} := \sup_{n\in\mathbb{N}} |\sigma_n^{\alpha} f| / (n+1)^{1/p-\alpha-1} \log^{(1+\alpha)[p+\alpha(1+\alpha)]}(n+1)$$

is bounded from the Hardy space H_p to the space L_p, where $0 < p \leq 1/(1+\alpha)$. Moreover, the rate of the weights $\left\{(n+1)^{1/p-\alpha-1} \log^{(1+\alpha)[p+\alpha(1+\alpha)]}(n+1)\right\}_{n=1}^{\infty}$ in nth Cesàro mean is given exactly.

It is well-known that Vilenkin systems do not form bases in the space $L_1(G_m)$. Moreover, there is a function in the Hardy space $H_1(G_m)$, such that the partial sums of f are not bounded in L_1-norm. Simon [21] (for $p=1$ see [1] and [7] and for $0 < p < 1$ it was shown in [24]) proved that there exists an absolute constant c_p, depending only on p, such that

$$\frac{1}{\log^{[p]} n} \sum_{k=1}^{n} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p, \quad (0 < p \leq 1) \tag{1}$$

for all $f \in H_p$ and $n \in \mathbb{N}_+$, where $[p]$ denotes the integer part of p. In [23] it was proved that sequence $\{1/k^{2-p}\}_{k=1}^{\infty}$ $(0 < p < 1)$ in (1) can not be improved.

In [5] it was proved that there exists an absolute constant c_p, depending only on p, such that

$$\frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p, \quad (0 < p \leq 1/2,\ n = 2,3,\ldots). \tag{2}$$

Analogical result for (C,α) $(0 < \alpha < 1)$ means when $p = 1/(1+\alpha)$ was generalized in [4] and the case $0 < p < 1/(1+\alpha)$ was proved in [25]. In particular the following inequality

$$\frac{1}{\log^{[\alpha/(1+\alpha)+p]} n} \sum_{k=1}^{n} \frac{\|\sigma_k^{\alpha} f\|_p^p}{k^{2-(1+\alpha)p}} \leq c_p \|f\|_{H_p}^p, \quad (0 < p \leq 1/(1+\alpha),\ n = 2,3,\ldots)$$

holds.

Móricz and Siddiqi [12] investigated the approximation properties of some special Nörlund means of L_p function in norm. For more information on Nörlund logarithmic means, see paper of Blahota and Gát [2] and Nagy [14] (see also [16] and [15]). In [3] there were proved strong convergence theorems of Nörlund means and boundedness of weighted maximal operators of Nörlund means

$$\widetilde{t}^{*} f := \sup_{n\in\mathbb{N}} |t_n f| / \log^{1+\alpha}(n+1)$$

from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha)}$, but in the case when sequence $\{q_n : n \geq 0\}$ is non-increasing, such that

$$n^{\alpha}/Q_n = O(1), \text{ as } n \to \infty, \tag{3}$$

and

$$(q_n - q_{n+1})/n^{\alpha-2} = O(1), \text{ as } n \to \infty, \tag{4}$$

where $Q_n := \sum_{k=0}^{n-1} q_k$.

In this paper we prove and discuss some new (H_p, L_p)-type inequalities of weighted maximal operators of Vilenkin – Nörlund means with non-increasing coefficients. As applications, both some well-known and new results are pointed out in the theory of strong convergence of Vilenkin – Nörlund means.

This paper is organized as follows: in order not to disturb our discussions later on some definitions and notations are presented in Section 2. The main results and some of its consequences can be found in Section 3. For the proofs of main results we need some auxiliary results. These results are presented in Section 4. The detailed proofs are given in Section 5.

2. Definitions and Notations

Denote by $\mathbb{N}_+$ the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of the positive integers not less than 2. Denote by $Z_{m_n} := \{0, 1, \ldots, m_n - 1\}$ the additive group of integers modulo m_n. Define the group G_m as the complete direct product of the groups Z_{m_n} with the product of the discrete topologies of Z_{m_n}'s.

In this paper we discuss bounded Vilenkin groups, i.e. the case when $\sup_n m_n < \infty$.

The direct product μ of the measures

$$\mu_n(\{j\}) := 1/m_n, \quad (j \in Z_{m_n})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \ldots, x_n, \ldots), \quad (x_n \in Z_{m_n}).$$

It is easy to give a base for the neighbourhood of G_m :

$$I_0(x) := G_m, \ I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$$

for $x \in G_m, \ n \in \mathbb{N}$.

Denote $I_n := I_n(0)$, for $n \in \mathbb{N}_+$ and

$$e_n := (0, \ldots, x_n = 1, 0, \ldots) \in G_m, \quad (n \in \mathbb{N}).$$

It is evident that

$$\overline{I_N} = \left(\bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l} \right) \bigcup \left(\bigcup_{k=1}^{N-1} I_N^{k,N} \right), \tag{5}$$

where

$$I_N^{k,l} :=$$

$$\begin{cases} I_N(0,\ldots,0,x_k\neq 0,0,\ldots,0,x_l\neq 0,x_{l+1},\ldots,x_{N-1},\ldots), & \text{for } k<l<N,\\ I_N(0,\ldots,0,x_k\neq 0,0,\ldots,x_{N-1}=0,x_N,\ldots), & \text{for } l=N. \end{cases}$$

If we define the so-called generalized number system based on m in the following way :

$$M_0 := 1,\ M_{n+1} := m_n M_n \quad (n\in\mathbb{N}),$$

then every $n\in\mathbb{N}$ can be uniquely expressed as $n=\sum_{k=0}^{\infty} n_k M_k$, where $n_k\in Z_{m_k}$ $(k\in\mathbb{N}_+)$ and only a finite number of n_k's differ from zero.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. At first, we define the complex-valued function $r_k\colon G_m\to\mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi\imath x_k/m_k), \quad \left(\imath^2=-1, x\in G_m, k\in\mathbb{N}\right).$$

Now, define the Vilenkin system $\psi := (\psi_n : n\in\mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n\in\mathbb{N}).$$

Specifically, we call this system the Walsh-Paley system, when $m\equiv 2$.

The norms (or quasi-norm) of the spaces $L_p(G_m)$ and $weak-L_p(G_m)$ $(0<p<\infty)$ are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p\, d\mu, \quad \|f\|_{weak-L_p}^p := \sup_{\lambda>0}\lambda^p\mu(f>\lambda)<\infty.$$

The Vilenkin systems are orthonormal and complete in $L_2(G_m)$ (see [26]).

Now we introduce analogues of the usual definitions in Fourier-analysis. If $f\in L_1(G_m)$ we can define Fourier coefficients, partial sums of the Fourier series, Dirichlet kernels with respect to the Vilenkin systems in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f\overline{\psi}_n d\mu,\ S_n f := \sum_{k=0}^{n-1}\widehat{f}(k)\psi_k,\ D_n := \sum_{k=0}^{n-1}\psi_k, \quad (n\in\mathbb{N}_+)$$

respectively.

The σ-algebra generated by the intervals $\{I_n(x) : x\in G_m\}$ will be denoted by $\digamma_n\,(n\in\mathbb{N})$. Denote by $f=\left(f^{(n)}, n\in\mathbb{N}\right)$ a martingale with respect to $\digamma_n\,(n\in\mathbb{N})$. (for details see e.g. [27]).

The maximal function of a martingale f is defined by

$$f^* := \sup_{n\in\mathbb{N}}\left|f^{(n)}\right|.$$

For $0<p<\infty$ the Hardy martingale spaces $H_p\ (G_m)$ consist of all martingales, for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty} \int_{G_m} f^{(k)} \overline{\psi}_i d\mu.$$

Let $\{q_n : n \geq 0\}$ be a sequence of non-negative numbers. The nth Nörlund mean is defined by

$$t_n f := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f,$$

where $Q_n := \sum_{k=0}^{n-1} q_k$.

It is well known that

$$t_n f(x) = \int_{G_m} f(t) F_n(x-t)\, dt, \quad F_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k.$$

We always assume that $q_0 > 0$ and $\lim_{n\to\infty} Q_n = \infty$. In this case (see [11]) the summability method generated by $\{q_n : n \geq 0\}$ is regular if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = 0.$$

If $q_n \equiv 1$, then we respectively get the usual nth Fejér mean and Fejér kernel

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k.$$

The (C, α)-means (Cesàro means) of the Vilenkin-Fourier series are defined by

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f,$$

where

$$A_0^\alpha := 0, \; A_n^\alpha := \frac{(\alpha+1)\ldots(\alpha+n)}{n!}, \quad \alpha \neq -1, -2, \ldots$$

We consider following maximal operators:

$$\tilde{t}{}_p^* f := \sup_{n\in\mathbb{N}} |t_n f| / (n+1)^{1/p-\alpha-1}, \; \widetilde{\sigma}_p^{\alpha,*} f := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha f| / (n+1)^{1/p-\alpha-1}.$$

A bounded measurable function a is called a p-atom, if there exists an interval I, such that

$$\int_I a\, d\mu = 0, \; \|a\|_\infty \leq \mu(I)^{-1/p}, \quad \operatorname{supp}(a) \subset I.$$

3. Formulation of main results

Theorem 1. *Let $f \in H_p$, where $0 < \alpha < 1$, $0 < p < 1/(1+\alpha)$ and $\{q_n : n \geq 0\}$, be a sequence of non-increasing numbers, satisfying conditions (3) and (4). Then there exists an absolute constant c_α, depending only on α and p, such that*

$$\left\| \widetilde{t}_p^{*} f \right\|_p \leq c_{\alpha,p} \left\| f \right\|_{H_p}.$$

Corollary 1 (Blahota, Tephnadze [4])**.** *Let $f \in H_p$, where $0 < \alpha < 1$ and $0 < p < 1/(1+\alpha)$. Then there exists an absolute constant $c_{\alpha,p}$, depending only on α and p, such that*

$$\left\| \widetilde{\sigma}_p^{\alpha,*} f \right\|_p \leq c_{\alpha,p} \left\| f \right\|_{H_p}.$$

Theorem 2. *Let $f \in H_p$, where $0 < \alpha < 1$, $0 < p < 1/(1+\alpha)$ and $\{q_n : n \geq 0\}$, be a sequence of non-increasing numbers, satisfying condition (3) and (4). Then there exists an absolute constant $c_{\alpha,p}$, depending only on α and p, such that*

$$\sum_{k=1}^{\infty} \frac{\left\| t_k f \right\|_{H_p}^p}{k^{2-(1+\alpha)p}} \leq c_{\alpha,p} \left\| f \right\|_{H_p}^p.$$

Corollary 2 (Blahota, Tephnadze [4])**.** *Let $f \in H_p$, where $0 < \alpha < 1$ and $0 < p < 1/(1+\alpha)$. Then there exists an absolute constant $c_{\alpha,p}$, depending only on α and p, such that*

$$\sum_{k=1}^{\infty} \frac{\left\| \sigma_k^{\alpha} f \right\|_{H_p}^p}{k^{2-(1+\alpha)p}} \leq c_{\alpha,p} \left\| f \right\|_{H_p}^p.$$

4. Auxiliary results

Lemma 1 (Weisz[27])**.** *A martingale $f = (f_n,\ n \in \mathbb{N})$ is in H_p $(0 < p \leq 1)$ if and only if there exists a sequence $(a_k,\ k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k,\ k \in \mathbb{N})$ of real numbers, such that for every $n \in \mathbb{N}$*

$$\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f_n, \tag{6}$$

$$\sum_{k=0}^{\infty} \left| \mu_k \right|^p < \infty.$$

Moreover, $\left\| f \right\|_{H_p} \backsim \inf \left(\sum_{k=0}^{\infty} \left| \mu_k \right|^p \right)^{1/p}$, where the infimum is taken over all decompositions of f of the form (6).

Lemma 2 (Weisz [29])**.** *Suppose that an operator T is σ-linear and for some $0 < p \leq 1$*

$$\int\limits_{\overline{I}} \left| Ta \right|^p d\mu \leq c_p < \infty,$$

for every p-atom a, where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then

$$\|Tf\|_p \leq c_p \|f\|_{H_p}.$$

Lemma 3 ([3])**.** *Let $0 < \alpha \leq 1$ and $\{q_n : n \geq 0\}$ be a sequence of non-increasing numbers, satisfying conditions (3) and (4). Then*

$$|F_n| \leq \frac{c_\alpha}{n^\alpha}\left\{\sum_{j=0}^{|n|} M_j^\alpha \left|K_{M_j}\right|\right\}.$$

Moreover, if $r \geq M_N$, then

$$\int_{I_N} |F_r(x-t)|\, d\mu(t) \leq \frac{c_\alpha M_l^\alpha M_k}{r^\alpha M_N}, \quad x \in I_N^{k,l},$$

where $k = 0, \ldots, N-2, l = k+2, \ldots, N-1$ and

$$\int_{I_N} |F_r(x-t)|\, d\mu(t) \leq \frac{c_\alpha M_k}{M_N}, \quad x \in I_N^{k,N},$$

where $k = 0, \ldots, N-1$.

5. Proofs of main results

Proof of Theorem 1. Since t_n is bounded from L_∞ to L_∞ (the boundedness follows from Lemma 3) according to Lemma 2 the proof of Theorem 1 will be complete if we show that

$$\int_{\overline{I_N}} \left|\widetilde{t}_p^* a\right|^p d\mu < \infty,$$

for every p-atoms a. We may assume that a is an arbitrary p-atom, with support I, $\mu(I) = M_N^{-1}$ and $I = I_N$. It is easy to see that $t_n(a) = 0$, when $n \leq M_N$. Therefore, we can suppose that $n > M_N$.

Let $x \in I_N$. Since $\|a\|_\infty \leq M_N^{1/p}$ we obtain that

$$\begin{aligned}|t_n a(x)| &\leq \int_{I_N} |a(t)|\,|F_n(x-t)|\, d\mu(t)\\ &\leq \|a\|_\infty \int_{I_N} |F_n(x-t)|\, d\mu(t) \leq M_N^{1/p} \int_{I_N} |F_n(x-t)|\, d\mu(t).\end{aligned}$$

Let $x \in I_N^{k,l}$, $0 \leq k < l < N$. Then from Lemma 3 we get that

$$|t_n a(x)| \leq \frac{c_{\alpha,p} M_N^{1/p-1} M_l^\alpha M_k}{n^\alpha}. \tag{7}$$

Let $x \in I_N^{k,N}$, $0 \leq k < N$. Then from Lemma 3 we have that

$$|t_n a(x)| \leq c_{\alpha,p} M_N^{1/p-1} M_k. \tag{8}$$

Since $n > M_N$, if we apply (5), (7) and (8) we obtain that

$$
\begin{aligned}
&\int_{\overline{I_N}} \sup_{n\in\mathbb{N}} \left|\frac{t_n a}{n^{1/p-1-\alpha}}\right|^p d\mu \\
=&\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{x_j=0, j\in\{l+1,\dots,N-1\}}^{m_j-1}\int_{I_N^{k,l}}\sup_{n>M_N}\left|\frac{t_n a}{n^{1/p-1-\alpha}}\right|^p d\mu \\
&+\sum_{k=0}^{N-1}\int_{I_N^{k,N}}\sup_{n>M_N}\left|\frac{t_n a}{n^{1/p-1-\alpha}}\right|^p d\mu \\
\leq&\frac{1}{M_N^{1-(1+\alpha)p}}\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{x_j=0, j\in\{l+1,\dots,N-1\}}^{1}\int_{I_N^{k,l}}\sup_{n>M_N}|t_n a|^p d\mu \\
&+\frac{1}{M_N^{1-(1+\alpha)p}}\sum_{k=0}^{M-1}\int_{I_N^{k,N}}\sup_{n>M_N}|t_n a|^p d\mu \\
\leq&\frac{c_{\alpha,p}}{M_N^{1-(1+\alpha)p}}\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{1}{M_l}\frac{M_N^{1-p}M_l^{\alpha p}M_k^p}{M_N^{\alpha p}}+\frac{c_{\alpha,p}}{M_N^{1-(1+\alpha)p}}\frac{1}{M_N}\sum_{k=0}^{N-1}M_N^{1-p}M_k^p \\
\leq&c_{\alpha,p}\sum_{k=0}^{N-2}M_k^p\sum_{l=k+1}^{N-1}\frac{1}{M_l^{1-\alpha p}}+\frac{c_{\alpha,p}}{M_N^{1-(1+\alpha)p}}\sum_{k=0}^{N-1}\frac{M_k^p}{M_N^p}\leq c_{\alpha,p}<\infty.
\end{aligned}
$$

□

Proof of Theorem 2. By Lemma 1 the proof of Theorem 2 will be complete, if we show that

$$\sum_{k=1}^{\infty}\frac{\|t_k a\|_p^p}{k^{2-(1+\alpha)p}}\leq c_{\alpha,p}<\infty,$$

for every p-atom a. Analogously to the proof of Theorem 1 we may assume that a be an arbitrary p-atom with support I, $\mu(I)=M_N^{-1}$ and $I=I_N$ and $n>M_N$.

Let $x\in I_N$. Since t_m is bounded from L_∞ to L_∞ (the boundedness follows from Lemma 3) and $\|a\|_\infty\leq M_N^{1/p}$, we obtain

$$\int_{I_N}|t_n a(x)|^p d\mu\leq\|a(x)\|_\infty^p M_N^{-1}\leq c_{\alpha,p}<\infty.$$

Hence

$$\sum_{k=M_N}^{\infty}\frac{\int_{I_N}|t_k a(x)|^{1/(1+\alpha)}d\mu}{k^{2-(1+\alpha)p}}\leq\sum_{k=1}^{\infty}\frac{1}{k^{2-(1+\alpha)p}}\leq c_{\alpha,p}<\infty.$$

By combining (5) and (7)-(8) we can conclude that

$$\begin{aligned}
&\sum_{k=M_N+1}^{\infty}\frac{\int_{\overline{I_N}}\left|t_k a\left(x\right)\right|^p d\mu\left(x\right)}{k^{2-(1+\alpha)p}}\\
&=\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{x_j=0,j\in\{l+1,\dots,N-1\}}^{m_j-1}\frac{\int_{I_N^{k,l}}\left|t_k a\left(x\right)\right|^p d\mu\left(x\right)}{k^{2-(1+\alpha)p}}\\
&\quad+\sum_{k=M_N+1}^{n}\sum_{k=0}^{N-1}\frac{\int_{I_N^{k,N}}\left|t_k a\left(x\right)\right|^p d\mu\left(x\right)}{k^{2-(1+\alpha)p}}\\
&\le c_{\alpha,p}\sum_{k=M_N+1}^{\infty}\left(\frac{M_N^{1-p}}{k^{2-p}}\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{M_l^{p\alpha}M_k^p}{M_l}+\frac{M_N^{1-p}}{k^{2-(1+\alpha)p}}\sum_{k=0}^{N-1}\frac{M_k^p}{M_N}\right)\\
&< c_{\alpha,p}M_N^{1-p}\sum_{k=M_N+1}^{\infty}\frac{1}{k^{2-p}}+c_{\alpha,p}\sum_{k=M_N+1}^{\infty}\frac{1}{k^{2-(1+\alpha)p}}\le c_{\alpha,p}<\infty.
\end{aligned}$$

which complete the proof of Theorem 2. □

References

[1] I. Blahota. On a norm inequality with respect to Vilenkin-like systems. *Acta Math. Hungar.*, 89(1-2):15–27, 2000.

[2] I. Blahota and G. Gát. Norm summability of Nörlund logarithmic means on unbounded Vilenkin groups. *Anal. Theory Appl.*, 24(1):1–17, 2008.

[3] I. Blahota, L.-E. Persson, and G. Tephnadze. On the Nörlund means of Vilenkin-Fourier series. *Czechoslovak Math. J.*, 65(140)(4):983–1002, 2015.

[4] I. Blahota and G. Tephnadze. On the (C,α)-means with respect to the Walsh system. *Anal. Math.*, 40(3):161–174, 2014.

[5] I. Blahota and G. Tephnadze. Strong convergence theorem for Vilenkin-Fejér means. *Publ. Math. Debrecen*, 85(1-2):181–196, 2014.

[6] N. Fujii. A maximal inequality for H^1-functions on a generalized Walsh-Paley group. *Proc. Amer. Math. Soc.*, 77(1):111–116, 1979.

[7] G. Gát. Investigations of certain operators with respect to the Vilenkin system. *Acta Math. Hungar.*, 61(1-2):131–149, 1993.

[8] U. Goginava. On the approximation properties of Cesàro means of negative order of Walsh-Fourier series. *J. Approx. Theory*, 115(1):9–20, 2002.

[9] U. Goginava. The maximal operator of Marcinkiewicz-Fejér means of the d-dimensional Walsh-Fourier series. *East J. Approx.*, 12(3):295–302, 2006.

[10] U. Goginava. The maximal operator of the (C,α) means of the Walsh-Fourier series. *Ann. Univ. Sci. Budapest. Sect. Comput.*, 26:127–135, 2006.

[11] C. N. Moore. *Summable series and convergence factors*. Dover Publications, Inc., New York, 1966.

[12] F. Móricz and A. H. Siddiqi. Approximation by Nörlund means of Walsh-Fourier series. *J. Approx. Theory*, 70(3):375–389, 1992.

[13] K. Nagy. Approximation by Cesàro means of negative order of Walsh-Kaczmarz-Fourier series. *East J. Approx.*, 16(3):297–311, 2010.

[14] K. Nagy. Approximation by Nörlund means of quadratical partial sums of double Walsh-Fourier series. *Anal. Math.*, 36(4):299–319, 2010.
[15] K. Nagy. Approximation by Nörlund means of Walsh-Kaczmarz-Fourier series. *Georgian Math. J.*, 18(1):147–162, 2011.
[16] K. Nagy. Approximation by Nörlund means of double Walsh-Fourier series for Lipschitz functions. *Math. Inequal. Appl.*, 15(2):301–322, 2012.
[17] J. Pál and P. Simon. On a generalization of the concept of derivative. *Acta Math. Acad. Sci. Hungar.*, 29(1-2):155–164, 1977.
[18] F. Schipp. Certain rearrangements of series in the Walsh system. *Mat. Zametki*, 18(2):193–201, 1975.
[19] P. Simon. Investigations with respect to the Vilenkin system. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 27:87–101 (1985), 1984.
[20] P. Simon. Cesaro summability with respect to two-parameter Walsh systems. *Monatsh. Math.*, 131(4):321–334, 2000.
[21] P. Simon. Strong convergence theorem for Vilenkin-Fourier series. *J. Math. Anal. Appl.*, 245(1):52–68, 2000.
[22] P. Simon and F. Weisz. Weak inequalities for Cesàro and Riesz summability of Walsh-Fourier series. *J. Approx. Theory*, 151(1):1–19, 2008.
[23] G. Tephnadze. A note on the Fourier coefficients and partial sums of Vilenkin-Fourier series. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, 28(2):167–176, 2012.
[24] G. Tephnadze. On the partial sums of Vilenkin-Fourier series. *Izv. Nats. Akad. Nauk Armenii Mat.*, 49(1):60–72, 2014.
[25] G. Tephnadze and T. R. Strong convergence theorem of cesàro means with respect to the walsh system. to appear.
[26] N. J. Vilenkin. On a class of complete orthonormal systems. *Amer. Math. Soc. Transl. (2)*, 28:1–35, 1963.
[27] F. Weisz. *Martingale Hardy spaces and their applications in Fourier analysis*, volume 1568 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.
[28] F. Weisz. Cesàro summability of one- and two-dimensional Walsh-Fourier series. *Anal. Math.*, 22(3):229–242, 1996.
[29] F. Weisz. Hardy spaces and Cesàro means of two-dimensional Fourier series. In *Approximation theory and function series (Budapest, 1995)*, volume 5 of *Bolyai Soc. Math. Stud.*, pages 353–367. János Bolyai Math. Soc., Budapest, 1996.
[30] F. Weisz. (C, α) summability of Walsh-Fourier series. *Anal. Math.*, 27(2):141–155, 2001.
[31] F. Weisz. θ-summability of Fourier series. *Acta Math. Hungar.*, 103(1-2):139–175, 2004.

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István Blahota,
Institute of Mathematics and Computer Sciences,
University of Nyíregyháza,
P.O. Box 166, Nyíregyháza, H-4400,
Hungary.
E-mail address: blahota.istvan@nye.hu

Giorgi Tephnadze,
Department of Mathematics,
Faculty of Exact and Natural Sciences,
Tbilisi State University,
Chavchavadze str. 1, Tbilisi 0128,
Georgia
and
Department of Engineering Sciences and Mathematics,
Luleå University of Technology,
SE-971 87 Luleå,
Sweden.
E-mail address: giorgitephnadze@gmail.com